
M1M1: Problem Sheet 6

Complex numbers

1. Put the following complex numbers into standard form, i.e., in the form $x + iy$ for some real x and y :

$$(a) \frac{1}{2+3i}; \quad (b) \frac{1}{2-3i}; \quad (c) \frac{1-i}{1+i}; \quad (d) \frac{1+i}{1-i}; \quad (e) \frac{1}{i^5};$$
$$(f) \frac{(1+i)(2+i)(3+i)}{(1-i)}; \quad (g) \frac{1}{1+\sqrt{3}i}; \quad (h) \sqrt{5+12i}.$$

2. Let $z_1 = -1 + 2i$ and $z_2 = 3 - 2i$. Find the standard form of the following complex numbers:

$$(a) 2z_1 - 3z_2; \quad (b) z_1z_2; \quad (c) \frac{z_1^2}{z_2}; \quad (d) |z_1^2z_2|.$$

Now let $z_1 = 1 - 3i$ and $z_2 = 3 - 2i$. Find the standard form of the following complex numbers:

$$(a) |z_1 + z_2|; \quad (b) |z_2|z_1; \quad (c) z_1 + |z_1|; \quad (d) \left| \frac{z_1}{z_2} \right|.$$

3. Write the following complex numbers in standard form:

$$(a) 2e^{i\pi/2}; \quad (b) 3e^{-i\pi}; \quad (c) 2e^{-i\pi/2}; \quad (d) 3e^{i\pi/4}; \quad (e) 2e^{i\pi/6}.$$

Write the following complex numbers in polar form:

$$(a) i; \quad (b) \frac{1+i}{\sqrt{2}}; \quad (c) -1 + i\sqrt{3}; \quad (d) 6 + 8i; \quad (e) -1.$$

4. By expressing $-1+i$ in polar form (i.e., in the form $re^{i\theta}$), find the standard form of the number $(-1+i)^{-8}$.

5. Show that

$$\frac{1 + \cos \theta + i \sin \theta}{1 - \cos \theta + i \sin \theta} = \cot \left(\frac{\theta}{2} \right) e^{i(\theta - \pi/2)}.$$

6. Given that $2 + i$ is a solution of the equation

$$z^4 - 2z^3 - z^2 + 2z + 10 = 0,$$

find the other solutions.

7. If $z = e^{i\theta}$, show that

$$\cos(n\theta) = \frac{1}{2} \left(z^n + \frac{1}{z^n} \right).$$

Hence, show that

$$\cos^6 \theta = \frac{1}{32} \cos(6\theta) + \frac{3}{16} \cos(4\theta) + \frac{15}{32} \cos(2\theta) + \frac{5}{16}.$$

8. Use the fact that

$$\cos(n\theta) = \operatorname{Re}[e^{in\theta}]$$

to establish the identity

$$1 + \cos(\theta) + \frac{\cos(2\theta)}{2!} + \frac{\cos(3\theta)}{3!} + \frac{\cos(4\theta)}{4!} + \dots = e^{\cos(\theta)} \cos(\sin(\theta)).$$

9. Sketch the following curves in the complex plane:

$$(a) |z - i| = |z - 1|; \quad (b) |z - i| = 2; \quad (c) \operatorname{Re}[z^2] = 1;$$

$$(d) z\bar{z} = 1; \quad (e) \arg \left[\frac{z+1}{z-1} \right] = \pm \frac{\pi}{2}, \quad z \neq \pm 1$$

10. Find the roots of the quadratic equation

$$z^{2i} + z^i + 1 = 0.$$

11. Three roots of a polynomial equation of degree 5 with real coefficients are $1, i \pm 1$. Find the equation.

12. Find all the values of $\log(1+i)$ and $\log(1+i)^{1/i}$.

13. Find all complex solutions to the following equations:

$$(a) e^z = -2; \quad (b) z^7 = -1; \quad (c) \cos z = \sqrt{2}.$$

14. Use de Moivre's theorem to show that

$$\begin{aligned} \cos(5\theta) &= 16 \cos^5(\theta) - 20 \cos^3(\theta) + 5 \cos(\theta), \\ \sin(5\theta) &= \sin \theta (16 \cos^4(\theta) - 12 \cos^2(\theta) + 1). \end{aligned}$$

15. Use the formula:

$$1 + z + z^2 + z^3 + z^4 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

to prove Lagrange's identity:

$$\sum_{n=0}^N \cos(n\theta) = \frac{1}{2} + \frac{\sin((N+1/2)\theta)}{2 \sin(\theta/2)}.$$