2 M3-4-5 A34 Assessed Problems # 2 Mar 2012

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 2.1. ***Adjoint and coadjoint actions of semidirect product*** $(S,T)\,\circledS\,\mathbb{R}$

Compute the adjoint and coadjoint actions for the semidirect-product group $(S,T)\,\circledS\,\mathbb{R}$ *obtained from the action of scaling* S *and translations* $T\in\mathbb{R}$ *on the real line.*

The group composition rule is

$$(\tilde{S},\tilde{v})(S,v)=(\tilde{S}S,\tilde{S}v+\tilde{v})\,, \tag{1}$$

which can be represented by multiplication of 2×2 *matrices. That is, the action of* G *on* $\mathbb{R}^2$ *has a matrix representation, given by*

$$(S,v)\mapsto\begin{pmatrix} S & v\\ 0 & 1\end{pmatrix}, \tag{2}$$

where $S\in\mathbb{R}$*. The matrix multiplication*

$$\begin{pmatrix} \tilde{S} & \tilde{v}\\ 0 & 1\end{pmatrix}\begin{pmatrix} S & v\\ 0 & 1\end{pmatrix}=\begin{pmatrix} \tilde{S}S & \tilde{S}v+\tilde{v}\\ 0 & 1\end{pmatrix} \tag{3}$$

agrees with the notation for $SE(2)=SO(2)\circledS\mathbb{R}^2$ *and* $SL(2,\mathbb{R})\circledS\mathbb{R}^2$*, except that rotations or* $SL(2,\mathbb{R})$ *actions on* $\mathbb{R}^2$ *in those cases are replaced here by a simple scaling of the real line. The inverse group element is given by*

$$(\tilde{S},\tilde{v})^{-1}=(\tilde{S}^{-1},-\tilde{S}^{-1}\tilde{v}) \tag{4}$$

and the identity element is $(S,v)_{\rm Id}=(1,0)$*.*

The semidirect-product group action of (S,T) *on* $\mathbb{R}$ *may be represented by the action of the* 2×2 *matrix representation of this group in (2) on an* extended vector $(r,1)^T\in$ *as*

$$\begin{pmatrix} S & v\\ 0 & 1\end{pmatrix}\begin{pmatrix} r\\ 1\end{pmatrix}=\begin{pmatrix} S\,r+v\\ 1\end{pmatrix},$$

The Lie group $G=(S,T)\circledS\mathbb{R}$ *has two parameters, the scale factor* $S\in\mathbb{R}$ *and the translation* $v\in\mathbb{R}$*.*

Problem statement

(a) *Derive the* AD, Ad *and* ad *actions for* $(S,T)\,\circledS\,\mathbb{R}$*. Use the notation* $(S'(0),v'(0))=(\sigma,\nu)$ *for Lie algebra elements.*

Answer

$$\begin{aligned}
{\rm AD}_{(\tilde{S},\tilde{v})}(S,v)&=(\tilde{S},\tilde{v})(S,v)(\tilde{S},\tilde{v})^{-1}\\
&=(\tilde{S}S,\tilde{S}v+\tilde{v})(\tilde{S}^{-1},-\tilde{S}^{-1}\tilde{v})\\
&=(\tilde{S}S\tilde{S}^{-1},-\tilde{S}S\tilde{S}^{-1}\tilde{v}+\tilde{S}v+\tilde{v})\\
&=(S,-S\tilde{v}+\tilde{S}v+\tilde{v})
\end{aligned}$$

Set $(S'(0),v'(0))=(\sigma,\nu)$. Then linearisation in un-tilde variables at the identity yields

$$\begin{aligned}
{\rm Ad}_{(\tilde{S},\tilde{v})}(\sigma,\nu)&=(\tilde{S},\tilde{v})(\sigma,\nu)(\tilde{S},\tilde{v})^{-1}\\
&=(\tilde{S}\sigma\tilde{S}^{-1},-\tilde{S}\sigma\tilde{S}^{-1}\tilde{v}+\tilde{S}\nu)\\
&=(\sigma,-\sigma\tilde{v}+\tilde{S}\nu)
\end{aligned}$$

Set $(\tilde{S}'(0),\tilde{v}'(0))=(\tilde{\sigma},\tilde{\nu})$. Then linearisation in tilde variables at the identity yields

$${\rm ad}_{(\tilde{\sigma},\tilde{\nu})}(\sigma,\nu)=(0,-\sigma\tilde{\nu}+\tilde{\sigma}\nu)$$

▲

(b) *Introduce a natural pairing in which to define the dual Lie algebra and derive its* Ad^* *and* ad^* *actions. Denote elements of the dual Lie algebra as* (α, β).

Answer

Elements (α, β) of the dual Lie algebra will be defined in terms of the natural pairing

$$\langle(\alpha, \beta),\, (\sigma, \nu)\rangle = \alpha\sigma + \beta\nu$$

Consequently, its Ad^* action is obtained as

$$\begin{aligned}\langle \mathrm{Ad}^*_{(\tilde{S},\tilde{v})}(\alpha, \beta),\, (\sigma, \nu)\rangle &= \langle(\alpha, \beta),\, \mathrm{Ad}_{(\tilde{S},\tilde{v})}(\sigma, \nu)\rangle \\ &= \langle(\alpha, \beta),\, (\sigma, -\sigma\tilde{v} + \tilde{S}\nu)\rangle = \alpha\sigma - \beta\sigma\tilde{v} + \beta\tilde{S}\nu \\ &= (\alpha - \beta\tilde{v})\sigma + \beta\tilde{S}\nu \\ &= \big\langle\big(\alpha - \beta\tilde{v}, \beta\tilde{S}\big),\, (\sigma, \nu)\big\rangle\end{aligned}$$

So $\mathrm{Ad}^*_{(\tilde{S},\tilde{v})}(\alpha, \beta) = (\alpha - \beta\tilde{v}, \beta\tilde{S})$
Likewise, its ad^* action is obtained as

$$\begin{aligned}\langle \mathrm{ad}^*_{(\tilde{\sigma},\tilde{\nu})}(\alpha, \beta),\, (\sigma, \nu)\rangle &= \langle(\alpha, \beta),\, \mathrm{ad}_{(\tilde{\sigma},\tilde{\nu})}(\sigma, \nu)\rangle \\ &= \langle(\alpha, \beta),\, (0, -\sigma\tilde{\nu} + \tilde{\sigma}\nu)\rangle = -\beta\sigma\tilde{\nu} + \beta\tilde{\sigma}\nu \\ &= \langle(-\beta\tilde{\nu}, \beta\tilde{\sigma}),\, (\sigma, \nu)\rangle\end{aligned}$$

So $\mathrm{ad}^*_{(\tilde{\sigma},\tilde{\nu})}(\alpha, \beta) = (-\beta\tilde{\nu}, \beta\tilde{\sigma})$

▲

(c) *Compute its coadjoint motion equations as Euler-Poincaré equations.*

Answer

With $\left(\frac{\partial \ell}{\partial \tilde{\sigma}}, \frac{\partial \ell}{\partial \tilde{\nu}}\right) = (\alpha, \beta)$ the coadjoint motion equation $\frac{d}{dt}(\alpha, \beta) = \mathrm{ad}^*_{(\tilde{\sigma},\tilde{\nu})}(\alpha, \beta) = (-\beta\tilde{\nu}, \beta\tilde{\sigma})$ implies the Euler-Poincaré equations

$$\left(\frac{d}{dt}\frac{\partial \ell}{\partial \tilde{\sigma}},\, \frac{d}{dt}\frac{\partial \ell}{\partial \tilde{\nu}}\right) = \mathrm{ad}^*_{(\tilde{\sigma},\tilde{\nu})}\left(\frac{\partial \ell}{\partial \tilde{\sigma}},\, \frac{\partial \ell}{\partial \tilde{\nu}}\right) = \left(-\frac{\partial \ell}{\partial \tilde{\nu}}\tilde{\nu},\, \frac{\partial \ell}{\partial \tilde{\nu}}\tilde{\sigma}\right)$$

▲

(d) *Legendre transform and identify the corresponding Lie-Poisson brackets*

Answer

The Legendre transform for this system is given by,

$$h(\alpha, \beta) = \langle \alpha, \tilde{\sigma}\rangle + \langle \beta, \tilde{\nu}\rangle - \ell(\tilde{\sigma}, \tilde{\nu})$$

whose differential yields

$$dh(\alpha, \beta) = \langle d\alpha, \tilde{\sigma}\rangle + \langle d\beta, \tilde{\nu}\rangle + \left\langle \alpha - \frac{\partial \ell}{\partial \tilde{\sigma}}, d\tilde{\sigma}\right\rangle + \left\langle \beta - \frac{\partial \ell}{\partial \tilde{\nu}}, d\tilde{\nu}\right\rangle$$

Consequently,

$$\left(\frac{\partial \ell}{\partial \tilde{\sigma}},\, \frac{\partial \ell}{\partial \tilde{\nu}}\right) = (\alpha, \beta) \quad \text{and} \quad \left(\frac{\partial h}{\partial \alpha},\, \frac{\partial h}{\partial \beta}\right) = (\tilde{\sigma}, \tilde{\nu}).$$

The Euler-Poincaré equations above then become the Hamiltonian equations

$$\left(\frac{d\alpha}{dt},\, \frac{d\beta}{dt}\right) = \mathrm{ad}^*_{\left(\frac{\partial h}{\partial \alpha}, \frac{\partial h}{\partial \beta}\right)}(\alpha, \beta) = \left(-\beta\frac{\partial h}{\partial \beta},\, \beta\frac{\partial h}{\partial \alpha}\right)$$

Thus, a function $f(\alpha,\beta)$ satisfies

$$\begin{aligned}
\frac{d}{dt}f(\alpha,\beta) &= \frac{\partial f}{\partial \beta}\frac{d\beta}{dt} + \frac{\partial f}{\partial \alpha}\frac{d\alpha}{dt} \\
&= \beta\left(\frac{\partial f}{\partial \beta}\frac{\partial h}{\partial \alpha} - \frac{\partial f}{\partial \alpha}\frac{\partial h}{\partial \beta}\right) \\
&= \left(\frac{\partial f}{\partial \log\beta}\frac{\partial h}{\partial \alpha} - \frac{\partial f}{\partial \alpha}\frac{\partial h}{\partial \log\beta}\right) \\
&= \{f, h\}
\end{aligned}$$

where the Lie-Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ for coadjoint motion turns out to be the canonical Poisson bracket in the variables $(\log\beta,\alpha)$. That is,

$$\frac{d\log\beta}{dt} = \{\log\beta,\, h\} = \frac{\partial h}{\partial \alpha} \quad\text{and}\quad \frac{d\alpha}{dt} = \{\alpha,\, h\} = -\,\frac{\partial h}{\partial \log\beta}\,.$$

▲

(e) Choose a Hamiltonian and solve its coadjoint motion equations.

Answer

Since we have reduced the problem to canonical equations in the variables $(\alpha, \log\beta)$, one may choose any Hamiltonian of interest in those variables. For example, one could choose the Hamiltonian for simple harmonic motion,

$$h = \frac{1}{2}\alpha^2 + \frac{1}{2}(\log\beta)^2$$

▲

Exercise 2.2. The Clebsch Momentum Map

Consider the Clebsch constrained variational principle

$$\delta S = 0\,, \qquad S = \int_a^b \ell(\xi) + \langle\!\langle\, p\,,\, \dot q - \Phi_\xi(q)\,\rangle\!\rangle\, dt\,,$$

where $\ell : \mathfrak{g}\times Q \to \mathbb{R}$ is the Lagrangian and $\Phi_\xi(q)\in TQ$ is the infinitesimal action of the Lie algebra $\mathfrak{g}$ on the manifold Q. The Lagrange multiplier $p\in T^*Q$ enforces the Clebsch constraint, $\dot q - \Phi_\xi(q) = 0$ at $q\in Q$, by using the pairing, $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle : T^*Q\times TQ\to\mathbb{R}$ (denoted with double brackets).

Define the momentum map $J(p,q) : T^*Q\to\mathfrak{g}$ by

$$J^\xi(p,q) : \langle\, J(p,q)\,,\,\xi\,\rangle = \langle\!\langle\, p_q\,,\, \Phi_\xi(q)\,\rangle\!\rangle$$

in terms of the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ (denoted with single brackets).

Problem statement:

(a) Derive the equations arising from the Clebsch approach with constrained variations of ξ, q and p in the action S above. Assume that the variation δq vanishes at the endpoints $t = a$ and $t = b$. Express your results in terms of $J(p,q)$ and $J^\xi(p,q)$.

Answer

$$
\begin{aligned}
\delta\xi : &\quad \frac{\partial \ell}{\partial \xi} = J(p,q) \\
\delta p : &\quad \dot q = \Phi_\xi(q) = \{ q\,,\, J^\xi \} \\
\delta q : &\quad \dot p = -\,\frac{\partial \Phi_\xi}{\partial q}^T \cdot p = \{ p\,,\, J^\xi \}
\end{aligned}
$$

▲

(b) Prove the canonical (p,q) Poisson bracket relation,

$$\left\{ J^\xi \,,\, J^\eta \right\} = J^{[\xi\,,\,\eta]}(p,q)\,,$$

where $[\xi\,,\,\eta]$ is the Lie algebra bracket.

Hint: Choose a 'Euclidean' basis for T^*Q in which it makes sense to write

$$J^\xi(p,q) = \langle\!\langle\, p_q\,,\, \Phi_\xi(q) \,\rangle\!\rangle = p_j \Phi^j_\xi(q)$$

Answer

The required Poisson bracket relation is expressed in the Euclidean basis as

$$\left\{ J^\xi \,,\, J^\eta \right\} = \left\{ p_j \Phi^j_\xi(q)\,,\, p_j \Phi^j_\eta(q) \right\} = p_j \Phi^j_{[\xi\,,\,\eta]}(q) = J^{[\xi\,,\,\eta]}(p,q)$$

The proof is based on the well-known expression for the commutator $[\Phi_\xi\,,\Phi_\eta]$ between two *vector fields* Φ_ξ and Φ_η.

$$
\begin{aligned}
\left\{ J^\xi \,,\, J^\eta \right\} &= \left\{ p_j \Phi^j_\xi(q)\,,\, p_j \Phi^j_\eta(q) \right\} \\
&= p_j \left(\frac{\partial \Phi^j_\xi}{\partial q^k} \Phi^k_\eta - \frac{\partial \Phi^j_\eta}{\partial q^k} \Phi^k_\xi \right) \\
&= -\, p_j \,[\Phi_\xi\,, \Phi_\eta]^j \\
&= p_j\, \Phi^j_{[\xi,\eta]} \\
&= J^{[\xi,\eta]}\,,
\end{aligned}
$$

in which we keep in mind that the commutator of VFs is *minus* the Jacobi-Lie bracket on their Lie algebra. The proof could also be accomplished by using the properties of *Hamiltonian* vector fields (HVFs), as follows. Define using $J^\xi(p,q) = p_j \Phi^j_\xi(q)$

$$
\begin{aligned}
X_{J^\xi} = \left\{ \,\cdot\,,\, J^\xi \right\} &= \dot q^k \frac{\partial}{\partial q^k} + \dot p_k \frac{\partial}{\partial q^k} \\
&= \frac{\partial J^\xi}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial J^\xi}{\partial q^k}\frac{\partial}{\partial q^k} \\
&= \Phi^k_\xi(q)\frac{\partial}{\partial q^k} - p_m \frac{\partial \Phi^m_\xi}{\partial q^k}\frac{\partial}{\partial p_k}
\end{aligned}
$$

This HVF is the cotangent lift of the VF $X_{\Phi_\xi} = \Phi^k_\xi(q)\partial/\partial q^k$. The properties of HVFs that we need are, first, that

$$
\begin{aligned}
[X_{J^\xi}\,,\, X_{J^\eta}] &= -\, X_{\{J^\xi, J^\eta\}} \\
&= -\, X_{J^{[\xi\,,\,\eta]}} = -\, X_{J^{\mathrm{ad}_\xi \eta}}
\end{aligned}
$$

and, second, that the commutator of HVFs is again *minus* their Jacobi-Lie bracket. (These minus signs may be irritating, but they *are* necessary.) The results of this part of the question will be useful in the next two parts.

▲

(c) Show that the momentum map $(q,p) \to J$ is Poisson. That is, $\big\{F \circ J, H \circ J\big\} = \big\{F, H\big\} \circ J$, for smooth functions F and H.

Answer

Suppose a Hamiltonian is defined on T^*Q by the composition $H \circ J = H(J(q,p))$. The induced dynamics for a smooth function $F \circ J = F(J(q,p))$ is given by,

$$\dot{F}(J) = \big\{F(J), H(J)\big\} = \frac{\partial F}{\partial J^\xi}\dot{J}^\xi = \frac{\partial F}{\partial J^\xi}\big\{J^\xi\,,\,J^\eta\big\}\frac{\partial H}{\partial J^\eta} = \frac{\partial F}{\partial J^\xi}J^{\mathrm{ad}_\xi \eta}\frac{\partial H}{\partial J^\eta}$$

$$= \frac{\partial F}{\partial J^\xi}\langle\, J\,,\, \mathrm{ad}_\xi \eta\,\rangle\frac{\partial H}{\partial J^\eta} = \Big\langle\, J\,,\, \mathrm{ad}_{\frac{\partial F}{\partial J}}\frac{\partial H}{\partial J}\,\Big\rangle$$

$$= \Big\langle\, J\,,\, \Big[\frac{\partial F}{\partial J}\,,\,\frac{\partial H}{\partial J}\Big]\,\Big\rangle =: \{F, H\}(J)$$

Thus, the momentum map $(q,p) \to J$ is Poisson and the canonical Poisson bracket induces the Lie-Poisson bracket. That is,

$$\big\{F \circ J, H \circ J\big\} = \big\{F, H\big\} \circ J\,.$$

This is what it means to say that the Clebsch momentum map is a Poisson map. ▲

(d) Derive the equation of motion for J from the canonical Hamiltonian equations for $J^\xi(q,p)$ given Hamiltonian $H(J(q,p))$ and express the $\dot{J}$ equation in Lie-Poisson Hamiltonian form. That is, derive the $\dot{J}$ equation expressed in terms of a Lie-Poisson bracket.

Answer

The time derivative of the momentum map component $J^\xi(q,p)$ may be computed using the canonical Poisson bracket of T^*Q as

$$\dot{J}^\xi = \big\{J^\xi, H(J)\big\} = \big\{J^\xi\,,\,J^\eta\big\}\frac{\partial H}{\partial J^\eta} = J^{\mathrm{ad}_\xi \eta}\frac{\partial H}{\partial J^\eta}$$

Hence,

$$\begin{aligned}
\dot{J}^\xi = \Big\langle\, \dot{J}\,,\,\xi\,\Big\rangle = \langle\, J\,,\,\mathrm{ad}_\xi\eta\,\rangle\frac{\partial H}{\partial J^\eta} &= -\,\langle\, J\,,\,\mathrm{ad}_\eta\xi\,\rangle\frac{\partial H}{\partial J^\eta}\\
&= -\,\Big\langle\, J\,,\,\mathrm{ad}_{\frac{\partial H}{\partial J}}\xi\,\Big\rangle\\
&= \Big\langle -\,\mathrm{ad}^*_{\frac{\partial H}{\partial J}}J\,,\,\xi\,\Big\rangle
\end{aligned}$$

and in Lie-Poisson form,

$$\dot{J} = -\,\mathrm{ad}^*_{\frac{\partial H}{\partial J}}J = \big\{J\,,\,H\big\}.$$

▲

(e) Use the Clebsch constrained variational principle to compute the momentum map when:

(i) $\Phi_\xi(q) = \xi q$ for a skew-symmetric 3×3 matrix $\xi \in \mathfrak{so}(3)$ and a vector $q \in \mathbb{R}^3$.

Answer $J = q \times p$ ▲

(ii) $\Phi_\xi(q) = [\,\xi,\,q\,]$ for skew-Hermitian $n\times n$ matrix $\xi \in \mathfrak{su}(n)$ and q an $n\times n$ Hermitian matrix.

Answer $J = [\, q,\, p\,]$ ▲

(iii) $\Phi_\xi(q) = -\mathcal{L}_\xi q$ for q in manifold Q and $\mathcal{L}_\xi q$ the Lie derivative of $q \in Q$ with respect to $\xi \in \mathfrak{g}$.

Answer $J = q \diamond p$ with operation $\diamond : T^*Q \times TQ \to \mathfrak{g}^*$ defined by

$$\langle\, p \diamond q \,,\, \xi \,\rangle = \langle\!\langle\, p \,,\, -\mathcal{L}_\xi q \,\rangle\!\rangle$$

▲

Exercise 2.3. (Clebsch approach for motion on $T^*(G\times V)$) versus EP for $T^*(G\circledS V)$)

In the first part of this problem, one assumes an action $G\times V\to V$ of the Lie group G on the vector space V that may represent a feature of the potential energy of the system. The Lagrangian then takes the form $L:TG\times V\to\mathbb{R}$. We assume that the Lagrangian is left invariant under the isotropy subgroup G_V that leaves invariant the elements of V.

One then computes the variations of the Clebsch-constrained action integral

$$S(\xi,q,\dot q,p)=\int_a^b\Big[l(\xi,q)+\Big\langle\!\!\Big\langle\,p\,,\,\dot q+\pounds_\xi q\,\Big\rangle\!\!\Big\rangle\Big]\,dt$$

for the reduced left-invariant Lagrangian $l(\xi,q):\mathfrak{g}\times V\to\mathbb{R}$. The point is to determine the effects of the presence of $q\in V$ in the Euler–Poincaré equations for motion on $T^*(G\times V)$) versus $T^*(G\circledS V)$).

These steps will lead you through the problem.

(a) Start by showing that stationarity of S implies the following set of equations:

$$\frac{\delta l}{\delta\xi}=p\diamond q\,,\quad \dot q=-\,\pounds_\xi q\,,\quad \dot p=\pounds_\xi^T p+\frac{\delta l}{\delta q}\,.$$

Answer

The variations of the action integral

$$S(\xi,q,\dot q,p)=\int_a^b\Big[l(\xi,q)+\Big\langle\!\!\Big\langle\,p\,,\,\dot q+\pounds_\xi q\,\Big\rangle\!\!\Big\rangle\Big]\,dt \tag{5}$$

are given (after an integration by parts in time) by

$$\begin{aligned}\delta S&=\int_a^b\Big\langle\,\frac{\delta l}{\delta\xi}\,,\,\delta\xi\,\Big\rangle+\Big\langle\!\!\Big\langle\,\frac{\delta l}{\delta p}\,,\,\delta p\,\Big\rangle\!\!\Big\rangle+\Big\langle\!\!\Big\langle\,\frac{\delta l}{\delta q}-\dot p+\pounds_\xi^T p\,,\,\delta q\,\Big\rangle\!\!\Big\rangle+\Big\langle\!\!\Big\langle\,p\,,\,\pounds_{\delta\xi}q\,\Big\rangle\!\!\Big\rangle\,dt\\&=\int_a^b\Big\langle\,\frac{\delta l}{\delta\xi}-p\diamond q\,,\,\delta\xi\,\Big\rangle+\Big\langle\!\!\Big\langle\,\delta p\,,\,\dot q+\pounds_\xi q\,\Big\rangle\!\!\Big\rangle-\Big\langle\!\!\Big\langle\,\dot p-\pounds_\xi^T p-\frac{\delta l}{\delta q}\,,\,\delta q\,\Big\rangle\!\!\Big\rangle\,dt\,.\end{aligned}$$

Thus, stationarity of this implicit variational principle implies the following set of equations:

$$\frac{\delta l}{\delta\xi}=p\diamond q\,,\quad \dot q=-\,\pounds_\xi q\,,\quad \dot p=\pounds_\xi^T p+\frac{\delta l}{\delta q}\,. \tag{6}$$

In these formulas, the notation distinguishes between the two types of pairings,

$$\langle\,\cdot\,,\,\cdot\,\rangle\,:\,\mathfrak{g}^*\times\mathfrak{g}\mapsto\mathbb{R}\quad\text{and}\quad\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle\,:\,T^*M\times TM\mapsto\mathbb{R}\,. \tag{7}$$

(The third pairing in the formula for δS is not distinguished because it is equivalent to the second one under integration by parts in time.)

▲

(b) Transform to the variable $\delta l/\delta\xi=p\diamond q$ and derive its equation of motion on the space $\mathfrak{g}^*\times V$,

Answer

The Euler–Poincaré equation emerges from elimination of (q,p) using these formulas and the properties of the diamond operation that arise from its definition, as follows, for any vector $\eta \in \mathfrak{g}$:

$$
\begin{aligned}
\Big\langle \frac{d}{dt}\frac{\delta l}{\delta \xi}\,,\,\eta\Big\rangle &= \frac{d}{dt}\Big\langle \frac{\delta l}{\delta \xi}\,,\,\eta\Big\rangle\,,\\
[\text{Definition of }\diamond\,] &= \frac{d}{dt}\Big\langle p\diamond q\,,\,\eta\Big\rangle = \frac{d}{dt}\Big\langle\!\!\Big\langle p\,,\,-\pounds_\eta q\Big\rangle\!\!\Big\rangle\,,\\
[\text{Equations (6)}] &= \Big\langle\!\!\Big\langle \pounds_\xi^T p + \frac{\delta l}{\delta q}\,,\,-\pounds_\eta q\Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle p\,,\,\pounds_\eta\pounds_\xi q\Big\rangle\!\!\Big\rangle\,,\\
[\text{Transpose, }\diamond\text{ and ad}\,] &= \Big\langle\!\!\Big\langle p\,,\,-\pounds_{[\xi,\,\eta]}q\Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle \frac{\delta l}{\delta q}\,,\,-\pounds_\eta q\Big\rangle\!\!\Big\rangle\\
&= \Big\langle p\diamond q\,,\,\mathrm{ad}_\xi\eta\Big\rangle + \Big\langle \frac{\delta l}{\delta q}\diamond q\,,\,\eta\Big\rangle\,,\\
[\text{Definition of ad}^*\,] &= \Big\langle\ \mathrm{ad}^*_\xi\frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta q}\diamond q\,,\,\eta\Big\rangle\,.
\end{aligned}
$$

The diamond term on the right side of the Euler–Poincaré equation represents the effects of potential energy for this system. Thus,

$$
\begin{aligned}
\frac{d}{dt}\frac{\delta l}{\delta \xi} &= \mathrm{ad}^*_\xi\frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta q}\diamond q\,,\\
\frac{dq}{dt} &= -\,\pounds_\xi q\,.
\end{aligned}
\tag{8}
$$

▲

(c) Perform the Legendre transformation to derive the Lie–Poisson Hamiltonian formulation corresponding to $l(\xi,q)$.

Answer

The Legendre transformation for this system,

$$h(\mu,q) = \langle \mu,\,\xi\rangle - l(\xi,q)$$

$$\delta h(\mu,q) = \langle \delta\mu,\,\xi\rangle + \Big\langle \mu - \frac{\delta l}{\delta\xi}\,,\,\delta\xi\Big\rangle - \Big\langle\!\!\Big\langle \frac{\delta l}{\delta q},\,\delta q\Big\rangle\!\!\Big\rangle,$$

will replace $l(\xi,q)$ by $h(\mu,q)$, with $\frac{\delta l}{\delta\xi}=\mu$, $\frac{\delta h}{\delta\mu}=\xi$ and $\frac{\delta l}{\delta q} = -\frac{\delta h}{\delta q}$. The equations will be rewritten in these variables as

$$
\begin{aligned}
\frac{d\mu}{dt} &= \mathrm{ad}^*_{\delta h/\delta\mu}\mu - \frac{\delta h}{\delta q}\diamond q\,,\\
\frac{dq}{dt} &= -\,\pounds_{\delta h/\delta\mu}q\,.
\end{aligned}
$$

A function $f(\mu,q):\mathfrak{g}\times V\to\mathbb{R}$ evolves as

$$
\begin{aligned}
\frac{df}{dt} &= \Big\langle \frac{\delta f}{\delta\mu},\,\frac{d\mu}{dt}\Big\rangle + \Big\langle\!\!\Big\langle \frac{\delta f}{\delta q},\,\frac{dq}{dt}\Big\rangle\!\!\Big\rangle\\
&= \Big\langle \frac{\delta f}{\delta\mu},\,\mathrm{ad}^*_{\delta h/\delta\mu}\mu - \frac{\delta h}{\delta q}\diamond q\Big\rangle + \Big\langle\!\!\Big\langle \frac{\delta f}{\delta q},\,-\,\pounds_{\delta h/\delta\mu}q\Big\rangle\!\!\Big\rangle\\
&= \Big\langle \mu,\,\mathrm{ad}_{\delta h/\delta\mu}\frac{\delta f}{\delta\mu}\Big\rangle - \Big\langle\!\!\Big\langle \frac{\delta h}{\delta q},\,-\,\pounds_{\delta f/\delta\mu}q\Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle \frac{\delta f}{\delta q},\,-\,\pounds_{\delta h/\delta\mu}q\Big\rangle\!\!\Big\rangle\\
&= \Big\langle \mu,\,\mathrm{ad}_{\delta h/\delta\mu}\frac{\delta f}{\delta\mu}\Big\rangle + \Big\langle\!\!\Big\langle \pounds^T_{\delta f/\delta\mu}\frac{\delta h}{\delta q},\,q\Big\rangle\!\!\Big\rangle - \Big\langle\!\!\Big\langle \pounds^T_{\delta h/\delta\mu}\frac{\delta f}{\delta q},\,q\Big\rangle\!\!\Big\rangle\\
&= \Big\langle \mu,\,\Big[\frac{\delta f}{\delta\mu},\,\frac{\delta h}{\delta\mu}\Big]\Big\rangle + \Big\langle\!\!\Big\langle q,\,\pounds^T_{\delta f/\delta\mu}\frac{\delta h}{\delta q} - \pounds^T_{\delta h/\delta\mu}\frac{\delta f}{\delta q}\Big\rangle\!\!\Big\rangle\\
&= \big\{f\,,\,h\big\}\,,
\end{aligned}
$$

where the Lie-Poisson bracket $\{f\,,\,h\}$ is defined on the dual $\mathfrak{s}^*$ of the Lie algebra $\mathfrak{s}$ of the semidirect-product Lie group $S = G{\circledS}V$. To see this, recall that the (left) Lie algebra of the semidirect-product Lie group S is the semidirect-product Lie algebra, $\mathfrak{s} = \mathfrak{g}\,{\circledS}\,V$, whose Lie bracket is expressed as

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2],\, \xi_1 v_2 - \xi_2 v_1)\,,$$

where the induced action of $\mathfrak{g}$ on V is denoted by concatenation, as in $\xi_1 v_2$.
(Bit of a sign difficulty still to be ironed out here.) ▲

(d) *Compute the Euler–Poincaré equations on the space $\mathfrak{g}^* \times V^*$ for the semidirect product group $S = G{\circledS}V$ and discuss the differences between those equations and the Euler–Poincaré equations in the previous part.*

Answer

In terms of variational derivatives, the Euler–Poincaré equations for the semidirect product group $S = G{\circledS}V$ differ from those in (8) in the last equation, namely, they are, as in equation (7.1.7) in the text,

$$\begin{aligned} \frac{d}{dt}\frac{\delta l}{\delta \xi} &= \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta q} \diamond q\,, \\ \frac{d}{dt}\frac{\delta l}{\delta q} &= -\,\pounds_\xi \frac{\delta l}{\delta q}\,. \end{aligned} \tag{9}$$

After the Legendre transformation,

$$h(\mu, \beta) = \langle \mu,\, \xi\rangle + \langle\!\langle \beta, q\rangle\!\rangle - l(\xi, q)$$

$$\delta h(\mu, \beta) = \langle \delta\mu,\, \xi\rangle + \langle\!\langle \delta\beta, q\rangle\!\rangle + \left\langle \mu - \frac{\delta l}{\delta \xi}\,,\, \delta\xi \right\rangle + \left\langle\!\!\left\langle \beta - \frac{\delta l}{\delta q},\, \delta q \right\rangle\!\!\right\rangle$$

these become

$$\begin{aligned} \frac{d\mu}{dt} &= \mathrm{ad}^*_{\delta h/\delta\mu}\mu + \beta \diamond \frac{\delta h}{\delta \beta}\,, \\ \frac{d\beta}{dt} &= -\,\pounds_{\delta h/\delta\mu}\beta\,. \end{aligned}$$

A function $f(\mu, \beta) : \mathfrak{s}^* \to \mathbb{R}$ evolves as

$$\begin{aligned} \frac{df}{dt} &= \left\langle \frac{\delta f}{\delta \mu},\, \frac{d\mu}{dt} \right\rangle + \left\langle\!\!\left\langle \frac{\delta f}{\delta \beta},\, \frac{d\beta}{dt} \right\rangle\!\!\right\rangle \\ &= \left\langle \frac{\delta f}{\delta \mu},\, \mathrm{ad}^*_{\delta h/\delta\mu}\mu + \beta \diamond \frac{\delta h}{\delta \beta} \right\rangle + \left\langle\!\!\left\langle \frac{\delta f}{\delta \beta},\, -\,\pounds_{\delta h/\delta\mu}\beta \right\rangle\!\!\right\rangle \\ &= \left\langle \mu,\, \mathrm{ad}_{\delta h/\delta\mu}\frac{\delta f}{\delta \mu} \right\rangle - \left\langle\!\!\left\langle \beta,\, \pounds_{\delta f/\delta\mu}\frac{\delta h}{\delta \beta} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \beta,\, \pounds_{\delta h/\delta\mu}\frac{\delta f}{\delta \beta} \right\rangle\!\!\right\rangle \\ &= -\left\langle \mu,\, \left[\frac{\delta f}{\delta \mu},\, \frac{\delta h}{\delta \mu}\right] \right\rangle - \left\langle\!\!\left\langle \beta,\, \pounds_{\delta f/\delta\mu}\frac{\delta h}{\delta \beta} - \pounds_{\delta h/\delta\mu}\frac{\delta f}{\delta \beta} \right\rangle\!\!\right\rangle \\ &= \{f\,,\,h\}\,. \end{aligned}$$

Once again, this is the Lie-Poisson bracket $\{f\,,\,h\}$ defined on the dual $\mathfrak{s}^*$ of the Lie algebra $\mathfrak{s}$ of the semidirect-product Lie group $S = G{\circledS}V$, although the variables have changed from the previous case!

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