M1M1: Problem Sheet 6: SOLUTIONS

Complex numbers

1.(a) $\dfrac{1}{2+3i} = \dfrac{1}{2+3i}\left(\dfrac{2-3i}{2-3i}\right) = \dfrac{2-3i}{13};$

(b) $\dfrac{1}{2-3i} = \dfrac{1}{\overline{2+3i}} = \dfrac{2+3i}{13};$

(c) $\dfrac{1-i}{1+i} = \dfrac{1-i}{1+i}\left(\dfrac{1-i}{1-i}\right) = -i;$

(d) $\dfrac{1+i}{1-i} = \dfrac{\overline{1-i}}{\overline{1+i}} = i;$

(e) $\dfrac{1}{i^5} = \dfrac{1}{i} = -i;$

(f) $\dfrac{(1+i)(2+i)(3+i)}{1-i} = \dfrac{(1+i)(2+i)(3+i)}{1-i}\left(\dfrac{1+i}{1+i}\right) = -5+5i;$

(g) $\dfrac{1}{1+i\sqrt{3}} = \dfrac{1}{1+i\sqrt{3}}\left(\dfrac{1-i}{1-i}\right) = \dfrac{1-i\sqrt{3}}{4}$

(h) $\sqrt{5+12i} = \left(13e^{i\tan^{-1}(12/5)}\right)^{1/2}$

$$= \sqrt{13}\left(\cos\left(\frac{1}{2}\tan^{-1}(12/5)\right) + i\sin\left(\frac{1}{2}\tan^{-1}(12/5)\right)\right)$$

2.

(a) $2z_1 - 3z_2 = 2(-1+2i) - 3(3-2i) = -11+10i;$

(b) $z_1z_2 = (-1+2i)(3-2i) = 1+8i;$

(c) $\dfrac{z_1^2}{z_2} = \dfrac{(-1+2i)^2}{(3-2i)}\dfrac{(-1+2i)^2}{(3-2i)}\left(\dfrac{3+2i}{3+2i}\right) = -\dfrac{1}{13} - \dfrac{18i}{13};$

(d) $|z_1^2z_2| = |z_1|^2|z_2| = 5\sqrt{13};$

(e) $|z_1+z_2| = |1-3i+3-2i| = \sqrt{41};$

(f) $|z_2|z_1 = (1-3i)\sqrt{13} = \sqrt{13} - 3\sqrt{13}i;$

(g) $z_1 + |z_1| = 1-3i+\sqrt{10} = 1+\sqrt{10}-3i$

(h) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|1-3i|}{|3-2i|} = \dfrac{\sqrt{10}}{\sqrt{13}}.$

3.

$$(a)\ 2e^{i\pi/2} = 2i; \quad (b)\ 3e^{-i\pi} = -3; \quad (c)\ 2e^{-i\pi/2} = -2i;$$

$$(d)\ 3e^{i\pi/4} = \frac{3}{\sqrt{2}}(1+i)\,; \quad (e)\ 2e^{i\pi/6} = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right);$$

$$(f) i = e^{i\pi/2+2k\pi i}, \quad k \text{ any integer;} \quad (g)\frac{1+i}{\sqrt{2}} = e^{i\pi/4+2k\pi i}, \quad k \text{ any integer;}$$

$$(h) -1+\sqrt{3}i = 2e^{2i\pi/3+2k\pi i}, \quad k \text{ any integer;}$$

$$(i) 6+8i = 10e^{i\tan^{-1}(8/6)+2k\pi i}, \quad k \text{ any integer;}$$

$$(j) -1 = e^{i\pi+2k\pi i}, \quad k \text{ any integer.}$$

4.

$$(a)\ -1+i = \sqrt{2}e^{3\pi i/4}; \quad (b)\ (-1+i)^8 = \left(\sqrt{2}e^{3\pi i/4}\right)^{-8} = \frac{1}{16}.$$

5.

$$\begin{aligned}(a)\frac{1+\cos\theta + i\sin\theta}{1-\cos\theta+i\sin\theta} &= \frac{2\cos^2(\theta/2)+2i\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)+2i\sin(\theta/2)\cos(\theta/2)} \\ &= \frac{\cos(\theta/2)e^{i\theta/2}}{i\sin(\theta/2)e^{-i\theta/2}} \\ &= \cot(\theta/2)e^{i\theta-i\pi/2}.\end{aligned}$$

6. Polynomial equation has real coefficients so another root is $2-i$. Therefore

$$(z-(2+i))(z-(2-i)) = z^2-4z+5$$

must be a factor of the given polynomial. In fact,

$$z^4-2z^3-z^2+2z+10 = (z^2-4z+5)(z^2+2z+2),$$

so the two other roots are roots of the quadratic $z^2+2z+2=0$ which are $-1\pm i$.

7. Let $z = e^{i\theta}$ where θ is real. For any integer n,

$$\begin{aligned} z^n &= e^{in\theta} = \cos n\theta + i\sin n\theta; \\ z^{-n} &= e^{-in\theta} = \cos n\theta - i\sin n\theta.\end{aligned}$$

Adding these two equations gives

$$\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right).$$

Now,

$$2\cos\theta = z + \frac{1}{z},$$

which implies that

$$\begin{aligned} 2^6\cos^6\theta &= \left(z + \frac{1}{z}\right)^6 \\ &= z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}. \end{aligned}$$

On use of the first result, this is

$$2^6\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20.$$

Finally, we get

$$\cos^6\theta = \frac{\cos 6\theta}{32} + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}.$$

8. Consider

$$e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta}\left(\cos(\sin\theta) + i\sin(\sin\theta)\right).$$

But, on use of the series expansion of the exponential,

$$e^{e^{i\theta}} = 1 + e^{i\theta} + \frac{e^{2i\theta}}{2!} + \frac{e^{3i\theta}}{3!} + ...$$

which implies that

$$\mathrm{Re}[e^{e^{i\theta}}] = 1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos(3\theta)}{3!} + ...$$

Therefore

$$e^{\cos\theta}\cos(\sin\theta) = 1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos(3\theta)}{3!} + ...$$

9. (a) The equation $|z - i| = |z - 1|$ corresponds, geometrically, to all those points z which are equidistant from the point 1 and the point i in the complex plane. This is clearly the line $y = x$.

(b) The equation $|z - i| = 2$ corresponds, geometrically, to all those points z which are distance 2 away from $z = i$. This is a circle, centred at i of radius 2.

(c) The equation $\text{Re}[z^2] = 1$ is equivalent to the equation $x^2 - y^2 = 1$ which is a rectangular hyperbola.

(d) The equation $z\bar{z} = 1$ is the same as $|z|^2 = 1$ which corresponds to all those points z which are unit distance from the origin. It is the unit circle in the z-plane.

(e) Let

$$w = \frac{z+1}{z-1}$$

so that the equation is equivalent to $\arg[w] = \pm\pi/2$. But this corresponds to all the points w which are on the imaginary axis (excluding $w = 0, \infty$). Thus,

$$\bar{w} = -w$$

or,

$$\frac{\bar{z}+1}{\bar{z}-1} = -\frac{z+1}{z-1}$$

On rearrangement, this equation becomes $|z|^2 = 1$, so the curve is again just the unit circle in the z-plane but with the points $z = \pm 1$ omitted.

10. Solving this quadratic for z^i yields

$$z^i = \frac{-1 \pm i\sqrt{3}}{2} = e^{\pm 2\pi i/3 + 2k\pi i},$$

where k is any integer. It follows that

$$z = e^{\pm 2\pi/3 + 2k\pi},$$

where k is any integer.

11. One factor of the degree 5 polynomial is

$$(z-1)(z-(i+1))(z-(i-1)) = (z-1)(z^2 - 2iz - 2)$$

from which we deduce, from the fact that the polynomial is real, that the full polynomial equation is

$$(z-1)(z^2-2iz-2)(z^2+2iz-2)=(z-1)(z^4+4)=0.$$

12. Let $z=\log(1+i)$ then

$$e^z=1+i=\sqrt{2}e^{i\pi/4+2k\pi i}=e^{\log 2+i\pi/4+2k\pi i},$$

where k is any integer. Therefore,

$$z=\log\sqrt{2}+i\frac{\pi}{4}+2k\pi i$$

where k is any integer. It is then clear that if $z=\log(1+i)^{1/i}$ then

$$z=-i\log\sqrt{2}+\frac{\pi}{4}+2k\pi$$

where k is any integer.

13. (a) Note that

$$e^z=-2=2e^{i\pi+2k\pi i}=e^{\log 2+i\pi+2k\pi i}$$

so that

$$z=\log 2+i\pi+2k\pi i$$

where k is any integer.

(b) Now,

$$z^7=-1=e^{i\pi+2k\pi i}$$

where k is any integer. So,

$$z=e^{i\pi/7+2k\pi i/7},\quad k=0,1,...,6.$$

(c) Finally,

$$\cos z=\frac{e^{iz}+e^{-iz}}{2}=\sqrt{2}$$

which implies the following quadratic for e^{iz},

$$e^{2iz}-2\sqrt{2}e^{iz}+1=0.$$

This has roots

$$e^{iz} = \sqrt{2} \pm 1 = e^{\log(\sqrt{2}\pm 1) + 2k\pi i}$$

where k is any integer. So,

$$z = 2k\pi - i\log(\sqrt{2} \pm 1)$$

where k is any integer.

14. Using De Moivre's theorem

$$\begin{aligned}\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + 5\cos^4\theta i\sin\theta - 10\cos^3\theta\sin^2\theta \\ &- 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\end{aligned}$$

Equating real and imaginary parts and making use of the identity $\sin^2\theta = 1 - \cos^2\theta$ yields the two results.

15. Let $z = e^{i\theta}$ where θ is real.

$$1 + z + z^2 + ... + z^N = \frac{1 - z^{N+1}}{1 - z}.$$

Taking the real part means that

$$1 + \cos\theta + \cos 2\theta + ... + \cos N\theta = \sum_{k=0}^{N} \cos k\theta = \mathrm{Re}\left[\frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}}\right].$$

But

$$\begin{aligned}\mathrm{Re}\left[\frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}}\right] &= \frac{1}{2}\left(\left[\frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}}\right] + \left[\frac{1 - e^{-i(N+1)\theta}}{1 - e^{-i\theta}}\right]\right) \\ &= \frac{1}{2}\left(\frac{(1 - e^{i(N+1)\theta})(1 - e^{-i\theta}) + (1 - e^{-i(N+1)\theta})(1 - e^{i\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})}\right) \\ &= \frac{1}{2}\left(\frac{1 - \cos\theta + \cos N\theta - \cos(N+1)\theta}{2(1 - \cos\theta)}\right).\end{aligned}$$

But

$$\begin{aligned}\cos N\theta - \cos(N+1)\theta &= -2\sin(-\theta/2)\sin((N + 1/2)\theta) \\ &= 2\sin(\theta/2)\sin((N + 1/2)\theta).\end{aligned}$$

On substitution of this identity, the result

$$\sum_{k=0}^{N} \cos k\theta = \frac{1}{2} + \frac{\sin((N+1/2)\theta)}{2\sin(\theta/2)}$$

follows after some algebraic manipulation.