# M1M1: Problem Sheet 7

Integration

**1.** Integrate the following functions of $x$:

$$(a)\ \frac{x+1}{x}; \quad (b)\ \frac{x}{x+1}; \quad (c)\ \frac{x+1}{x-1};$$

**2.** Use partial fractions to find the integrals of

$$(a)\ \frac{1}{x(2-3x)}; \quad (b)\ \frac{x}{x^2-1}; \quad (c)\ \frac{x^2}{x^3-1}; \quad (d)\ \frac{1}{x(x^2+1)}.$$

**3.** Use the method of integration by substitution to integrate:

$$(a)\ \frac{e^x}{e^x+1}; \quad (b)\ \sin^2 x \cos x; \quad (c)\ \frac{\sin x}{1+\cos x}; \quad (d)\ \frac{1}{\sqrt{1-x^2}};$$

$$(e)\ \frac{1}{\sqrt{x^2-1}}; \quad (f)\ \cos x e^{\sin x}.$$

**4.** Use integration by parts to integrate:

$$(a)\ e^x \cos x; \quad (b)\ \log x; \quad (c)\ x^2 \cos x; \quad (d)\ \cos^{-1} x.$$

**5.** Integrate the following functions (using any appropriate method):

$$(a)\ \sqrt{x^2-1}; \quad (b)\ \frac{\sqrt{x}}{1+x}; \quad (c)\ \text{cosec}\ x; \quad (d)\ \frac{\tan^{-1}(x)}{1+x^2}$$

Evaluate the following definite integrals:

$$(e)\ \int_0^{\pi/2} \frac{dx}{5+4\cos x}; \quad (f)\ \int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)^2};$$

$$(g)\ \int_0^1 \frac{dx}{(1+x^2)^{3/2}}; \quad (h)\ \int_1^2 \frac{dx}{x^2+3x+1}.$$

**6.** Show that

$$\int_0^\pi \frac{xdx}{1+\cos^2 x} = \frac{\pi}{2}\int_0^\pi \frac{dx}{1+\cos^2 x} = \frac{\pi^2}{2^{3/2}}.$$

**7.** Show that

$$\int_0^\pi \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2-1}}, \qquad \alpha > 1.$$

**8.** If

$$F_n = \int_0^1 x^n e^x dx$$

show that

$$F_n = e - nF_{n-1}, \quad n = 1, 2, ...$$

Hence, evaluate

$$\int_0^1 x^4 e^x dx.$$

**9.** If

$$I_n = \int_0^\infty x^n e^{-x^2} dx$$

where $n$ is a positive integer, show that

$$I_{n+1} = \frac{n}{2} I_{n-1}.$$

Hence, evaluate

$$\int_0^\infty x^5 e^{-x^2} dx.$$

**10.** Given that

$$u_n(x) = \int x^n \cos x dx, \qquad v_n(x) = \int x^n \sin x dx,$$

by performing a single integration, show that

$$u_n(x) = x^n \sin x - n v_{n-1}(x)$$
$$v_n(x) = -x^n \cos x + n u_{n-1}(x)$$

Hence, evaluate

$$\int x^4 \sin x dx.$$

**11.** If

$$I_n = \int_0^{\pi/4} \tan^n x dx,$$

show that

$$I_n = \frac{1}{n-1} - I_{n-2},$$

Hence, evaluate $I_5$.

**12.** The integrals $I(t_0)$ and $J(t_0)$ are defined as

$$I(t_0) = \int_{-\infty}^{\infty} \frac{dt}{\cosh t + \cosh t_0}, \qquad t_0 > 0,$$
$$J(t_0) = \int_{-\infty}^{\infty} \frac{dt}{\cosh t + \cos t_0}, \qquad 0 < t_0 < \pi/2.$$

Using the substitution $p = e^t$, or otherwise, show that

$$I(t_0) = \frac{2t_0}{\sinh t_0}, \quad J(t_0) = \frac{2t_0}{\sin t_0}.$$

**13.** If

$$I_n = \int_0^1 x^n \sqrt{1+x}\ dx, \quad n = 0, 1, 2, ....$$

show that

$$\frac{1}{n+1} < I_n < \frac{\sqrt{2}}{n+1},$$

and derive the recurrence relation

$$(3+2n)I_n = 2^{5/2} - 2nI_{n-1}.$$

Show that

$$I_n > \frac{\sqrt{2}}{n+3/2},$$

and hence that

$$nI_n \to \sqrt{2}.$$