Variational principles for Deterministic Fluid Dynamics

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What are you going to learn in these lectures?

1. Tutorial on the Infinite-Dimensional Geometry of Fluid Dynamics
2. Flows, Pull-backs, Differential k-forms, Lie derivatives and all that
3. What is advection, mathematically? Familiar fluid examples!
4. Deterministic Advection in Kelvin's Circulation Theorem
5. Hamilton's principle for Deterministic Advection by Lie Transport (DALT)
6. How does variational calculus produce fluid flow equations?
7. The Hamiltonian side
8. If time remains: SALT (Stochastic Advection by Lie Transport)

Pull-back dynamics – Overview

- DALT employs the action on functions $f \in \Lambda^0$ and k-forms $\alpha \in \Lambda^k$ of time-dependent smooth maps ϕ_t via pull-back, denoted ϕ_t^*.
- Pull-back is defined as the composition of functional dependence from the right. For example, the expression

$$\phi_t^* f := f \circ \phi_t \,, \quad \text{or} \quad \phi_t^* f(x) = f(\phi_t(x)) \,,$$

  is called the *pull-back* of the function f by the smooth map ϕ_t.
- This notation will also be applied to k-forms and vector fields.
- Push-forward is the pull-back by the inverse map. For a function f

$$\phi_{t*} f := f \circ \phi_t^{-1} \,, \quad \text{so} \quad \phi_{t*} f(\phi_t(x)) = f(x) \,,$$

  which means the inverse of the pull-back is the push-forward.

What is a differential k-form on a manifold, $\Lambda^k(M)$?

Definition

Manifolds are spaces on which the rules of calculus apply.

Differential forms are objects which you can integrate.

A k-form $\alpha \in \Lambda^k$ on a smooth manifold M is defined by the antisymmetric wedge product $\wedge$ of k differential basis elements dx^{i_j}, $j = 1, \dots, k$, as

$$\alpha = \alpha_{i_1 \dots i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k(M)\,.$$

we sum on repeated indices, $dx^i \wedge dx^j = -dx^j \wedge dx^i$ and we take $i_1 < i_2 < \cdots < i_k$, so $\alpha_{i_1 \dots i_k}(x)$ is *antisymmetric* in neighbouring indices.
If $\alpha \in \Lambda^k(M)$ and $\beta \in \Lambda^l(M)$, then $\alpha \wedge \beta \in \Lambda^{k+l}(M)$, for $k + l \le \dim(M)$.

Theorem

The wedge product is natural under pull-back. That is,

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^* \alpha \wedge \phi_t^* \beta.$$

What is a vector field?

Definition (Vector field)

A **vector field** $X \in \mathfrak{X}(M)$ is a map : $M \to TM$ from a manifold M to its tangent space which assigns a vector $X(x) \in T_xM$ at any point $x \in M$.

Definition (Local basis of a vector field)

A **basis** of the vector space T_xM may be obtained by using the gradient operator, with **components** $X^j(x)$ given by (summing on repeated indices)

$$X = X^j(x)\frac{\partial}{\partial x^j} =: X^j(x)\partial_j \,, \quad \text{with} \quad j = 1, \dots, \dim(M)\,.$$

Definition (Vector fields & differentials have dual basis elements)

The differential of a function $f \in \Lambda^0$ is given by $df = \frac{\partial f}{\partial x^k}\, dx^k \in \Lambda^1$.

Three more natural operations on differential k-forms (Λ^k)

Three more basic operations are commonly applied to differential forms.

They are: exterior derivative (d), contraction ($\rfloor$) and Lie derivative ($\mathcal{L}_X$) in the direction of a vector field X.

(1) Exterior derivative ($d\alpha$) raises the degree:

$$d\Lambda^k \mapsto \Lambda^{k+1} \quad \text{and} \quad d^2 \to 0\,.$$

(2) Contraction ($X \rfloor \alpha$) with a vector field X lowers degree:

$$X \rfloor \Lambda^k \mapsto \Lambda^{k-1} \quad \text{and} \quad \frac{\partial}{\partial x^j} \rfloor dx^k = \delta^k_j \quad \text{(duality)}\,.$$

(3) Lie derivative ($\mathcal{L}_X\alpha$) by vector field X preserves degree:

$$\mathcal{L}_X\Lambda^k \mapsto \Lambda^k.$$

Geometrically, $\mathcal{L}_X\alpha := X \rfloor \mathrm{d}\alpha + \mathrm{d}(X \rfloor \alpha)$ (Cartan's formula)

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How pull-back dynamics leads to Lie derivatives

Under the action of a smooth invertible map ϕ_t on k-forms α, $\beta \in \Lambda^k(M)$, at a point $\mathbf{x} \in M$, the pull-back ϕ_t^* is natural for d, $\wedge$ and $\lrcorner$. That is,

$$
\begin{aligned}
\mathrm{d}(\phi_t^*\alpha) &= \phi_t^*\mathrm{d}\alpha\,,\\
\phi_t^*(\alpha\wedge\beta) &= \phi_t^*\alpha\wedge\phi_t^*\beta\,,\\
\phi_t^*(X\lrcorner\,\alpha) &= \phi_t^*X\lrcorner\,\phi_t^*\alpha\,.
\end{aligned}
$$

In addition, the Lie derivative $\mathcal{L}_X\alpha$ of a k-form $\alpha \in \Lambda^k(M)$ by the vector field X tangent to the flow ϕ_t on M with $\phi_t|_{t=0} = Id$ may be defined either dynamically or geometrically (by Cartan's formula) as

$$
\mathcal{L}_X\alpha = \frac{d}{dt}\bigg|_{t=0}(\phi_t^*\alpha) = X\lrcorner\,\mathrm{d}\alpha + \mathrm{d}(X\lrcorner\,\alpha)\,.
$$

Equality of dynamical and geometrical definitions of $\mathcal{L}_X$!
(This is how differential geometry becomes dynamical!)

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Equality of dynamical and geometrical definitions of $\mathcal{L}_X$

For example, in the case $\alpha(\mathbf{x}) = u_i(\mathbf{x})\mathrm{d}x^i \in \Lambda^1(\mathbb{R}^3)$, $i = 1, 2, 3$, this equivalence implies a well-known vector calculus identity, namely

$$
\begin{aligned}
\mathcal{L}_X(u_i(x)\mathrm{d}x^i) &:= \frac{d}{dt}\Big|_{t=0} \phi_t^*(u_i(x)\mathrm{d}x^i) \\
&= \left[\frac{\partial u_i(\phi_t(x))}{\partial \phi_t^j(x)}\frac{d\phi_t^j}{dt}\right]_{t=0} dx^i + u_i(x)\mathrm{d}\left[\frac{d}{dt}\phi_t^j(x)\right]_{t=0} \\
&= \left[\frac{\partial u_i(x)}{\partial x^j}X^j + u_j(x)\frac{\partial X^j(x)}{\partial x^i}\right]\mathrm{d}x^i \\
&= \left[(\mathbf{X}\cdot\nabla)\boldsymbol{u} + u_j\nabla X^j\right]\cdot \mathrm{d}\mathbf{x}\,, \\
&= \left[-\mathbf{X}\times \mathrm{curl}\boldsymbol{u} + \nabla(\mathbf{X}\cdot\boldsymbol{u})\right]\cdot \mathrm{d}\mathbf{x} \\
&= X \,\lrcorner\, \mathrm{d}(\boldsymbol{u}\cdot \mathrm{d}\mathbf{x}) + \mathrm{d}\big(X \,\lrcorner\, (\boldsymbol{u}\cdot \mathrm{d}\mathbf{x})\big)\,.
\end{aligned}
$$

This calculation yields the fundamental vector calculus identity of fluid dynamics and it is the basis of the Kelvin circulation theorem.

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What is advection, mathematically?

Definition (An advected quantity is invariant along a flow trajectory.)

When the pull-back relation is applied to this definition for $x_t = \phi_t(x_0)$

$$\alpha_0(x_0) = \alpha_t(x_t) = (\phi_t^* \alpha_t)(x_0)\,,$$

one finds that advected quantities satisfy the transport formula,

$$0 = \frac{d}{dt}\alpha_0(x_0) = \frac{d}{dt}(\phi_t^* \alpha_t)(x_0) = \phi_t^*(\partial_t + \mathcal{L}_X)\alpha_t(x_0) = (\partial_t + \mathcal{L}_X)\alpha_t(x_t)\,,$$

where the vector field $X = \dot{\phi}_t \phi_t^{-1}$ generates the flow map ϕ_t.
Equivalently, via the push-forward relation,

$$\alpha_t(x_t) = (\alpha_0 \circ \phi_t^{-1})(x_t) = ((\phi_t)_* \alpha_0)(x_t)\,,$$

implies the same transport formula,

$$\frac{d}{dt}\alpha_t(x_t) = \frac{d}{dt}(\phi_t)_* \alpha_0 = -(\mathcal{L}_X \alpha_t)(x_t)\,.$$

Examples of Deterministic Advection by Lie Transport

Definition (Lie derivative)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\phi_t^* K)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} K(\phi_t(x)) =: \mathcal{L}_u K(x)$$

$$\text{with } \frac{\mathrm{d}\phi_t(x)}{\mathrm{d}t}\Big|_{t=0} = u(x).$$

Example (Familiar examples from fluid dynamics)

(Functions) $(\partial_t + \mathcal{L}_u)\theta(\mathbf{x}) = \partial_t\theta + \mathbf{u}\cdot\nabla\theta\,,$

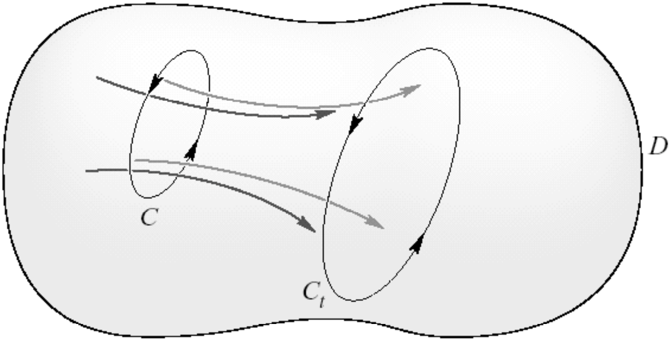
(1-forms) $$(\partial_t + \mathcal{L}_u)(\mathbf{v}(\mathbf{x})\cdot d\mathbf{x}) = \big(\partial_t\mathbf{v} + \mathbf{u}\cdot\nabla\,\mathbf{v} + v_j\nabla u^j\big)\cdot d\mathbf{x} = \big(\partial_t\mathbf{v} - \,\mathbf{u}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x}\,,$$

(2-forms) $(\partial_t + \mathcal{L}_u)(\boldsymbol{\omega}(\mathbf{x})\cdot d\mathbf{S}) = \big(\partial_t\boldsymbol{\omega} - \,\mathrm{curl}\,(\mathbf{u}\times\boldsymbol{\omega}) + \mathbf{u}\,\mathrm{div}\,\boldsymbol{\omega}\big)\cdot d\mathbf{S}\,,$

(3-forms) $(\partial_t + \mathcal{L}_u)(\rho(\mathbf{x})\,d^{\,3}x) = (\partial_t\rho + \mathrm{div}\,\rho\,\mathbf{u})\,d^{\,3}x\,.$

The **deterministic** Kelvin circulation theorem follows from Newton's law for the evolution of momentum/mass $\mathbf{v}$ concentrated on an **advecting material loop**, $c_t = \phi_t c_0$

$$\frac{d}{dt}\oint_{c_t}\mathbf{v}\cdot \mathrm{d}\mathbf{x} = \oint_{c_t}(\partial_t + \mathcal{L}_{u(t,\mathbf{x})})(\mathbf{v}\cdot d\mathbf{x}) = \oint_{c_t}\underbrace{\mathbf{f}\cdot \mathrm{d}\mathbf{x}}_{\text{Newton's Law}}$$



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Proof of the deterministic Kelvin's theorem

Proof.

Consider a closed loop moving with the material flow $c_t = \phi_t c_0$ with Eulerian velocity $\frac{d}{dt}\phi_t(x) = \phi_t^* u(t,x) = u(t,\phi_t(x))$.

Compute the time derivative of the loop momentum/mass

$$
\begin{aligned}
\frac{d}{dt}\oint_{c_t} \mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x} &= \oint_{c_0} \frac{d}{dt}\Big(\phi_t^*\big(\mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x}\big)\Big) \\
&= \oint_{c_0} \underbrace{\phi_t^*\Big(\big(\partial_t + \mathcal{L}_{u(t,\mathbf{x})}\big)(\mathbf{v}\cdot d\mathbf{x})\Big)}_{\text{Defines Lie derivative via product rule}} \\
&= \oint_{\phi_t c_0 = c_t} \big(\partial_t + \mathcal{L}_{u(t,\mathbf{x})}\big)(\mathbf{v}\cdot d\mathbf{x}) \\
&= \oint_{c_t} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}} = \oint_{c_0} \phi_t^*\Big(\mathbf{f}\cdot d\mathbf{x}\Big)
\end{aligned}
$$

□

Lagrangian v Eulerian Kelvin's theorem

Lagrangian Kelvin's theorem: In moving coordinates, $x_t = \phi_t(x)$,

$$\oint_{c_0} \frac{d}{dt}\Big(\phi_t^*\big(\mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x}\big)\Big) = \oint_{c_0} \phi_t^*\Big(\mathbf{f}\cdot d\mathbf{x}\Big)$$

Thus, in the moving frame,

$$\frac{d}{dt}\Big(\mathbf{v}(t,\phi_t(\mathbf{x}))\cdot d\phi_t(\mathbf{x})\Big) = \phi_t^*\Big(\mathbf{f}\cdot d\mathbf{x} + \nabla p\cdot d\mathbf{x}\Big)$$

Eulerian Kelvin's theorem: In spatially fixed coordinates, x, after using the pull-back formula for Lie derivative, $\mathcal{L}_u$,

$$\oint_{c_t} (\partial_t + \mathcal{L}_{u(t,\mathbf{x})})(\mathbf{v}\cdot d\mathbf{x}) = \oint_{c_t} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}}$$

Thus, in the fixed frame,

$$\Big(\partial_t\mathbf{v} + (\mathbf{u}\cdot\nabla)\,\mathbf{v} + v_j\nabla u^j\Big)\cdot d\mathbf{x} = \big(\mathbf{f} + \nabla p\big)\cdot d\mathbf{x}\,.$$

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Newton's Law for fluids implies Kelvin's theorem

Kelvin's theorem reveals the geometry of Newton's Law for fluids.

Namely, as we have seen in the last step of the proof of Kelvin's theorem,

$$\frac{d}{dt}\oint_{c_t}\mathbf{v}\cdot d\mathbf{x} = \oint_{c_t}(\partial_t + \mathcal{L}_{u(t,\mathbf{x})})(\mathbf{v}\cdot d\mathbf{x}) = \oint_{c_t}\underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}}$$

and the second relation implies that (modulo an exact differential dp)

$$(\partial_t + \mathcal{L}_{u(t,\mathbf{x})})(\mathbf{v}\cdot d\mathbf{x}) = \mathbf{f}\cdot d\mathbf{x} + dp$$

or, in coordinates and noting that $\mathcal{L}_u dx^j = d(\mathcal{L}_u x^j) = du^j = \nabla u^j \cdot d\mathbf{x}$,

$$\partial_t\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + v_j\nabla u^j = \mathbf{f} + \nabla p\,.$$

As we will see, $\mathbf{v} = \delta\ell/\delta\mathbf{u}$ arises from the physics of Hamilton's principle as the *momentum density associated with the Eulerian transport velocity* $\mathbf{u}(t,\mathbf{x})$ *which carries the material loop.*

For Hamilton's principle, we need the diamond operation

Definition

The operation $\diamond : V \times V^* \to \mathfrak{X}^*$ between tensor space elements $a \in V^*$ and $b \in V$ produces an element of $\mathfrak{X}(\mathcal{D})^*$, a one-form density, defined by

$$\Big\langle b \diamond a, u \Big\rangle_{\mathfrak{X}} = -\int_{\mathcal{D}} b \cdot \mathcal{L}_u \, a =: \Big\langle b \,,\, -\mathcal{L}_u \, a \Big\rangle_V \,,$$

where $\langle \cdot \,,\, \cdot \rangle_{\mathfrak{X}}$ denotes the symmetric, non-degenerate L^2 pairing between vector fields and one-form densities, which are dual with respect to this pairing. Likewise, $\langle \cdot \,,\, \cdot \rangle_V$ represents the corresponding L^2 pairing between dual elements of V and V^*.

Also, $\mathcal{L}_u a$ stands for the Lie derivative of an element $a \in V^*$ with respect to a vector field $u \in \mathfrak{X}(\mathcal{D})$, and $b \cdot \mathcal{L}_u \, a$ denotes the contraction between elements $b \in V$ and elements $a \in V^*$.

What does diamond ($\diamond$) mean operationally?

Example (Calculating diamond ($\diamond$) with vector calculus)

In $\mathbb{R}^3$, for $a = \theta \in \Lambda^0, A = \mathbf{A}\cdot d\mathbf{x} \in \Lambda^1, B = \mathbf{B}\cdot d\mathbf{S} \in \Lambda^2, \mathrm{D} = Dd^3x \in \Lambda^3$ and $\mathrm{S} = S_{ab}dx^a \otimes dx^b \in T^0_2(\mathrm{sym})$

$$
\begin{aligned}
\left(\frac{\delta \ell}{\delta a}\diamond a\right) = \Big[& -\frac{\delta \ell}{\delta \theta}\nabla\theta \cdot d\mathbf{x} + D\nabla\left(\frac{\delta \ell}{\delta D}\right)\cdot d\mathbf{x} \\
& + \left(\frac{\delta \ell}{\delta \mathbf{A}} \times \mathrm{curl}\,\mathbf{A} - \mathbf{A}\,\mathrm{div}\frac{\delta \ell}{\delta \mathbf{A}}\right)\cdot d\mathbf{x} \\
& + \left(\mathrm{curl}\frac{\delta \ell}{\delta \mathbf{B}} \times \mathbf{B} - \frac{\delta \ell}{\delta \mathbf{B}}\mathrm{div}\mathbf{B}\right)\cdot d\mathbf{x} \\
& + \left(\frac{\delta \ell}{\delta S_{ab}}S_{ab,k} - \left(\frac{\delta \ell}{\delta S_{ab}}S_{kb}\right)_{,a} - \left(\frac{\delta \ell}{\delta S_{ab}}S_{ka}\right)_{,b}\right)dx^k\Big] \otimes d^3x\,.
\end{aligned}
$$

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The variational derivative, what is it?

A functional $F(\rho)$ is defined as a map $F : \rho \in C^\infty(M) \to \mathbb{R}$.

The functional derivative of $F(\rho)$, denoted $\delta F/\delta\rho$, is defined by

$$\int \frac{\delta F}{\delta \rho}(x)\phi(x)\, dx = \lim_{\varepsilon \to 0} \frac{F[\rho + \varepsilon\phi] - F[\rho]}{\varepsilon} = \left[\frac{d}{d\varepsilon} F[\rho + \varepsilon\phi] \right]_{\epsilon = 0},$$

where ϕ is an arbitrary function.

The quantity $\varepsilon\phi$ is called the variation of ρ.

Since the variation is a linear operator, we can write the variation operationally as

$$\delta F(\rho) = \left\langle \frac{\delta F}{\delta \rho}\,,\, \delta\rho \right\rangle .$$

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Hamilton's principle for deterministic advection

Theorem (Variational principle for deterministic continuum dynamics)

Consider a deterministic path $x_t = \phi_t X$ with $\phi_t \in \mathrm{Diff}(\mathcal{D})$. The following two statements are equivalent:

(i) *The Clebsch-constrained Hamilton's variational principle holds on $\mathfrak{X}(\mathcal{D}) \times V^*$,*

$$\delta S := \delta \int_{t_1}^{t_2} \ell(\phi_t^* u, \phi_t^* a) + \left\langle \phi_t^* b \,,\, \frac{d}{dt}(\phi_t^* a) \right\rangle_V dt = 0\,,$$

(ii) *The Euler–Poincaré equations for continua hold, in the form*

$$\frac{d}{dt}\left(\phi_t^* \frac{\delta \ell}{\delta u}\right) = \phi_t^* \left(\partial_t \frac{\delta \ell}{\delta u} + \mathcal{L}_u \frac{\delta \ell}{\delta u}\right) = \phi_t^* \left(\frac{\delta \ell}{\delta a} \diamond a\right)\,,$$
$$\frac{d}{dt}(\phi_t^* a_t) = \phi_t^* \Big(\partial_t a_t + \mathcal{L}_u a_t\Big) = 0\,.$$

Proof.

Evaluating the variational derivatives at fixed time t and coordinate X yields the following relations when e.g. $\delta(\phi_t^* b) := \partial_\epsilon|_{\epsilon=0}(\phi_{t,\epsilon}^* b)$:

$$
\begin{aligned}
\delta(\phi_t^* b) \ &: 0 = \frac{d}{dt}(\phi_t^* a) = \phi_t^* \Big(\partial_t a_t + \mathcal{L}_u a_t \Big), \\
\delta(\phi_t^* a) \ &: 0 = -\frac{d}{dt}(\phi_t^* b) + \phi_t^* \left(\frac{\delta \ell}{\delta a} \right) \\
& \quad\ \ 0 = -\partial_t b_t + \mathcal{L}_u^T b_t + \phi_t^* \left(\delta \ell / \delta a \right) \\
\delta(\phi_t^* u) \ &: 0 = \frac{\delta \ell}{\delta(\phi_t^* u)} - (\phi_t^* b) \diamond (\phi_t^* a) \,.
\end{aligned}
$$

One then computes the motion equation via

$$
\frac{d}{dt} \frac{\delta \ell}{\delta(\phi_t^* u)} = \frac{d}{dt} \, (\phi_t^* b) \diamond (\phi_t^* a) + (\phi_t^* b) \diamond \frac{d}{dt} \, (\phi_t^* a) \,,
$$

$$
\text{leading to} \quad \phi_t^* \left(\partial_t \frac{\delta \ell}{\delta u} + \mathcal{L}_u \frac{\delta \ell}{\delta u} \right) = \phi_t^* \left(\frac{\delta \ell}{\delta a} \diamond a \right) \mathrm{d}t \,,
$$

Details of proof

Proof.

With notation $\phi_t^* a =: a_t$, one computes the motion equation in detail as

$$\left\langle \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u_t} \,,\, \xi \right\rangle_{\mathfrak{X}} = \langle \partial_t b_t \diamond a_t + b_t \diamond \partial_t a_t \,,\, \xi \rangle_{\mathfrak{X}}$$

$$\begin{aligned}
\left\langle \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u_t} - \frac{\delta \ell}{\delta a_t} \diamond a_t \,,\, \xi \right\rangle &= \left\langle \mathcal{L}_u^T b_t \diamond a_t \,,\, \xi \right\rangle + \langle b_t \diamond (-\mathcal{L}_u a_t) \,,\, \xi \rangle \\
&= \left\langle \mathcal{L}_u^T b_t \,,\, -\mathcal{L}_\xi a_t \right\rangle + \langle b_t \,,\, \mathcal{L}_\xi \mathcal{L}_u a_t \rangle \\
&= \langle b_t \,,\, -\mathcal{L}_u \mathcal{L}_\xi a_t \rangle + \langle b_t \,,\, \mathcal{L}_\xi \mathcal{L}_u a_t \rangle \\
&= \left\langle b_t \,,\, \mathcal{L}_{[\xi,u]} a_t \right\rangle = \langle b_t \diamond a_t \,,\, [u,\xi] \rangle \\
&= \langle b_t \diamond a_t \,,\, -\mathrm{ad}_u \xi \rangle = \langle -\mathrm{ad}_u^* (b_t \diamond a_t) \,,\, \xi \rangle
\end{aligned}$$

$b_t \diamond a_t = \delta\ell/\delta u_t$, $\mathrm{ad}_u^*(\delta\ell/\delta u_t) = \mathcal{L}_u(\delta\ell/\delta u_t)$ & pull-back formula yields

$$\left\langle \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} + \mathcal{L}_u \frac{\delta \ell}{\delta u} \,,\, \xi \right\rangle_{\mathfrak{X}} = \left\langle \frac{\delta \ell}{\delta a} \diamond a \,,\, \xi \right\rangle_{\mathfrak{X}} .$$

Hamiltonian formulation

We reach the Hamiltonian side via the Legendre transform.

$$\mu = \delta\ell/\delta u \quad \text{and} \quad h(\mu, a) = \langle \mu\,,\, u\rangle - \ell(u, a)\,.$$

Note that $\delta h/\delta u = 0$, by definition.

Taking variations of $h(\mu, a)$ yields

$$\begin{aligned}
\delta h(\mu, a) &= \Big\langle \delta\mu\,,\, \frac{\delta h}{\delta\mu}\Big\rangle + \Big\langle \frac{\delta h}{\delta a}\,,\, \delta a\Big\rangle \\
&= \langle \delta\mu\,,\, u\rangle - \Big\langle \frac{\delta\ell}{\delta a}\,,\, \delta a\Big\rangle + \Big\langle \mu - \frac{\delta\ell}{\delta u}\,,\, \delta u\Big\rangle\,.
\end{aligned}$$

Then, upon identifying corresponding terms, one verifies $\mu = \delta\ell/\delta u$ and finds the following variational derivatives of the Hamiltonian,

$$\frac{\delta h}{\delta\mu} = u \quad \text{and} \quad \frac{\delta h}{\delta a} = -\,\frac{\delta\ell}{\delta a}\,.$$

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Lie–Poisson bracket for DALT

With $\delta h/\delta\mu = u$ and $\delta h/\delta a = -\,\delta\ell/\delta a$, the EP equation

$$\frac{\partial}{\partial t}\frac{\delta\ell}{\delta u} + \mathcal{L}_u\frac{\delta\ell}{\delta u} = \frac{\delta\ell}{\delta a}\diamond a\,,$$

translates into the Lie–Poisson Hamiltonian form, as

$$\frac{\partial}{\partial t}\begin{bmatrix}\mu\\ a\end{bmatrix} = -\begin{bmatrix}\mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,)\diamond a\\ \mathcal{L}_{(\,\cdot\,)}a & 0\end{bmatrix}\begin{bmatrix}\delta h/\delta\mu\\ \delta h/\delta a\end{bmatrix}.$$

The definition of the diamond operator $(\diamond)$ will ensure that the Lie–Poisson matrix operator is skew-symmetric in L^2 pairing under integration by parts.

$$\frac{d}{dt}f(\mu,a) = -\left\langle \begin{bmatrix}\delta f/\delta\mu\\ \delta f/\delta a\end{bmatrix}^T, \begin{bmatrix}\mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,)\diamond a\\ \mathcal{L}_{(\,\cdot\,)}a & 0\end{bmatrix}\begin{bmatrix}\delta h/\delta\mu\\ \delta h/\delta a\end{bmatrix}\right\rangle =: \Big\{f,h\Big\},$$

where $\{f,h\}$ is the “Lie–Poisson” bracket.

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Casimirs

A functional $C[\mu, a]$ whose variational derivatives $[\delta C/\delta\mu, \delta C/\delta a]^T$ comprise a null eigenvector of the Lie–Poisson matrix operator is called a Casimir functional for that Lie–Poisson system.

Casimir functionals satisfy

$$\frac{d}{dt}C(\mu, a) = \left\langle \begin{bmatrix} \delta h/\delta\mu \\ \delta h/\delta a \end{bmatrix}^T, \begin{bmatrix} \mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,) \diamond a \\ \mathcal{L}_{(\,\cdot\,)}a & 0 \end{bmatrix} \begin{bmatrix} \delta C/\delta\mu \\ \delta C/\delta a \end{bmatrix} \right\rangle =: \Big\{ f, h \Big\} = 0,$$

so that $C[\mu_t, a_t] = C[\mu_0, a_0]$ is conserved for any Hamiltonian $h[\mu, a]$.

RSW motion

RSW motion is governed by the following nondimensional equations for horizontal fluid velocity $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$ with $\mathrm{curl}\mathbf{R}(\mathbf{x}) = 2\Omega(\mathbf{x})\hat{\mathbf{z}}$ and depth D,

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{u} \times \mathrm{curl}\mathbf{v} + \nabla\psi = 0\,, \qquad \frac{\partial D}{\partial t} + \nabla \cdot D\mathbf{u} = 0\,,$$

with notation

$$\psi = \frac{D - B}{\epsilon\mathcal{F}} + \frac{\epsilon}{2}D|\mathbf{u}|^2\,,$$

and variable Coriolis parameter $2\Omega(\mathbf{x})$, bottom topography $B = B(\mathbf{x})$, Rossby number ϵ and rotational Froude number $\mathcal{F}$,

$$\epsilon = \frac{\mathcal{U}_0}{f_0 L} \ll 1 \quad \text{and} \quad \mathcal{F} = \frac{f_0^2 L^2}{gB_0} = O(1)\,.$$

The dimensional scales $(B_0, L, \mathcal{U}_0, f_0, g)$ denote equilibrium fluid depth, horizontal length scale, horizontal fluid velocity, reference Coriolis parameter, and gravitational acceleration, respectively.

Homework #1

(a) Show that the RSW equations arise as Euler–Poincaré equations from Hamilton's principle with action integral,

$$S_{\rm RSW} = \int \Big[D\mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - \frac{(D-B)^2}{2\epsilon\mathcal{F}} + \frac{\epsilon}{2}D|\mathbf{u}|^2 \Big]\, dx^1\wedge dx^2\, {\rm d}t\,,$$

where $(\partial_t D + \nabla\cdot(D\mathbf{u}))dx^1\wedge dx^2 = 0$.
Hint: first identify the momentum and advected quantity, so $(\diamond)$ may be computed.

(b) Write the Kelvin circulation theorem for RSW.

(c) Legendre transform to compute the Hamiltonian.

(d) Compute the Lie–Poisson form of the RSW equations.

(e) Compute the Casimirs for the Lie–Poisson bracket.

(f) Explain how the Casimirs are related to PV and depth.

SALT (Stochastic Advection by Lie Transport) comes next

However, first we needed to understand the calculus of differential forms.

As I said today, Flows, Pull-backs, k-forms, Lie derivatives and all that.

The diamond operator appearing in the δu equation defined the cotangent lift (Clebsch) momentum map, $T^*V \to \mathfrak{X}^*$ which is the route to the Hamiltonian formulation of the symmetry reduced dynamics.

For example,

$$b_t \diamond a_t = \delta\ell/\delta u_t \in \mathfrak{X}^* \,.$$

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What's next? Over to you! Any questions?

[CGH17],[CFH17],[AGH17],[HT16a],[HT16b],[ACH16],[Hol15]

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