M1M1: Problem Sheet 8: SOLUTIONS

First-order ordinary differential equations

1. (a) The equation,

$$\frac{dy}{dx} = \frac{2x}{y+1}$$

is separable, so solution is

$$\int (y+1)dy = \int 2x dx$$

or

$$y^2/2 + y = x^2 + c.$$

(b) Equation is separable so solution is

$$\int \frac{dy}{1+y} = \int (1+x)dx$$

or

$$\log(1+y) = x + \frac{x^2}{2} + c.$$

(c) Equation is separable so solution is

$$\int \frac{dy}{1+y} = \int \frac{dx}{2+x}$$

or

$$\log(1+y) = \log(2+x) + c.$$

(d) Equation is homogeneous so use substitution $u = y/x$ in

$$\frac{dy}{dx} = \frac{1+(y/x)}{1-2(y/x)}$$

and equation becomes

$$u + x\frac{du}{dx} = \frac{1+u}{1-2u},$$

which is separable so solution is

$$\int \frac{1-2u}{1+2u^2} du = \int \frac{dx}{x}$$

or

$$\int \left(\frac{1}{1+2u^2} - \frac{2u}{1+2u^2} \right) du = \int \frac{dx}{x}.$$

Second integral on left hand side is a logarithmic derivative. Now consider the substitution $\sqrt{2}u = t$ in the integral

$$\int \frac{du}{1+2u^2}$$

and one obtains

$$\int \frac{du}{1+2u^2} = \int \frac{dt}{\sqrt{2}} \frac{1}{1+t^2} = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}u) + c$$

so final solution is

$$\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}y/x) - \frac{1}{2} \log(1 + 2y^2/x^2) = \log x + c.$$

(e) Equation can be written

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

which is homogeneous. On use of the substitution $u = y/x$, equation becomes

$$u + \frac{xdu}{dx} = u + \frac{1}{u}$$

or

$$x\frac{du}{dx} = \frac{1}{u},$$

which is separable so solution is

$$\int udu = \int \frac{dx}{x}$$

which is separable so solution is

$$\frac{u^2}{2} = \frac{y^2}{2x^2} = \log x + c.$$

(f) Equation can be written

$$\frac{dy}{dx} = \frac{x^2}{y^2} + \frac{y}{x}$$

so it is homogeneous. Use $u = y/x$,

$$u + x\frac{du}{dx} = \frac{1}{u^2} + u$$

which is separable so solution is

$$\int u^2 dx = \int \frac{dx}{x}$$

or

$$\frac{u^3}{3} = \frac{y^3}{3x^3} = \log x + c.$$

(g) Equation can be written

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

so it is homogeneous. Use $u = y/x$,

$$u + x\frac{du}{dx} = u + e^u$$

or

$$\int e^{-u} du = \int \frac{dx}{x}.$$

Solution is therefore

$$-e^{-y/x} = \log x + c.$$

(h) Equation is homogeneous and can be written

$$\frac{dy}{dx} = \frac{1}{u}e^{-u^2} + u$$

where $u = y/x$. Therefore

$$u + x\frac{du}{dx} = u + \frac{e^{-u^2}}{u}$$

which is separable so solution is

$$\int ue^{u^2} dx = \int \frac{dx}{x}$$

or

$$\frac{1}{2}e^{y^2/x^2} = \log x + c.$$

(i) Equation is separable, so solution is

$$\int y dy = \int 4 \log x dx$$

or

$$\frac{y^2}{2} = 4\left(x \log x - x\right) + c,$$

where we have used integration by parts to integrate the right hand side.

2. (a) Let $y = t - x$ so that $x = t - y$. Then

$$\frac{dx}{dt} = 1 - \frac{dy}{dt} = \frac{y+2}{y+3}$$

so that

$$\frac{dy}{dt} = 1 - \frac{y+2}{y+3} = \frac{1}{y+3}.$$

Integration yields

$$\int (y+3) dy = \int dt$$

or

$$\frac{y^2}{2} + 3y = t + c$$

or

$$\frac{(t-x)^2}{2} + 3(t-x) = t + c.$$

(b) Let $y = t + 2x$ so that $x = (1/2)(y - t)$ then

$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{dy}{dt} - 1\right) = \frac{1-y}{2y}$$

or

$$\frac{dy}{dt} = \frac{1}{y}$$

thus

$$\frac{y^2}{2} = t + c.$$

Hence

$$\frac{(t+2x)^2}{2} = t + c.$$

3. (a) Equation is separable so

$$\int \frac{dx}{2x-1} = \int 2t dt$$

or

$$\frac{1}{2}\log(2x-1) = t^2 + c$$

Applying condition $x(0) = 0$ gives $c = 0$.
(b) Equation is first-order and linear so use integrating factor e^{5t}:

$$\frac{d(e^{5t}x)}{dt} = te^{5t} + e^{3t}.$$

Integration by parts on the right hand side gives

$$xe^{5t} = \frac{te^{5t}}{5} - \frac{e^{5t}}{25} + \frac{e^{3t}}{3} + c$$

while condition that $x(-1) = 0$ implies

$$c = \frac{6}{25e^5} - \frac{1}{3e^3}.$$

In summary,

$$x(t) = \frac{t}{5} - \frac{1}{25} + \frac{e^{-2t}}{3} + \left(\frac{6}{25e^5} - \frac{1}{3e^3}\right) e^{-5t}.$$

(c) Equation is linear and integrating factor is

$$\exp\left(\int \cot t dt\right) = \sin t.$$

Equation becomes

$$\frac{d}{dt}\left(x(t)\sin t\right) = \frac{\sin 2t}{2}.$$

Integration yields

$$x \sin t = -\frac{\cos 2t}{4} + c.$$

Condition $x(0) = 0$ gives $c = 1/4$ so

$$x(t) = -\frac{\cos 2t}{4\sin t} + \frac{1}{4\sin t}.$$

(d) Equation is linear with integrating factor $(1+t^2)^{3/2}$. Multiplying by this factor gives

$$\frac{d}{dt}\left((1+t^2)^{3/2}x(t)\right) = 5t(1+t^2)^{3/2}.$$

Integration yields

$$x(t)(1+t^2)^{3/2} = (1+t^2)^{5/2} + c$$

Condition $x(1) = 2$ gives $c = 0$. Hence

$$x(t) = (1+t^2).$$

4. Note that equation can be rewritten as

$$\frac{dx}{dy} = \frac{x+e^y}{2}$$

which is linear in the dependent variable x. It can be written as

$$\frac{dx}{dy} - \frac{x}{2} = \frac{e^y}{2}$$

or

$$\frac{d}{dy}\left(xe^{-y/2}\right) = \frac{e^{y/2}}{2}.$$

Integration yields

$$x = e^y + ce^{y/2}.$$

5. On use of the substitution $u = yx$,

$$\frac{dy}{dx} = \frac{1}{x}\frac{du}{dx} - \frac{u}{x^2}$$

so that

$$y(u+1) + x(1+u+u^2)\left(\frac{1}{x}\frac{du}{dx} - \frac{u}{x^2}\right) = 0.$$

On rearrangement,

$$u^3 = (1 + u + u^2)x\frac{du}{dx}.$$

This is separable with solution

$$\int \frac{dx}{x} = \int \frac{1 + u + u^2}{u^3} du$$

or

$$\log x = -\frac{1}{2x^2y^2} - \frac{1}{xy} + \log(xy) + c.$$

6. Equation is

$$\frac{r \tan\theta}{a^2 - r^2}\frac{dr}{d\theta} = 1$$

which can be rewritten as

$$\int \frac{r dr}{a^2 - r^2} = \int \frac{\cos\theta}{\sin\theta} d\theta.$$

Integration yields

$$-\frac{1}{2}\log(a^2 - r^2) = \log\sin\theta + c.$$

But $r(\pi/4) = 0$ so

$$c = \log(\sqrt{2}/a)$$

so final solution is

$$\frac{1}{\sqrt{a^2 - r^2}} = \frac{\sqrt{2}\sin\theta}{a}.$$

7. Let

$$u = \frac{dR}{dt}$$

then

$$\frac{du}{dt} - \frac{2u}{t} = t^4$$

which is linear. Integrating factor is t^{-2}. So

$$\frac{d}{dt}\left(\frac{u}{t^2}\right) = t^2.$$

On integration,

$$\frac{u}{t^2} = \frac{t^3}{3} + c.$$

Therefore

$$\frac{dR}{dt} = \frac{t^5}{3} + ct^2.$$

Another integration gives

$$R(t) = \frac{t^6}{18} + et^3 + f$$

where e and f are constants.

8. First note that

$$y\frac{dx}{dt} - x\frac{dy}{dt} = 2ty^2.$$

But

$$\frac{d}{dt}\left(\frac{x}{y}\right) = \frac{1}{y}\frac{dx}{dt} - \frac{x}{y^2}\frac{dy}{dt}$$

or

$$y^2\frac{d}{dt}\left(\frac{x}{y}\right) = y\frac{dx}{dt} - x\frac{dy}{dt}.$$

The equation is therefore equivalent to

$$-\frac{d}{dt}\left(\frac{x}{y}\right) = 2t$$

which integrates to

$$-\frac{x}{y} = t^2 + c$$

where $x(0) = 0, y(0) = 1$ implies $c = 0$. Hence $x = -t^2 y$. We therefore have

$$\frac{dy}{dt} = 1 + t^2$$

which means

$$y = \frac{t^3}{3} + t + \hat{c}$$

But $y(0) = 1$ implies $\hat{c} = 1$ so that

$$y(t) = \frac{t^3}{3} + t + 1;$$

$$x(t) = -\frac{t^5}{3} - t^3 - t^2.$$

9. First note that, by the chain rule,

$$\frac{d^2R}{dt^2} = \frac{dR}{dt}\frac{d}{dR}\left(\frac{dR}{dt}\right) = \frac{d}{dR}\left(\frac{1}{2}\left(\frac{dR}{dt}\right)^2\right).$$

On use of this, the equation becomes

$$1 = R\frac{d}{dR}\left(\frac{1}{2}\left(\frac{dR}{dt}\right)^2\right) + \frac{1}{2}\left(\frac{dR}{dt}\right)^2.$$

Now let

$$u = \frac{1}{2}\left(\frac{dR}{dt}\right)^2.$$

Then

$$1 = R\frac{du}{dR} + u = \frac{d}{dR}(uR).$$

Integrating,

$$R = uR + c$$

where $c = 1$ because $u = 0$ when $R = 1$. Therefore

$$u = 1 - \frac{1}{R}$$

or

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 = \frac{R-1}{R}.$$

This can be written

$$\frac{dR}{dt} = \sqrt{\frac{2(R-1)}{R}}$$

which is separable. Solution is

$$\int \sqrt{\frac{R}{R-1}}dR = \int \sqrt{2}dt = \sqrt{2}t + c.$$

To do integral on left hand side, let

$$R = \sec^2\theta$$

so that $R - 1 = \sec^2\theta - 1 = \tan^2\theta$ and

$$\frac{dR}{d\theta} = 2\sec^2\theta\tan\theta.$$

Then left hand integral becomes

$$\int \sqrt{\frac{R}{R-1}} dR = \int 2 \sec^3 \theta d\theta.$$

But, on integration by parts,

$$\begin{aligned} \int \sec^3 \theta d\theta = \int \sec\theta \sec^2\theta d\theta = \\ = \Big[\sec\theta \tan\theta \Big] - \int \sec\theta \tan^2\theta d\theta \\ = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1) d\theta \end{aligned}$$

which, on rearrangement, gives

$$2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \log(\sec\theta + \tan\theta).$$

Therefore

$$\sqrt{R(R-1)} + \log(\sqrt{R} + \sqrt{R-1}) = \sqrt{2}t + c$$

But when $t = 0$, $R = 1$ so $c = 0$. Final solution is

$$\sqrt{R^2 - R} + \log(\sqrt{R} + \sqrt{R-1}) = \sqrt{2}t.$$