
M4A32: Vortex Dynamics

Problem Sheet 4

(Point vortex dynamics on a sphere)

1. Let ζ be the usual stereographically-projected complex coordinate, i.e.,

$$\zeta = \cot\left(\frac{\theta}{2}\right) e^{i\phi}$$

If the Laplace-Beltrami operator is defined as

$$\nabla_{\Sigma}^2 \psi \equiv \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2}$$

show that

$$\nabla_{\Sigma}^2 \psi = (1 + \zeta\bar{\zeta})^2 \frac{\partial^2\psi}{\partial\zeta\partial\bar{\zeta}}$$

2. If u_{ϕ} and u_{θ} denote the zonal and meridional components of the velocity field, show that

$$u_{\phi} - iu_{\theta} = \frac{2\zeta}{\sin\theta} \psi_{\zeta}$$

3. The streamfunction $\psi(\theta, \phi; \theta', \phi')$ associated with a unit circulation point vortex situated at the point $(1, \theta', \phi')$ is known to be

$$\psi(\theta, \phi; \theta', \phi') = -\frac{1}{4\pi} \log(1 - \cos\gamma)$$

where

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')$$

Show that

$$1 - \cos\gamma = \frac{2(\zeta - \zeta')(\bar{\zeta} - \bar{\zeta}')}{(1 + \zeta\bar{\zeta})(1 + \zeta'\bar{\zeta}')}$$

where ζ and ζ' are the stereographically-projected points corresponding to $(1, \theta, \phi)$ and $(1, \theta', \phi')$ respectively.

4. N equal point vortices of strength κ are placed at equal distances around the latitude circle at $\theta = \theta_0$. The configuration rotates with a constant angular velocity Ω_N . Find Ω_N as a function of N , κ and θ_0 . Show that no matter how strong the vortices are, the configuration is completely stationary when situated at the equator.

5. Suppose that the fluid on the surface of the sphere is in pure solid body rotation about an axis through the north and south pole with constant angular velocity Ω . Show that the associated vorticity distribution is $-2\Omega \cos\theta$.

6. An equilibrium vorticity distribution on a sphere consists of the northern hemisphere having constant uniform vorticity ω_N and the southern hemisphere having constant uniform vorticity ω_S . Write down the streamfunction associated with the equilibrium as a function of the stereographically-projected coordinates ζ and $\bar{\zeta}$. What is the associated velocity field? Can ω_N and ω_S be chosen arbitrarily?