M1M1: Problem Sheet 8

First-order ordinary differential equations

1. Solve the following differential equations:

$$(a)\ \frac{dy}{dx} = \frac{2x}{(y+1)}; \qquad (b)\ \frac{dy}{dx} = (1+x)(1+y); \qquad (c)\ \frac{dy}{dx} = \frac{(1+y)}{(2+x)};$$

$$(d)\ \frac{dy}{dx} = \frac{(x+y)}{(x-2y)}; \qquad (e)\ xy\frac{dy}{dx} = x^2 + y^2; \qquad (f)\ y^2\frac{dy}{dx} = \frac{x^3+y^3}{x};$$

$$(g)\ x\frac{dy}{dx} = y + xe^{y/x}; \qquad (h)\ xy\frac{dy}{dx} = x^2 e^{-\frac{y^2}{x^2}} + y^2; \qquad (i)\ \frac{dy}{dx} = \frac{4\ln x}{y^2};$$

2. By making a substitution of the form $y = at + bx + c$, solve the following differential equations:

$$(a)\ \frac{dx}{dt} = \frac{t-x+2}{t-x+3}; \qquad (b)\ \frac{dx}{dt} = \frac{1-2x-t}{4x+2t}$$

3. Find the solutions of the following initial value problems:

$$(a)\ \frac{dx}{dt} - 2t(2x-1) = 0, \quad x(0) = 0; \quad (b)\ \frac{dx}{dt} + 5x - t = e^{-2t}, \quad x(-1) = 0;$$

$$(c)\ \frac{dx}{dt} + x\cot t = \cos t, \quad x(0) = 0; \quad (d)\ (1+t^2)\frac{dx}{dt} + 3xt = 5t, \quad x(1) = 2;$$

4. Solve

$$\frac{dy}{dx} = \frac{2}{x+e^y}.$$

5. By using a suitable substitution (or otherwise), find the solution of

$$y(xy+1) + x(1+xy+x^2y^2)\frac{dy}{dx} = 0.$$

6. Solve

$$\frac{r\tan\theta}{a^2 - r^2}\left(\frac{dr}{d\theta}\right) = 1, \quad r\left(\frac{\pi}{4}\right) = 0.$$

7. Find the general solution $R(t)$ of

$$\frac{d^2R}{dt^2} - \frac{2}{t}\frac{dR}{dt} = t^4.$$

8. Find $x(t)$ and $y(t)$ satisfying the coupled system of first-order differential equations given by

$$\frac{dy}{dt} + \frac{x}{y} = 1,$$
$$y\frac{dx}{dt} - x\frac{dy}{dt} = 2ty^2.$$

with $x(0) = 0$ and $y(0) = 1$.

9. (Difficult). Find the solution $R(t)$ of the nonlinear second-order equation

$$1 = R\frac{d^2R}{dt^2} + \frac{1}{2}\left(\frac{dR}{dt}\right)^2.$$

satisfying the conditions $R(0) = 1$ and $\frac{dR}{dt}(0) = 0$.
Hint: try finding $\frac{dR}{dt}$ as a function of $R(t)$.