M1M1: Problem Sheet 7: SOLUTIONS

Integration

1. (a) Integrand is rational function, so put into partial fraction form:

$$\int \frac{x+1}{x}\, dx = \int \left(1 + \frac{1}{x}\right) dx = x + \log x + c.$$

(b) Similarly,

$$\int \frac{x}{x+1} dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \log(x+1) + c.$$

(c) Also,

$$\int \frac{x+1}{x-1} dx = \int \left(1 + \frac{2}{x-1}\right) dx = x + 2\log(x-1) + c.$$

2. (a) Put integrand into partial fraction form:

$$\int \frac{dx}{x(2-3x)} = \int \left(\frac{1}{2x} + \frac{3}{2(2-3x)}\right) = \frac{1}{2}\log x - \frac{1}{2}\log(2-3x) + c.$$

(b) This is a logarithmic derivative, so

$$\int \frac{x dx}{x^2-1} = \frac{1}{2}\log(x^2-1) + c.$$

(c) Another logarithmic derivative,

$$\int \frac{x^2 dx}{x^3-1} = \frac{1}{3}\log(x^3-1) + c.$$

(d) Put integrand into partial fraction form:

$$\int \frac{dx}{x(x^2+1)} = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) = \log x - \frac{1}{2}\log(x^2+1) + c.$$

3. (a) Use substitution $u = e^x$,

$$\int \frac{e^x}{e^x+1} dx = \int \frac{du}{u+1} = \log(u+1) + c = \log(e^x+1) + c.$$

(b) Let $u = \sin x$ then

$$\int \sin^2 x \cos x dx = \int u^2 du = \frac{u^3}{3} + c = \frac{\sin^3 x}{3} + c.$$

(c) Let $u = \cos x$ then

$$\int \frac{\sin x}{1+\cos x} dx = -\int \frac{du}{1+u} = -\log(1+u) + c = -\log(1+\cos x) + c.$$

(d) Let $x = \sin\theta$ then

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \theta + c = \sin^{-1}(x) + c.$$

(e) Let $x = \cosh\theta$ then

$$\int \frac{dx}{\sqrt{x^2-1}} = \int d\theta = \theta + c = \cosh^{-1}(x) + c.$$

(f) Let $u = \sin x$ then

$$\int \cos x e^{\sin x} dx = \int e^u du = e^u + c = e^{\sin x} + c.$$

4. (a) On integration by parts twice,

$$\begin{aligned} I = \int e^x \cos x dx &= \Big[e^x \cos x\Big] + \int e^x \sin x dx \\ &= \Big[e^x \cos x\Big] + \left\{\Big[e^x \sin x\Big] - \int e^x \cos x dx\right\} \\ &= \Big[e^x \cos x\Big] + \Big[e^x \sin x\Big] - I \end{aligned}$$

Rearranging,

$$I = \frac{1}{2} e^x \Big[\cos x + \sin x\Big].$$

(b) Similarly,

$$\int \log x dx = \Big[x \log x\Big] - \int \frac{x}{x} dx = x\left(\log x - 1\right) + c.$$

(c) On integrating by parts twice,

$$\begin{aligned} I = \int x^2 \cos x dx &= \left[x^2 \sin x\right] - \int 2x \sin x dx \\ &= \left[x^2 \sin x\right] - \left[-2x \cos x\right] - 2\int \cos x dx \\ &= x^2 \sin x + 2x \cos x - 2\sin x + c. \end{aligned}$$

(d) On integrating by parts,

$$\int \cos^{-1} x dx = \left[x \cos^{-1} x\right] + \int \frac{x}{\sqrt{1-x^2}} dx = x\cos^{-1} x - \sqrt{1-x^2} + c.$$

5. (a) Let $x = \cosh\theta$,

$$\int \sqrt{x^2-1} dx = \int \sinh^2\theta d\theta = \int \frac{\cosh 2\theta - 1}{2} d\theta = \frac{\sinh 2\theta}{4} - \frac{\theta}{2} + c.$$

(b) Let $u^2 = x$,

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x} dx = \int \frac{2u^2}{1+u^2} du = \int \left(2 - \frac{2}{1+u^2}\right) du &= 2u - 2\tan^{-1} u + c \\ &= 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + c. \end{aligned}$$

(c) Use $t = \tan(x/2)$,

$$\begin{aligned} \int \operatorname{cosec} x dx = \int \frac{dx}{\sin x} = \int \frac{1+t^2}{2t} \frac{2dt}{1+t^2} = \int \frac{dt}{t} &= \log t + c \\ &= \log\tan(x/2) + c. \end{aligned}$$

(d) By inspection

$$\int \frac{\tan^{-1} x}{1+x^2} dx = \frac{1}{2}\left(\tan^{-1} x\right)^2 + c.$$

(e) Use $t = \tan(x/2)$ and then $t = 3u$,

$$\int_0^{\pi/2} \frac{dx}{5+4\cos x} = \int_0^1 \frac{2dt}{9+t^2} = \int_0^{1/3} \frac{6du}{9(1+u^2)} = \left[\frac{2}{3}\tan^{-1} u\right]_0^{1/3} = \frac{2}{3}\tan^{-1}(1/3).$$

(f) Either use t-substitution, or note that

$$\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$$

so that

$$\int_0^{\pi/2} \frac{dx}{(\cos x + \sin x)^2} = \int_0^{\pi/2} \frac{1}{2}\sec^2(x - \pi/4)dx = \left[\frac{1}{2}\tan(x - \pi/4)\right]_0^{\pi/2} = 1.$$

(g) Let $x = \tan\theta$,

$$\int_0^1 \frac{dx}{(1+x^2)^{3/2}} = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^3\theta}d\theta = \int_0^{\pi/4} \cos\theta d\theta = \Big[\sin\theta\Big]_0^{\pi/4} = \frac{1}{\sqrt{2}}.$$

(h) Completing the square in the denominator,

$$\int_1^2 \frac{dx}{x^2+3x+1} = \int_1^2 \frac{dx}{(x+3/2)^2 - 5/4}.$$

Now let $x + 3/2 = \frac{\sqrt{5}}{2}u$, then integral becomes

$$\frac{2}{\sqrt{5}}\int_{5/\sqrt{5}}^{7/\sqrt{5}} \frac{du}{u^2-1} = \frac{2}{\sqrt{5}}\int_{5/\sqrt{5}}^{7/\sqrt{5}} \frac{du}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right) = \frac{1}{\sqrt{5}}\log\left(\frac{7-\sqrt{5}}{5+\sqrt{5}}\right).$$

6. Substitute $y = \pi - x$ then integral I becomes

$$I = \int_0^{\pi} \frac{xdx}{1+\cos^2 x} = \int_{\pi}^{0} -\frac{(\pi - y)dy}{1+\cos^2 y} = \pi\int_0^{\pi} \frac{dx}{1+\cos^2 x} - I.$$

Therefore

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{1+\cos^2 x}.$$

Now use t-substitution $t = \tan(x/2)$. This yields

$$\frac{\pi}{2}\int_0^{\infty} \frac{dt(1+t^2)}{1+t^4}$$

But, putting into partial fraction form yields,

$$\frac{\pi}{2}\int_0^{\infty} \frac{1}{2}\left(\frac{1}{t^2 - \sqrt{2}t + 1} + \frac{1}{t^2 + \sqrt{2}t + 1}\right)dt.$$

Completing the squares in the denominators gives

$$\frac{\pi}{2}\int_0^\infty \frac{1}{2}\left(\frac{1}{(t-1/\sqrt{2})^2+1/2}+\frac{1}{(t+1/\sqrt{2})^2+1/2}\right)dt$$

Now substitute $u=\sqrt{2}(t-1/\sqrt{2})$ in first integral and $u=\sqrt{2}(t+1/\sqrt{2})$ in the second integral to get

$$\frac{\pi}{2}\int_0^\infty \frac{1}{2}\left(\frac{\sqrt{2}du}{1+u^2}+\frac{\sqrt{2}du}{1+u^2}\right) == \frac{\pi}{2}\left[\sqrt{2}\tan^{-1}u\right]_0^\infty = \frac{\pi^2}{2^{3/2}}.$$

7. Use t-substitution, $t=\tan(x/2)$ to get

$$\int_0^\infty \frac{2dt}{\alpha(1+t^2)-1+t^2}=\int_0^\infty \frac{2dt}{(\alpha+1)t^2+(\alpha-1)}.$$

Now use substitution

$$\sqrt{\frac{\alpha-1}{\alpha+1}}=t$$

so that integral becomes

$$\left(\frac{2}{\alpha+1}\right)\sqrt{\frac{\alpha-1}{\alpha+1}}\int_0^\infty \frac{\alpha+1}{\alpha-1}\frac{du}{1+u^2}=\frac{\pi}{\sqrt{\alpha^2-1}}.$$

8. Define

$$F_n=\int_0^1 x^n e^x dx=\left[x^n e^x\right]_0^1-\int_0^1 nx^{n-1}e^x dx = e-nF_{n-1}.$$

Repeated use of this gives $F_4=-48+33e$ where we have exploited the fact that

$$F_0=\left[e^x\right]_0^1=(e-1).$$

9. Define

$$\begin{aligned} I_n=\int_0^\infty x^n e^{-x^2}dx &= \left[-\frac{x^{n-1}}{2}e^{-x^2}\right]_0^\infty-\frac{n-1}{2}\int_0^\infty x^{n-1}e^{-x^2}dx \\ &= \frac{n-1}{2}I_{n-2}. \end{aligned}$$

On use of this,

$$I_5 = 2I_3 = 2I_1 = 1.$$

10. Define

$$w_n = u_n + iv_n$$

Then, on integration by parts,

$$w_n = \int x^n e^{ix} dx = \left[-ie^{ix} x^n \right] + in \int x^{n-1} e^{ix} dx$$
$$= -ix^n \cos x + x^n \sin x + inu_{n-1} - nv_{n-1}.$$

Equating real and imaginary parts gives

$$u_n = x^n \sin x - nv_{n-1}; \quad v_n = -x^n \cos x + nu_{n-1}.$$

On repeated use of this,

$$v_4 = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x.$$

11. Define

$$I_n = \int_0^{\pi/4} \tan^n x dx = \int_0^{\pi/4} \tan^{n-2} \left(\sec^2 x - 1\right) x dx = \left[\frac{\tan^{n-1} x}{n-1} \right] - I_{n-2}.$$

On repeated use of this,

$$I_5 = \log \sqrt{2} - \frac{1}{4}.$$

12. Making use of the substitution $p = e^t$,

$$I(t_0) = \int_0^{\infty} \frac{dp}{p} \frac{2}{(p + p^{-1} + 2\cosh t_0)} = \int_0^{\infty} \frac{2dp}{(p + \cosh t_0)^2 - \sinh^2 t_0}.$$

Now let

$$u \sinh t_0 = p + \cosh t_0$$

then

$$\int_{\coth t_0}^{\infty} \frac{2du}{\sinh t_0 (u^2 - 1)} = \frac{2}{\sinh t_0} \left[-\coth^{-1}(u) \right]_{\coth t_0}^{\infty} = \frac{2t_0}{\sinh t_0}.$$

In a similar way,

$$J(t_0) = \int_0^\infty \frac{2dp}{p^2 + 2p\cos t_0 + 1} = \int_0^\infty \frac{2dp}{(p + \cos t_0)^2 + \sin^2 t_0}$$

The substitution

$$u \sin t_0 = p + \cos t_0$$

leads to

$$\int_{\cot t_0}^\infty \frac{2dp}{\sin^2 t_0 (1 + u^2)} = -\frac{2}{\sin t_0}\Big[\cot^{-1} u\Big]_{\cot t_0}^\infty = \frac{2t_0}{\sin t_0}.$$

13. For $0 < x < 1$,

$$1 < \sqrt{1+x} < 2$$

so that

$$x^n < x^n\sqrt{1+x} < \sqrt{2}x^n.$$

Integrating this inequality gives

$$\int_0^1 x^n dx < \int_0^1 x^n\sqrt{1+x}dx < \int_0^1 \sqrt{2}x^n dx,$$

or

$$\frac{1}{n+1} < I_n < \frac{\sqrt{2}}{n+1}.$$

Now, on integration by parts,

$$\begin{aligned} I_n &= \left[\frac{2}{3}(1+x)^{3/2}x^n\right]_0^1 - \int_0^1 \frac{2n}{3}x^n(1+x)^{3/2}dx \\ &= \frac{2^{5/2}}{3} - \frac{2n}{3}\int_0^1 (x^{n-1} - x^n)\sqrt{1+x}dx. \end{aligned}$$

Rearranging

$$(3+2n)I_n = 2^{5/2} - 2nI_{n-1}.$$

Since $I_{n-1} > 0$,

$$I_n = \frac{2^{5/2} - 2nI_{n-1}}{3+2n} > \frac{2^{5/2}}{3+2n} > \frac{2^{3/2}}{3+2n} = \frac{\sqrt{2}}{n+3/2}$$

Therefore,

$$\frac{\sqrt{2}}{n+3/2} < I_n < \frac{\sqrt{2}}{n+1}$$

or

$$\frac{\sqrt{2}n}{n+3/2} < nI_n < \frac{\sqrt{2}n}{n+1}.$$

But the limit, as $n \to \infty$, of both upper and lower bounds is $\sqrt{2}$ so

$$\lim_{n\to\infty} nI_n = \sqrt{2}.$$