~~Stochastic~~ Deterministic Advective Lie Transport (~~SALT~~ DALT) in Geophysical Fluid Dynamics (GFD)

Notes of LMS Research School, "Mathematics of Climate" Held 8 – 12 July 2019 at University of Reading

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8 – 12 July 2019

Abstract

These are notes of lectures given at the LMS Research School, entitled "Mathematics of Climate", held 8-12 July 2019 at the University of Reading, UK. The course is intended for PhD students, postdocs and other researchers interested in the variational principles for deriving the partial differential equations (PDE) underlying meteorology and climate science. Its first 5 lectures summarise the concepts and notation underlying the mathematical description of ideal fluid dynamics using the geometric theory of flows of smooth invertible maps described by deterministic PDE. It then shows how to use these concepts to derive fluid equations with Stochastic Advection by Lie Transport (SALT). The SALT equations answer the question, "How would Kelvin's circulation theorem change in the presence of noise?" The last lecture derives stochastic equations for the interaction of climate and weather, by using Lagrangian averaging from the perspective of the old adage, "Climate is what you expect. Weather is what you get." These are the Lagrangian averaged (LA) SALT equations. They have the potential for modelling the spatially integrated dynamics of the risk of extreme weather events.

Contents

1 Transformation Theory

motion	linearisation	differential, d
motion equation	infinitesimal transformation	differential k-form
vector field	pull-back	wedge product, $\wedge$
diffeomorphism	push-forward	Lie derivative, $\pounds_Q$
flow	Jacobian matrix	product rule
fixed point	directional derivative	fluid dynamics
equilibrium	commutator	other flows

1.1 Motions, pull-backs, push-forwards, vector fields, commutators & differentials

- A ***motion*** is defined as a smooth curve $q(t) \in M$ parameterised by $t \in \mathbb{R}$ that solves the ***motion equation***, which is a system of differential equations

$$\dot{q}(t) = \frac{dq}{dt} = f(q) \in TM\,, \tag{1}$$

or in components

$$\dot{q}^i(t) = \frac{dq^i}{dt} = f^i(q) \quad i = 1, 2, \ldots, n\,, \tag{2}$$

- The map $f : q \in M \to f(q) \in T_qM$ is a ***vector field***.

  According to standard theorems about differential equations that are not proven in this course, the solution, or integral curve, $q(t)$ exists, provided f is sufficiently smooth, which will always be assumed to hold.

- Vector fields can also be defined as ***differential operators*** that act on functions, as

$$\frac{d}{dt}G(q) = \dot{q}^i(t)\frac{\partial G}{\partial q^i} = f^i(q)\frac{\partial G}{\partial q^i} \quad i = 1, 2, \ldots, n, \quad \text{(sum on repeated indices)} \tag{3}$$

  for any smooth function $G(q) : M \to \mathbb{R}$.

- To indicate the dependence of the solution of its initial condition $q(0) = q_0$, we write the motion as a smooth transformation

$$q(t) = \phi_t(q_0)\,.$$

  Because the vector field f is independent of time t, for any fixed value of t we may regard ϕ_t as mapping from M into itself that satisfies the composition law

$$\phi_t \circ \phi_s = \phi_{t+s}$$

  and

$$\phi_0 = \mathrm{Id}\,.$$

Setting $s = -t$ shows that ϕ_t has a smooth inverse. A smooth mapping that has a smooth inverse is called a ***diffeomorphism***. Geometric mechanics deals with diffeomorphisms.

- The smooth mapping $\phi_t : \mathbb{R} \times M \to M$ that determines the solution $\phi_t \circ q_0 = q(t) \in M$ of the motion equation (1) with initial condition $q(0) = q_0$ is called the ***flow*** of the vector field Q.

  A point $q^\star \in M$ at which $f(q^\star) = 0$ is called a ***fixed point*** of the flow ϕ_t, or an ***equilibrium***.

  Vice versa, the vector field f is called the ***infinitesimal transformation*** of the mapping ϕ_t, since

$$\frac{d}{dt}\bigg|_{t=0} (\phi_t \circ q_0) = f(q)\,.$$

  That is, $f(q)$ is the ***linearisation*** of the flow map ϕ_t at the point $q \in M$.

  More generally, the ***directional derivative*** of the function h along the vector field f is given by the action of a differential operator, as

$$\frac{d}{dt}\bigg|_{t=0} h \circ \phi_t = \left[\frac{\partial h}{\partial \phi_t}\frac{d}{dt}(\phi_t \circ q_0)\right]_{t=0} = \frac{\partial h}{\partial q^i}\dot{q}^i = \frac{\partial h}{\partial q^i} f^i(q) =: Qh\,.$$

- Under a smooth change of variables $q = c(r)$ the vector field Q in the expression Qh transforms as

$$Q = f^i(q)\frac{\partial}{\partial q^i} \quad \mapsto \quad R = g^j(r)\frac{\partial}{\partial r^j} \quad \text{with} \quad g^j(r)\frac{\partial c^i}{\partial r^j} = f^i(c(r)) \quad \text{or} \quad g = c_r^{-1} f \circ c\,, \tag{4}$$

where c_r is the ***Jacobian matrix*** of the transformation. That is, since $h(q)$ is a function of q,

$$(Qh) \circ c = R(h \circ c)\,.$$

We express the transformation between the vector fields as $R = c^*Q$ and write this relation as

$$(Qh) \circ c =: c^*Q(h \circ c)\,. \tag{5}$$

The expression c^*Q is called the ***pull-back*** of the vector field Q by the map c.

Two vector fields are equivalent under a map c, if one is the pull-back of the other, and fixed points are mapped into fixed points.

The inverse of the pull-back is called the ***push-forward***. It is the pull-back by the inverse map.

- The ***commutator***

$$QR - RQ =: \big[Q,\, R\big]$$

of two vector fields Q and R defines another vector field. Indeed, if

$$Q = f^i(q)\frac{\partial}{\partial q^i} \quad \text{and} \quad R = g^j(q)\frac{\partial}{\partial q^j}$$

then

$$\big[Q,\, R\big] = \left(f^i(q)\frac{\partial g^j(q)}{\partial q^i} - g^i(q)\frac{\partial f^j(q)}{\partial q^i} \right) \frac{\partial}{\partial q^j}$$

because the second-order derivative terms cancel. By the pull-back relation (5) we have

$$c^*\big[Q,\, R\big] = \big[c^*Q,\, c^*R\big] \tag{6}$$

under a change of variables defined by a smooth map, c. This means the definition of the vector field commutator is independent of the choice of coordinates. As we shall see, the ***tangent*** to the relation $c_t^*\big[Q,\, R\big] = \big[c_t^*Q,\, c_t^*R\big]$ at the identity $t = 0$ is the ***Jacobi condition*** for the vector fields to form an algebra.

- The ***differential*** of a smooth function $f : M \to M$ is defined as

$$df = \frac{\partial f}{\partial q^i} dq^i\,.$$

- Under a smooth change of variables $s = \phi \circ q = \phi(q)$ the differential of the composition of functions $d(f \circ \phi)$ transforms according to the chain rule as

$$df = \frac{\partial f}{\partial q^i} dq^i \,, \quad d(f \circ \phi) = \frac{\partial f}{\partial \phi^j(q)} \frac{\partial \phi^j}{\partial q^i} dq^i = \frac{\partial f}{\partial s^j} ds^j \quad \Longrightarrow \quad d(f \circ \phi) = (df) \circ \phi \quad (7)$$

That is, the differential d commutes with the pull-back ϕ^* of a smooth transformation ϕ,

$$d(\phi^* f) = \phi^* df \,. \quad (8)$$

In a moment, this pull-back formula will give us the rule for transforming differential forms of any order.

1.2 Wedge products

- Differential k-forms on an n-dimensional manifold are defined in terms of the differential d and the antisymmetric ***wedge product*** $(\wedge)$ satisfying

$$dq^i \wedge dq^j = -\, dq^j \wedge dq^i \,, \quad \text{for} \quad i, j = 1, 2, \dots, n \quad (9)$$

By using wedge product, any k-form $\alpha \in \Lambda^k$ on M may be written locally at a point $q \in M$ in the differential basis dq^j as

$$\alpha_m = \alpha_{i_1 \dots i_k}(m) dq^{i_1} \wedge \cdots \wedge dq^{i_k} \in \Lambda^k \,, \quad i_1 < i_2 < \cdots < i_k \,, \quad (10)$$

where the sum over repeated indices is ordered, so that it must be taken over all i_j satisfying $i_1 < i_2 < \cdots < i_k$. Roughly speaking differential forms Λ^k are objects that can be integrated. As we shall see, vector fields also act on differential forms in interesting ways.

- Pull-backs of other differential forms may be built up from their basis elements, the dq^{i_k}. By equation (8), we have

  Theorem 1 (Pull-back of a wedge product)**.** *The pull-back of a wedge product of two differential forms is the wedge product of their pull-backs:*

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^*\alpha \wedge \phi_t^*\beta \,. \tag{11}$$

1.3 Contraction

1.4 Examples of contraction, or interior product

Definition 2 (Contraction, or interior product). *Let $\alpha \in \Lambda^k$ be a k-form on a manifold M,*

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k \,, \quad \textit{with} \quad i_1 < i_2 < \cdots < i_k \,,$$

and let $X = X^j \partial_j$ be a vector field. The contraction or interior product $X \lrcorner\, \alpha$ of a vector field X with a k-form α is defined by

$$X \lrcorner\, \alpha = X^j \alpha_{j i_2 \dots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k} \,. \tag{12}$$

Note that

$$\begin{aligned} X \lrcorner\, (Y \lrcorner\, \alpha) &= X^l Y^m \alpha_{m l i_3 \dots i_k} dx^{i_3} \wedge \cdots \wedge dx^{i_k} \\ &= -Y \lrcorner\, (X \lrcorner\, \alpha) \,, \end{aligned}$$

by antisymmetry of $\alpha_{m l i_3 \dots i_k}$, particularly in its first two indices.

Remark 3 (Examples of contraction).

1. A mnemonic device for keeping track of signs in contraction or substitution of a vector field into a differential form is to sum the substitutions of $X = X^j\partial_j$ over the permutations that bring the corresponding dual basis element into the leftmost position in the k-form α. For example, in two dimensions, contraction of the vector field $X = X^j\partial_j = X^1\partial_1 + X^2\partial_2$ into the two-form $\alpha = \alpha_{jk}dx^j \wedge dx^k$ with $\alpha_{21} = -\alpha_{12}$ yields

$$X \,\lrcorner\, \alpha = X^j\alpha_{ji_2}dx^{i_2} = X^1\alpha_{12}dx^2 + X^2\alpha_{21}dx^1 \,.$$

   Likewise, in three dimensions, contraction of the vector field $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3$ into the three-form $\alpha = \alpha_{123}dx^1 \wedge dx^2 \wedge dx^3$ with $\alpha_{213} = -\alpha_{123}$, etc. yields

$$\begin{aligned} X \,\lrcorner\, \alpha &= X^1\alpha_{123}dx^2 \wedge dx^3 + \text{cyclic permutations} \\ &= X^j\alpha_{ji_2i_3}dx^{i_2} \wedge dx^{i_3} \quad \text{with} \quad i_2 < i_3 \,. \end{aligned}$$

2. The rule for contraction of a vector field with a differential form develops from the relation

$$\partial_j \,\lrcorner\, dx^k = \delta^k_j \,,$$

in the coordinate basis $e_j = \partial_j := \partial/\partial x^j$ and its dual basis $e^k = dx^k$. Contraction of a vector field with a one-form yields the dot product, or inner product, between a covariant vector and a contravariant vector is given by

$$X^j\partial_j \mathbin{\lrcorner} v_k dx^k = v_k \delta^k_j X^j = v_j X^j \,,$$

or, in vector notation,

$$X \mathbin{\lrcorner} \mathbf{v}\cdot d\mathbf{x} = \mathbf{v}\cdot\mathbf{X}\,.$$

This is the ***dot product of vectors*** $\mathbf{v}$ and $\mathbf{X}$.

3. By the linearity of its definition (12), contraction of a vector field X with a differential k-form α satisfies

$$(hX) \mathbin{\lrcorner} \alpha = h(X \mathbin{\lrcorner} \alpha) = X \mathbin{\lrcorner} h\alpha\,.$$

Our previous calculations for two-forms and three-forms provide the following additional expressions for contraction of a vector field with a differential form, which may be written in vector notation as:

$$\begin{aligned} X \mathbin{\lrcorner} \mathbf{B}\cdot d\mathbf{S} &= -\mathbf{X}\times\mathbf{B}\cdot d\mathbf{x}\,, \\ X \mathbin{\lrcorner} d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S}\,, \\ d(X \mathbin{\lrcorner} d^{\,3}x) &= d(\mathbf{X}\cdot d\mathbf{S}) = (\mathrm{div}\,\mathbf{X})\,d^{\,3}x\,. \end{aligned}$$

Remark 4 (Physical examples of contraction)**.**
The first of these contraction relations represents the Lorentz, or Coriolis force, when $\mathbf{X}$ is particle velocity and $\mathbf{B}$ is either magnetic field, or rotation rate, respectively. The second contraction relation is the flux of the vector $\mathbf{X}$ through a surface element. The third is the exterior derivative of the second, thereby yielding the divergence of the vector $\mathbf{X}$.

Exercise. Show that

$$X \lrcorner \, (X \lrcorner \, \mathbf{B} \cdot d\mathbf{S}) = 0$$

and

$$(X \lrcorner \, \mathbf{B} \cdot d\mathbf{S}) \wedge \mathbf{B} \cdot d\mathbf{S} = 0 \, ,$$

for any vector field X and two-form $\mathbf{B} \cdot d\mathbf{S}$. ★

Proposition 5 (Contracting through wedge product)**.** *Let α be a k-form and β be a one-form on a manifold M and let $X = X^j \partial_j$ be a vector field. Then the contraction of X through the wedge product $\alpha \wedge \beta$ satisfies*

$$X \lrcorner \, (\alpha \wedge \beta) = (X \lrcorner \, \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \, \beta) \, . \tag{13}$$

Proof. The proof is a straightforward calculation using the definition of contraction. The exponent k in the factor $(-1)^k$ counts the number of exchanges needed to get the one-form β to the left most position through the k-form α. □

Proposition 6. *[Contraction is natural under pull-back]*
That is,

$$\phi^*(X(m) \,\lrcorner\, \alpha) = X(\phi(m)) \,\lrcorner\, \phi^*\alpha = \phi^* X \,\lrcorner\, \phi^*\alpha \,. \tag{14}$$

Proof. Direct verification using the relation between pull-back of forms and push-forward of vector fields.

Note the implication, $\pounds_X(Y \,\lrcorner\, \alpha) = [X\,,\,Y] \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X \alpha)$. □

Definition 7 (Alternative notations for contraction)**.** *Besides the hook notation with* $\lrcorner$ *, one also finds in the literature the following two alternative notations for contraction of a vector field X with k-form $\alpha \in \Lambda^k$ on a manifold M:*

$$X \,\lrcorner\, \alpha = i_X \alpha = \alpha(X, \underbrace{\,\cdot\,,\,\cdot\,,\,\dots\,,\,\cdot\,}_{k-1 \text{ slots}}) \in \Lambda^{k-1} \,. \tag{15}$$

In the last alternative, one leaves a dot $(\,\cdot\,)$ in each remaining slot of the form that results after contraction. For example, contraction of the Hamiltonian vector field $X_H = \{\,\cdot\,,\, H\}$ with the symplectic two-form $\omega \in \Lambda^2$ produces the one-form

$$X_H \,\lrcorner\, \omega = \omega(X_H,\,\cdot\,) = -\,\omega(\,\cdot\,,\, X_H) = dH \,.$$

In this alternative notation, the proof of formula (14) in Proposition 6 may be written, as follows.

Proof. Since forms are multilinear maps to the real numbers, one may define the pull-back of a k-form, α, by

$$\phi^*\alpha(X_1, X_2, ...) := \alpha(\phi_* X_1, \phi_* X_2, ...) \,.$$

Therefore, we are able to use the following proof.

$$\begin{aligned} \phi^* X \mathbin{\lrcorner} \phi^*\alpha(X_1, X_2, ...) &= \phi^*\alpha(\phi^* X, X_1, X_2, ...) \\ &= \alpha(\phi_*\phi^* X, \phi_* X_1, \phi_* X_2, ...) \\ &= \alpha(X, \phi_* X_1, \phi_* X_2, ...) \\ &= (X \mathbin{\lrcorner} \alpha)(\phi_* X_1, \phi_* X_2, ...) \\ &= \phi^*(X \mathbin{\lrcorner} \alpha)(X_1, X_2, ...) \end{aligned}$$

Now, if we allow $X_1, X_2, \ldots$ to be arbitrary, then formula (14) in Proposition 6 follows. □

1.5 Exercises in exterior calculus operations

Vector notation for differential basis elements One denotes differential basis elements dx^i and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for $i, j, k = 1, 2, 3$ in vector notation as

$$
\begin{aligned}
d\mathbf{x} &:= \left(dx^1, dx^2, dx^3\right), \\
d\mathbf{S} &= \left(dS_1, dS_2, dS_3\right) \\
&:= \left(dx^2 \wedge dx^3,\ dx^3 \wedge dx^1,\ dx^1 \wedge dx^2\right), \\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k, \\
d^{\,3}x &= dVol := dx^1 \wedge dx^2 \wedge dx^3 \\
&= \frac{1}{6}\epsilon_{ijk}dx^i \wedge dx^j \wedge dx^k.
\end{aligned}
$$

Exercise. (Vector calculus operations) Show that contraction $\,\lrcorner\, : \mathfrak{X} \times \Lambda^k \to \Lambda^{k-1}$ of the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ with the differential basis elements $d\mathbf{x}$, $d\mathbf{S}$ and $d^{\,3}x$ recovers the following familiar operations among vectors:

$$
\begin{aligned}
X \,\lrcorner\, d\mathbf{x} &= \mathbf{X}\,, \\
X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,, \\
(\text{or,}\quad X \,\lrcorner\, dS_i &= \epsilon_{ijk} X^j dx^k) \\
Y \,\lrcorner\, X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,, \\
X \,\lrcorner\, d^{\,3}x &= \mathbf{X} \cdot d\mathbf{S} = X^k dS_k\,, \\
Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y} \cdot d\mathbf{x} = \epsilon_{ijk} X^i Y^j dx^k\,, \\
Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y} \cdot \mathbf{Z}\,.
\end{aligned}
$$

★

Exercise. (Exterior derivatives in vector notation) Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation:

$$
\begin{aligned}
df &= f_{,j}\, dx^j =: \nabla f \cdot d\mathbf{x}\,, \\
0 = d^2 f &= f_{,jk}\, dx^k \wedge dx^j\,, \\
df \wedge dg &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \\
&=: (\nabla f \times \nabla g) \cdot d\mathbf{S}\,, \\
df \wedge dg \wedge dh &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \wedge h_{,l}\, dx^l \\
&=: (\nabla f \cdot \nabla g \times \nabla h)\, d^{\,3}x\,.
\end{aligned}
$$

★

Exercise. (Vector calculus formulas) Show that the exterior derivative yields the following vector calculus formulas:

$$
\begin{aligned}
df &= \nabla f \cdot d\mathbf{x}\,, \\
d(\mathbf{v} \cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathbf{v}) \cdot d\mathbf{S}\,, \\
d(\mathbf{A} \cdot d\mathbf{S}) &= (\mathrm{div}\,\mathbf{A})\, d^{\,3}x\,.
\end{aligned}
$$

The compatibility condition $d^2 = 0$ is written for these forms as

$$
\begin{aligned}
0 = d^2 f = d(\nabla f \cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathrm{grad}\, f) \cdot d\mathbf{S}\,, \\
0 = d^2(\mathbf{v} \cdot d\mathbf{x}) = d\big((\mathrm{curl}\,\mathbf{v}) \cdot d\mathbf{S}\big) &= (\mathrm{div}\,\mathrm{curl}\,\mathbf{v})\, d^{\,3}x\,.
\end{aligned}
$$

The product rule is written for these forms as

$$
\begin{aligned}
d\big(f(\mathbf{A}\cdot d\mathbf{x})\big) &= df\wedge \mathbf{A}\cdot d\mathbf{x} + f\,\mathrm{curl}\,\mathbf{A}\cdot d\mathbf{S}\\
&= \big(\nabla f\times \mathbf{A} + f\,\mathrm{curl}\,\mathbf{A}\big)\cdot d\mathbf{S}\\
&= \mathrm{curl}\,(f\mathbf{A})\cdot d\mathbf{S}\,,
\end{aligned}
$$

$$
\begin{aligned}
d\big((\mathbf{A}\cdot d\mathbf{x})\wedge(\mathbf{B}\cdot d\mathbf{x})\big) &= (\mathrm{curl}\,\mathbf{A})\cdot d\mathbf{S}\wedge \mathbf{B}\cdot d\mathbf{x} - \mathbf{A}\cdot d\mathbf{x}\wedge(\mathrm{curl}\,\mathbf{B})\cdot d\mathbf{S}\\
&= \big(\mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B}\big)d^{\,3}x\\
&= d\big((\mathbf{A}\times\mathbf{B})\cdot d\mathbf{S}\big)\\
&= \mathrm{div}(\mathbf{A}\times\mathbf{B})\,d^{\,3}x\,.
\end{aligned}
$$

These calculations yield familiar formulas from vector calculus for quantities curl(grad), div(curl), curl($f\mathbf{A}$) and div($\mathbf{A}\times\mathbf{B}$). ★

1.6 Integral calculus formulas

Exercise. (Integral calculus formulas) Show that the Stokes' theorem for the vector calculus formulas yields the following familiar results in $\mathbb{R}^3$:

– The ***fundamental theorem of calculus***, upon integrating df along a curve in $\mathbb{R}^3$ starting at point a and ending at point b:

$$\int_a^b df \;=\; \int_a^b \nabla f \cdot d\mathbf{x} = f(b) - f(a)\,.$$

– The ***classical Stokes theorem***, for a compact surface S with boundary ∂S:

$$\int_S (\mathrm{curl}\,\mathbf{v}) \cdot d\mathbf{S} \;=\; \oint_{\partial S} \mathbf{v} \cdot d\mathbf{x}\,.$$

(For a planar surface $S \in \mathbb{R}^2$, this is ***Green's theorem***.)

– The ***Gauss divergence theorem***, for a compact spatial domain D with boundary ∂D:

$$\int_D (\mathrm{div}\,\mathbf{A})\, d^{\,3}x \;=\; \oint_{\partial D} \mathbf{A} \cdot d\mathbf{S}\,.$$

★

These exercises illustrate the following,

Theorem 8 (Stokes' theorem)**.** *Suppose M is a compact oriented k-dimensional manifold with boundary ∂M and α is a smooth $(k-1)$-form on M. Then*

$$\int_M d\alpha = \oint_{\partial M} \alpha\,.$$

1.7 Summary of natural operations on differential forms

Besides the wedge product, three basic operations are commonly applied to differential forms. These are contraction, exterior derivative and Lie derivative.

- Contraction $\lrcorner$ with a vector field X lowers the degree:
$$X \lrcorner \Lambda^k \mapsto \Lambda^{k-1}\,.$$

- Exterior derivative d raises the degree:
$$d\Lambda^k \mapsto \Lambda^{k+1}\,.$$

- Lie derivative $\pounds_X$ by vector field X preserves the degree:
$$\pounds_X\Lambda^k \mapsto \Lambda^k\,, \quad \hbox{where} \quad \pounds_X\Lambda^k = \frac{d}{dt}\Big|_{t=0}\phi_t^*\Lambda^k\,,$$
in which ϕ_t is the flow of the vector field X. In analogy with fluids one may write $\pounds_X\Lambda^k = \frac{d}{dt}\Lambda^k$ along $\frac{dx}{dt} = X$.

- Lie derivative $\pounds_X$ satisfies ***Cartan's formula***: (The proof is a direct calculation.)
$$\pounds_X\alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha) \quad \hbox{for} \quad \alpha = \alpha_{i_1\dots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} \in \Lambda^k\,.$$

Remark 9.
Note also that the Lie derivative commutes with the exterior derivative. That is,

$$d(\pounds_X \alpha) = \pounds_X d\alpha \,, \quad \text{for} \quad \alpha \in \Lambda^k(M) \quad \text{and} \quad X \in \mathfrak{X}(M) \,.$$

1.8 Lie derivatives

Definition 10 (Lie derivative of a differential k-form)**.** *The **Lie derivative** of a differential k-form Λ^k by a vector field $Q \in \mathfrak{X}$ is defined by linearising its flow ϕ_t around the identity $t = 0$,*

$$\pounds_Q \Lambda^k = \frac{d}{dt}\bigg|_{t=0} \phi_t^* \Lambda^k \quad \textit{maps} \quad \mathfrak{X} \times \Lambda^k \mapsto \Lambda^k \,.$$

Hence, by equation (11), the Lie derivative satisfies the product rule for the wedge product.

Corollary 11 (Product rule for the Lie derivative of a wedge product)**.**

$$\pounds_Q(\alpha \wedge \beta) = \pounds_Q \alpha \wedge \beta + \alpha \wedge \pounds_Q \beta \,. \tag{16}$$

- Pullbacks of vector fields lead to Lie derivative expressions, too.

Definition 12 (Lie derivative of a vector field)**.** *The* ***Lie derivative*** *of a vector field $Y \in \mathfrak{X}$ by another vector field $X \in \mathfrak{X}$ is defined by linearising the flow ϕ_t of X around the identity $t = 0$,*

$$\pounds_X Y = \frac{d}{dt}\Big|_{t=0} \phi_t^* Y \quad maps \quad \pounds_X \in \mathfrak{X} \mapsto \mathfrak{X}\,.$$

Theorem 13. *The Lie derivative $\pounds_X Y$ of a vector field Y by a vector field X satisfies*

$$\pounds_X Y = \frac{d}{dt}\Big|_{t=0} \phi_t^* Y = [X,\, Y]\,, \tag{17}$$

where $[X,\, Y] = XY - YX$ is the commutator of the vector fields X and Y.

Proof. Denote the vector fields in components as

$$X = X^i(q)\frac{\partial}{\partial q^i} = \frac{d}{dt}\Big|_{t=0} \phi_t^* \quad \text{and} \quad Y = Y^j(q)\frac{\partial}{\partial q^j}\,.$$

Then, by the pull-back relation (5) a direct computation yields, on using the matrix identity $dM^{-1} = -\,M^{-1}dM M^{-1}$,

$$
\begin{aligned}
\pounds_X Y &= \frac{d}{dt}\bigg|_{t=0} \phi_t^* Y = \frac{d}{dt}\bigg|_{t=0} \left(Y^j(\phi_t q) \frac{\partial}{\partial (\phi_t q)^j} \right) \\
&= \frac{d}{dt}\bigg|_{t=0} \left(Y^j(\phi_t q) \left[\frac{\partial (\phi_t q)}{\partial q}^{-1} \right]^k_j \frac{\partial}{\partial q^k} \right) \\
&= \left(X^j \frac{\partial Y^k}{\partial q^j} - Y^j \frac{\partial X^k}{\partial q^j} \right) \frac{\partial}{\partial q^k} \\
&= [X,\, Y]\,.
\end{aligned}
$$

□

Corollary 14. *The Lie derivative of the relation (6) for the pull-back of the commutator* $c_t^*\big[Y,\, Z\big] = \big[c_t^* Y,\, c_t^* Z\big]$ *yields the* ***Jacobi condition*** *for the vector fields to form an algebra.*

Proof. By the product rule and the definition of the Lie bracket (17) we have

$$
\frac{d}{dt}\bigg|_{t=0} \phi_t^* \big[Y,\, Z\big] = \big[X, \big[Y,\, Z\big]\big] = \big[[X, Y],\, Z\big] + \big[Y,\, [X, Z]\big] = \frac{d}{dt}\bigg|_{t=0} \big[\phi_t^* Y,\, \phi_t^* Z\big]
$$

This is the ***Jacobi identity*** for vector fields. □

1.9 Summary and an exercise

Summary

The pull-back ϕ_t^* of a smooth flow ϕ_t generated by a smooth vector field X on a smooth manifold M commutes with the exterior derivative d, wedge product $\wedge$ and contraction $\lrcorner$.

That is, for k-forms α, $\beta \in \Lambda^k(M)$, and $m \in M$, the pull-back ϕ_t^* satisfies

$$
\begin{aligned}
d(\phi_t^*\alpha) &= \phi_t^* d\alpha\,, \\
\phi_t^*(\alpha \wedge \beta) &= \phi_t^*\alpha \wedge \phi_t^*\beta\,, \\
\phi_t^*(X \lrcorner\, \alpha) &= \phi_t^* X \lrcorner\, \phi_t^*\alpha\,.
\end{aligned}
$$

In addition, the Lie derivative $\pounds_X\alpha$ of a k-form $\alpha \in \Lambda^k(M)$ by the vector field X tangent to the flow ϕ_t on M is defined either dynamically or geometrically (by Cartan's formula) as

$$
\pounds_X\alpha \;=\; \frac{d}{dt}\bigg|_{t=0}(\phi_t^*\alpha) = X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha), \tag{18}
$$

in which the last equality is Cartan's geometric formula in (27) for the Lie derivative.

Definition 15. (Lie derivative pull-back formula)
The tangent to the pull-back $\phi_t^*\alpha$ of a differential form α is the pull-back of the Lie derivative of α wrt the vector field X that generates the flow, ϕ_t:

$$\frac{d}{dt}(\phi_t^*\alpha) = \phi_t^*\big(\pounds_X\alpha\big)\,.$$

Likewise, for the push-forward, which is the pull-back by the inverse, we have

$$\frac{d}{dt}((\phi_t^{-1})^*\alpha) = -(\phi_t^{-1})^*\big(\pounds_X\alpha\big)\,.$$

Definition 16. (Advected quantity)
A quantity which is invariant along a flow trajectory satisfies $\alpha_0(x_0) = \alpha_t(x_t) = (\phi_t^*\alpha_t)(x_0)$, so that

$$0 = \frac{d}{dt}\alpha_0(x_0) = \frac{d}{dt}(\phi_t^*\alpha_t)(x_0) = \phi_t^*(\partial_t + \pounds_X)\alpha_t(x_0) = (\partial_t + \pounds_X)\alpha_t(x_t)$$

Or vice versa

$$\alpha_t(x_t) = (\alpha_0 \circ \phi_t^{-1})(x_t) = ((\phi_t)_*\alpha_0)(x_t)$$

satisfies

$$\frac{d}{dt}\alpha_t(x_t) = \frac{d}{dt}(\phi_t)_*\alpha_0 = -\pounds_X\alpha_t\,.$$

2 Exercises

Exercise.

(a) Verify the formula $[X\,,\,Y]\,\lrcorner\,\alpha = \pounds_X(Y\,\lrcorner\,\alpha) - Y\,\lrcorner\,(\pounds_X\alpha)$.

(b) Use (a) to verify $\pounds_{[X\,,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$.

(c) Use (b) to verify the Jacobi identity.

(d) Use (c) to verify that the divergence-free vector fields are closed under commutation.

(e) For a top-form α show that divergence-free vector fields satisfy

$$[X\,,\,Y]\,\lrcorner\,\alpha = d\big(X\,\lrcorner\,(Y\,\lrcorner\,\alpha)\big)\,. \qquad (19)$$

(f) Write the equivalent of equation (19) as a formula in vector calculus.

★

Answer.

(a) The required formula follows immediately from the product rule for the dynamical definition of the Lie derivative. Since pull-back commutes with contraction, insertion of a vector field into a k-form transforms under the flow ϕ_t of a smooth vector field Y as

$$\phi_t^*(Y \mathbin{\lrcorner} \alpha) = \phi_t^* Y \mathbin{\lrcorner} \phi_t^* \alpha \,.$$

A direct computation using the dynamical definition of the Lie derivative $\pounds_Y \alpha = \frac{d}{dt}\big|_{t=0}(\phi_t^* \alpha)$, then yields

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \phi_t^* (Y \mathbin{\lrcorner} \alpha) &= \Big(\frac{d}{dt}\Big|_{t=0} \phi_t^* Y\Big) \mathbin{\lrcorner} \alpha \\ &\quad + Y \mathbin{\lrcorner} \Big(\frac{d}{dt}\Big|_{t=0} \phi_t^* \alpha\Big) \,. \end{aligned}$$

Hence, we recognise that the desired formula is the ***product rule*** met earlier for the Lie derivative

$$\pounds_X (Y \mathbin{\lrcorner} \alpha) = (\pounds_X Y) \mathbin{\lrcorner} \alpha + Y \mathbin{\lrcorner} (\pounds_X \alpha) \,.$$

(b) Insert $\pounds_X Y = [X\,,\,Y]$ into the product rule formula in part (a). Then

$$[X\,,\,Y] \mathbin{\lrcorner} \alpha = \pounds_X (Y \mathbin{\lrcorner} \alpha) - Y \mathbin{\lrcorner} (\pounds_X \alpha).$$

Now use Cartan's formula in (27)

$$\pounds_X\alpha \;=\; \frac{d}{dt}\Big|_{t=0}(\phi_t^*\alpha) = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha),$$

to compute the required result, as

$$\begin{aligned}
\pounds_{[X\,,\,Y]}\alpha &= d([X\,,\,Y] \,\lrcorner\, \alpha) + [X\,,\,Y] \,\lrcorner\, d\alpha \\
&= d\Big(\,\pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)\,\Big) \\
&\quad + \pounds_X(Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, (\pounds_X d\alpha) \\
&= \pounds_X d(Y \,\lrcorner\, \alpha) - d(Y \,\lrcorner\, (\pounds_X\alpha) \\
&\quad + \pounds_X(Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, d(\pounds_X\alpha) \\
&= \pounds_X(\pounds_Y\alpha) - \pounds_Y(\pounds_X\alpha)\,.
\end{aligned}$$

Can you think of an alternative proof based on the dynamical definition of the Lie derivative?

(c) Applying part (b), ($\pounds_{[X\,,\,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$) to $\alpha = d^3x$ proves that $\pounds_{[X\,,\,Y]}d^3x = 0$; since both $\pounds_Y d^3x = 0 = \pounds_X d^3x$, because, e.g., $\pounds_Y d^3x = (\mathrm{div}Y)\, d^3x$.

(d) As a consequence of part (b),

$$
\begin{aligned}
\pounds_{[Z,[X,Y]]}\alpha &= \pounds_Z(\pounds_X\pounds_Y - \pounds_Y\pounds_X)\alpha - (\pounds_X\pounds_Y - \pounds_Y\pounds_X)\pounds_Z\alpha \\
&= \pounds_Z\pounds_X\pounds_Y\alpha - \pounds_Z\pounds_Y\pounds_X\alpha \\
&\quad - \pounds_X\pounds_Y\pounds_Z\alpha + \pounds_Y\pounds_X\pounds_Z\alpha\,,
\end{aligned}
$$

and summing over cyclic permutations verifies that

$$
\pounds_{[Z,[X,Y]]}\,\alpha + \pounds_{[X,[Y,Z]]}\,\alpha + \pounds_{[Y,[Z,X]]}\,\alpha = 0\,.
$$

This is the ***Jacobi identity for the Lie derivative***.

(e) Substituting the relation $\pounds_X Y = [X,\, Y]$ into the product rule above in part (b) and rearranging yields

$$
[X,\, Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)\,, \tag{20}
$$

as required, for an arbitrary k-form α.

From formula (20), we have

$$
\begin{aligned}
[X\,,\, Y] \,\lrcorner\, \alpha &= \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha) \\
&= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha) + X \,\lrcorner\, d(Y \,\lrcorner\, \alpha)\big) - Y \,\lrcorner\, (\pounds_X\alpha) \\
&= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + X \,\lrcorner\, (\pounds_Y\alpha - Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, (\pounds_X\alpha) \\
&= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + X \,\lrcorner\, (\pounds_Y\alpha) - Y \,\lrcorner\, (\pounds_X\alpha) \\
[X\,,\, Y] \,\lrcorner\, \alpha &= d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + (\mathrm{div}\,\mathbf{Y})X \,\lrcorner\, \alpha - (\mathrm{div}\,\mathbf{X})Y \,\lrcorner\, \alpha. \qquad (21)
\end{aligned}
$$

The last two steps to obtain (21) follow, because $d\alpha = 0$ and $\pounds_X \alpha = (\operatorname{div}\mathbf{X})\,\alpha$ for a top-form α.

For divergence-free vectors $\mathbf{X}$ and $\mathbf{Y}$, the last result takes the elegant form,

$$[X\,,\,Y] \,\lrcorner\, \alpha = d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big), \tag{22}$$

when $\operatorname{div}\mathbf{X} = 0 = \operatorname{div}\mathbf{Y}$.

(f) The vector calculus formula to which equation (21) is equivalent may be found by writing its left and right sides in a coordinate basis, as

$$
\begin{aligned}
[X\,,\,Y] \,\lrcorner\, \alpha &= (\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X})\cdot d\mathbf{S} \\
d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) + X \,\lrcorner\, (\pounds_Y \alpha) - Y \,\lrcorner\, (\pounds_X \alpha) &= -\operatorname{curl}(\mathbf{X}\times\mathbf{Y})\cdot d\mathbf{S} \\
&\quad + (\operatorname{div}\mathbf{Y})\,\mathbf{X}\cdot d\mathbf{S} - (\operatorname{div}\mathbf{X})\,\mathbf{Y}\cdot d\mathbf{S}
\end{aligned}
$$

Thus, equation (21) for a top-form $\alpha\, d^n x$ is equivalent to the well-known vector calculus identity

$$(\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X}) = -\operatorname{curl}(\mathbf{X}\times\mathbf{Y}) + (\operatorname{div}\mathbf{Y})\,\mathbf{X} - (\operatorname{div}\mathbf{X})\,\mathbf{Y}\,.$$

▲

Exercise.

(a) Starting from

$$[u\,,\,v]\,\lrcorner\,\alpha = \pounds_u(v\,\lrcorner\,\alpha) - v\,\lrcorner\,(\pounds_u\alpha)$$

prove the following

$$\begin{aligned}\pounds_u(v\,\lrcorner\,\alpha) - \pounds_v(u\,\lrcorner\,\alpha) &= 2[u,v]\,\lrcorner\,\alpha + v\,\lrcorner\,\pounds_u\alpha - u\,\lrcorner\,\pounds_v\alpha \\ &= [u,v]\,\lrcorner\,\alpha - u\,\lrcorner\,(v\,\lrcorner\,\alpha) + d(u\,\lrcorner\,(v\,\lrcorner\,\alpha))\end{aligned}$$

(b) Evaluate the last equation for a k-form α with $k = 3, 2, 1$, in terms of vector calculus expressions.

★

Answer.

(a)

$$
\begin{aligned}
[u,v] \lrcorner\, \alpha &= \pounds_u(v \lrcorner\, \alpha) - v \lrcorner\, \pounds_u \alpha \\
&= d(u \lrcorner\, (v \lrcorner\, \alpha)) + u \lrcorner\, d(v \lrcorner\, \alpha) - v \lrcorner\, \pounds_u \alpha \\
&= d(u \lrcorner\, (v \lrcorner\, \alpha)) + u \lrcorner\, (\pounds_v \alpha - v \lrcorner\, d\alpha) - v \lrcorner\, \pounds_u \alpha \\
&= d(u \lrcorner\, (v \lrcorner\, \alpha)) + u \lrcorner\, \pounds_v \alpha - u \lrcorner\, (v \lrcorner\, d\alpha) - v \lrcorner\, \pounds_u \alpha \\
[u,v] \lrcorner\, \alpha + u \lrcorner\, (v \lrcorner\, \alpha) &= d(u \lrcorner\, (v \lrcorner\, \alpha)) \\
&\quad + (u \lrcorner\, \pounds_v \alpha - v \lrcorner\, \pounds_u \alpha)
\end{aligned}
$$

$$
\begin{aligned}
u \lrcorner\, \pounds_v \alpha - v \lrcorner\, \pounds_u \alpha &= [u,v] \lrcorner\, \alpha + u \lrcorner\, (v \lrcorner\, \alpha) - d(u \lrcorner\, (v \lrcorner\, \alpha)) \\
v \lrcorner\, \pounds_u \alpha - u \lrcorner\, \pounds_v \alpha &= [v,u] \lrcorner\, \alpha + v \lrcorner\, (u \lrcorner\, \alpha) - d(v \lrcorner\, (u \lrcorner\, \alpha)) \\
&= -[u,v] \lrcorner\, \alpha - u \lrcorner\, (v \lrcorner\, \alpha) + d(u \lrcorner\, (v \lrcorner\, \alpha))
\end{aligned}
$$

$$
\begin{aligned}
\pounds_u(v \lrcorner\, \alpha) - \pounds_v(u \lrcorner\, \alpha) &= 2[u,v] \lrcorner\, \alpha + v \lrcorner\, \pounds_u \alpha - u \lrcorner\, \pounds_v \alpha \\
&= [u,v] \lrcorner\, \alpha - u \lrcorner\, (v \lrcorner\, d\alpha) + d(u \lrcorner\, (v \lrcorner\, \alpha))
\end{aligned}
$$

(b) For a 3-form $\alpha = d^3x$ (top form in 3D) one again finds the vector calculus identity in the previous exercise which is antisymmetric under exchange of $\mathbf{u}$ and $\mathbf{v}$,

$$
(\mathrm{div}\mathbf{v})\mathbf{u} - (\mathrm{div}\mathbf{u})\mathbf{v} - \mathrm{curl}(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u} =: [\mathbf{u}, \mathbf{v}]
$$

For a 2-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{S}$ in 3D one finds the vector calculus identity

$$-\mathbf{u} \times \mathrm{curl}(\boldsymbol{\alpha} \times \mathbf{v}) + \mathbf{v} \times \mathrm{curl}(\boldsymbol{\alpha} \times \mathbf{u}) = \boldsymbol{\alpha} \times [\mathbf{u}, \mathbf{v}] + (\mathrm{div}\boldsymbol{\alpha})(\mathbf{u} \times \mathbf{v}) + \nabla(\boldsymbol{\alpha} \cdot \mathbf{u} \times \mathbf{v})$$

in which we denote as in the previous vector calculus identity

$$[\mathbf{u}, \mathbf{v}] := (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u} = -\,\mathrm{curl}(\mathbf{u} \times \mathbf{v}) - (\mathrm{div}\mathbf{u})\mathbf{v} + (\mathrm{div}\mathbf{v})\mathbf{u}$$

After the substitution of this expression for $[\mathbf{u}, \mathbf{v}]$ obtained in the case of the 3-form $\alpha = d^3x$, one sees that the vector calculus identity for a 2-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{S}$ has cyclic permutation symmetry

$$\begin{aligned} \mathbf{u} \times \mathrm{curl}(\mathbf{v} \times \boldsymbol{\alpha}) + \mathbf{v} \times \mathrm{curl}(\boldsymbol{\alpha} \times \mathbf{u}) + \boldsymbol{\alpha} \times \mathrm{curl}(\mathbf{u} \times \mathbf{v}) \\ = (\mathrm{div}\mathbf{u})(\mathbf{v} \times \boldsymbol{\alpha}) + (\mathrm{div}\mathbf{v})(\boldsymbol{\alpha} \times \mathbf{u}) \\ + (\mathrm{div}\boldsymbol{\alpha})(\mathbf{u} \times \mathbf{v}) + \nabla(\boldsymbol{\alpha} \cdot \mathbf{u} \times \mathbf{v}) \end{aligned}$$

Also, in the divergence-free case this reduces to

$$\mathrm{curl}\Big(\mathbf{u} \times \mathrm{curl}(\mathbf{v} \times \boldsymbol{\alpha}) + \mathbf{v} \times \mathrm{curl}(\boldsymbol{\alpha} \times \mathbf{u}) + \boldsymbol{\alpha} \times \mathrm{curl}(\mathbf{u} \times \mathbf{v})\Big) = 0.$$

For a 1-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{x}$, the result turns out to be trivial.

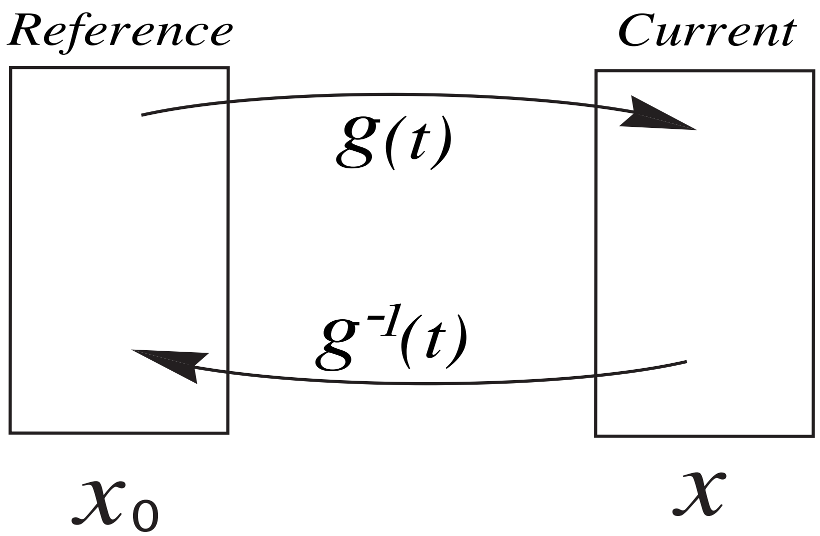
▲

3 Hamilton's principle for fluid dynamics

3.1 Advected quantities in fluid dynamics

We regard fluid flow as a smooth invertible time-dependent transformation of initial conditions x_0 regarded as fluid labels taking values in a configuration manifold M acted on by smooth invertible maps $\mathrm{Diff}(M)$. Thus, we lift the motion of fluid parcels $x_t \in M$ with initial condition $x_0 \in M$ to the manifold of diffeomorphisms by identifying it with a time-dependent curve $g_t \in \mathrm{Diff}(M)$ with $g_0 = Id$, whose action from the left generates the motion x_t,

$$x_t = g_t x_0 \quad \text{with} \quad \dot{x}_t = \dot{g}_t x_0 = (u_t \circ g_t) x_0$$



Advected quantity. A quantity $a_t(x_t) = a_0(x_0)$ which remains invariant under the flow is said to be *advected* by the flow. In terms of the group action, advected quantities satisfy

$$a_0(x_0) = a_t(x_t) = (a_t \circ g_t)(x_0) = (g_t^* a_t)(x_0)$$

where $g_t^* a_t$ is the *pull-back* of a_t by g_t. Invariance of an advected quantity implies an evolution equation

$$0 = \frac{d}{dt} a_0(x_0) = \frac{d}{dt}(g_t^* a_t)(x_0) = g_t^*\big((\partial_t + \pounds_u)a_t\big)(x_0) = (\partial_t + \pounds_u)a_t(x_t)$$

where $\pounds_u$ denotes the Lie derivative with respect to the vector field $u = \dot{g}g^{-1}$ which generates the flow g_t.

Vice versa, we have the *push-forward* relation

$$\frac{d}{dt}a_t(x_t) = \frac{d}{dt}(a_0 g_t^{-1})(x_t) = \frac{d}{dt}\big((g_t)_* a_0\big)(x_t) = -(\pounds_u a_t)(x_t)\,.$$

The previous formula will be useful in taking variations of advected quantities in Hamilton's principle, since it implies the following formula for the variation of an advected quantity, a_t at fixed t,

$$\delta a_t(x_t) = a_t'(x_t) := \Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0} a_{t,\epsilon}\Big)(x_t) = -(\pounds_{v_t} a_t)(x_t)$$

where

$$v_t = \Big(\frac{dg_{t,\epsilon}}{d\epsilon} g_{t,\epsilon}^{-1}\Big)\Big|_{\epsilon=0} = g' g^{-1}$$

and $\pounds_v$ denotes the Lie derivative with respect to the vector field $v = [{g_\epsilon}' g_\epsilon^{-1}]_{\epsilon=0}$ which generates the flow g_ϵ.

Equality of cross derivatives in t and ϵ implies the following pair of relations

$$\begin{aligned}(\dot g)' \circ g^{-1} &= (u \circ g)' \circ g^{-1} = (\partial_x u) g' \circ g^{-1} + (u' \circ g) \circ g^{-1}\\ &= (\partial_x u) v + u'\\ (g')^{\cdot} \circ g^{-1} &= (\partial_x v) u + \dot v\,,\end{aligned}$$

from which we conclude upon substituting $u = \dot{g}g^{-1}$ that

$$\delta(\dot{g}g^{-1}) = (\dot{g}g^{-1})' = \dot{v} + (\partial_x v)u - (\partial_x u)v = \dot{v} - [u, v] = \dot{v} - \mathrm{ad}_u v = \dot{v} - \mathrm{ad}_{\dot{g}g^{-1}} v$$

Now we are ready to compute the Euler-Poincaré equations for fluid dynamics.

3.2 Euler-Poincaré equations for fluid dynamics

We shall compute the compute the Euler-Poincaré equations for fluid dynamics using the Hamilton-Pontryagin principle,

$$\begin{aligned}
0 = \delta S = \delta \int_0^T & \ell(u, a_0 g_t^{-1}) + \langle m\,,\, \dot{g}g^{-1} - u\rangle \, dt \\
= \int_0^T & \left\langle \frac{\delta \ell}{\delta u} - m\,,\, \delta u \right\rangle + \left\langle \frac{\delta \ell}{\delta a}\,,\, -\pounds_v a \right\rangle \\
& + \left\langle m\,,\, \partial_t v - \mathrm{ad}_{\dot{g}g^{-1}} v \right\rangle + \left\langle \delta m\,,\, \dot{g}g^{-1} - u \right\rangle dt \\
= \int_0^T & \left\langle \frac{\delta \ell}{\delta u} - m\,,\, \delta u \right\rangle + \left\langle \frac{\delta \ell}{\delta a} \diamond a - \partial_t m - \mathrm{ad}^*_{\dot{g}g^{-1}} m\,,\, v \right\rangle \\
& + \left\langle \delta m\,,\, \dot{g}g^{-1} - u \right\rangle dt + \langle m\,,\, v\rangle \Big|_0^T ,
\end{aligned}$$

where we have used $\delta(a_0 g_t^{-1}) = -\pounds_v a$ and have defined the diamond operator $(\diamond)$ as

$$\diamond : V^* \times V \to \mathfrak{X}^* \quad \text{defined by} \quad \left\langle \frac{\delta \ell}{\delta a} \diamond a \,,\, v \right\rangle := \left\langle \frac{\delta \ell}{\delta a} \,,\, -\pounds_v a \right\rangle$$

and the ad^* operation as

$$\mathrm{ad}^* : \mathfrak{X} \times \mathfrak{X}^* \to \mathfrak{X}^* \quad \text{defined by} \quad \langle \mathrm{ad}^*_u m \,,\, v \rangle = \Big\langle m \,,\, \mathrm{ad}_u v \Big\rangle$$

In particular, $\mathrm{ad}^*_u m = \pounds_u m$, so that the fluid motion equation for $m = \mathbf{m} \cdot d\mathbf{x} \otimes d^3x$ and advection equations become

$$(\partial_t + \pounds_u) m = \frac{\delta \ell}{\delta a} \diamond a \quad \text{and} \quad (\partial_t + \pounds_u) a = 0$$

In general, fluid motion advects mass, so that $D_t(x_t) d^3x_t = D_0(x_0) d^3x_0$, which implies the continuity equation

$$0 = (\partial_t + \pounds_u)(D_t(x_t) d^3x_t) = \big(\partial_t D + \mathrm{div}(D\mathbf{u})\big) d^3x$$

Consequently, the motion equation may be rewritten as

$$\big(\partial_t + \pounds_u\big)(D^{-1}\mathbf{m} \cdot d\mathbf{x}) = \frac{1}{D}\frac{\delta \ell}{\delta a} \diamond a$$

in which $\frac{1}{D}\frac{\delta\ell}{\delta a}\diamond a$ is a 1-form. Integrating this relation around a material loop c_t moving with the fluid yields

$$\frac{d}{dt}\oint_{c_t}(D^{-1}\mathbf{m}\cdot d\mathbf{x}) = \oint_{c_t}\frac{1}{D}\frac{\delta\ell}{\delta a}\diamond a$$

This is the Kelvin-Noether theorem, which arises from relabelling symmetry of the Lagrangian fluid parcels.

3.3 Euler's fluid equations

Euler's equations for the incompressible motion of an ideal flow of a fluid of unit density and velocity $\mathbf{u}$ satisfying $\operatorname{div}\mathbf{u} = 0$ in a rotating frame with Coriolis parameter $\operatorname{curl}\mathbf{R} = 2\boldsymbol{\Omega}$ are given in the form of Newton's law of force by

$$\underbrace{\partial_t\,\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{\text{acceleration}} = \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}}\ . \tag{23}$$

Exercise. Prove that Euler's equations in a rotating frame arise as Euler-Poincaré equations from Hamilton's variational principle for the following action integral.

$$0 = \delta S = \int_0^T \frac{1}{2}D|\mathbf{u}|^2 + D\mathbf{u}\cdot\mathbf{R} - p(D-1)\,d^3x\,dt$$

★

The Newton's law equation for Euler fluid motion in (23) may be rearranged into an alternative form,

$$\partial_t \mathbf{v} - \mathbf{u} \times \boldsymbol{\omega} + \nabla\Big(p + \frac{1}{2}|\mathbf{u}|^2\Big) = 0\,, \tag{24}$$

by denoting

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{R}\,, \quad \boldsymbol{\omega} = \operatorname{curl}\mathbf{v} = \operatorname{curl}\mathbf{u} + 2\boldsymbol{\Omega}\,, \tag{25}$$

and using the fundamental vector calculus identity of fluid dynamics

$$\mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j = -\,\mathbf{u}\times\operatorname{curl}\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\,. \tag{26}$$

This identity follows from equality of the dynamic and geometricdefinitions of the Lie derivative $\pounds_u\alpha$ of a k-form $\alpha \in \Lambda^k(M)$ by the vector field $u = \dot{g}g^{-1}$ tangent to the flow g_t on M as

$$\pounds_u\alpha \;=\; \frac{d}{dt}\bigg|_{t=0}(g_t^*\alpha) = u \,\lrcorner\, d\alpha + d(u \,\lrcorner\, \alpha), \tag{27}$$

in which the last equality is Cartan's geometric formula for the Lie derivative.

For the case of the circulation 1-form $\alpha = \mathbf{v}\cdot d\mathbf{x}$, this becomes

$$
\begin{aligned}
\pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= \big(\mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j\big)\cdot d\mathbf{x} \\
&= u \,\lrcorner\, d(\mathbf{v}\cdot d\mathbf{x}) + d(u \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) \\
&= u \,\lrcorner\, (\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) + d(\mathbf{u}\cdot\mathbf{v}) \\
&= \big(-\ \mathbf{u}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x}\,,
\end{aligned}
\tag{28}
$$

and the identity (25) emerges. This identity and the calculation (28) recasts Euler's fluid motion equation into the following geometric form:

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x}) &= (\partial_t\mathbf{v} -\ \mathbf{u}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v}))\cdot d\mathbf{x} \\
&= -\,\nabla\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right)\cdot d\mathbf{x} \\
&= -\,d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right).
\end{aligned}
\tag{29}
$$

Requiring preservation of the divergence-free (volume-preserving) constraint $\nabla\cdot\mathbf{u} = 0$ results in a Poisson equation for pressure p, which may be written in several equivalent forms,

$$
\begin{aligned}
-\Delta p &= \operatorname{div}\big(\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{u}\times 2\boldsymbol{\Omega}\big) \\
&= u_{i,j}u_{j,i} - \operatorname{div}\big(\mathbf{u}\times 2\boldsymbol{\Omega}\big) \\
&= \operatorname{tr}\mathsf{S}^2 - \frac{1}{2}|\operatorname{curl}\mathbf{u}|^2 - \operatorname{div}\big(\mathbf{u}\times 2\boldsymbol{\Omega}\big)\,,
\end{aligned}
\tag{30}
$$

where $\mathsf{S} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the ***strain-rate tensor***.

We introduce the ***Lamb vector***,

$$
\boldsymbol{\ell} := -\,\mathbf{u}\times\boldsymbol{\omega}\,, \tag{31}
$$

which represents the nonlinearity in Euler's fluid equation (24). The Poisson equation (30) for pressure p may now be expressed in terms of the divergence of the Lamb vector,

$$
-\,\Delta\left(p + \frac{1}{2}|\mathbf{u}|^2\right) = \operatorname{div}(-\,\mathbf{u}\times\operatorname{curl}\mathbf{v}) = \operatorname{div}\boldsymbol{\ell}\,. \tag{32}
$$

Remark 17 (***Boundary conditions).***

Because the velocity $\mathbf{u}$ must be tangent to any fixed boundary, the normal component of the motion equation must vanish. This requirement produces a Neumann condition for pressure given by

$$
\partial_n\left(p + \frac{1}{2}|\mathbf{u}|^2\right) + \hat{\mathbf{n}}\cdot\boldsymbol{\ell} = 0\,, \tag{33}
$$

at a fixed boundary with unit outward normal vector $\hat{\mathbf{n}}$.

Remark 18 (***Helmholtz vorticity dynamics).***
Taking the curl of the Euler fluid equation (24) yields the ***Helmholtz vorticity equation***

$$\partial_t \boldsymbol{\omega} - \mathrm{curl}(\mathbf{u} \times \boldsymbol{\omega}) = 0\,, \tag{34}$$

whose geometrical meaning will emerge in discussing Stokes' Theorem 29 for the vorticity of a rotating fluid.

The rotation terms have now been fully integrated into both the dynamics and the boundary conditions. In this form, the ***Kelvin circulation theorem*** and the ***Stokes vorticity theorem*** will emerge naturally together as geometrical statements.

3.4 Kelvin's circulation theorem

Theorem 19 (***Kelvin's circulation theorem).*** *The Euler equations (23) preserve the circulation integral $I(t)$ defined by*

$$I(t) = \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x}\,, \tag{35}$$

where $c(\mathbf{u})$ is a closed circuit moving with the fluid at velocity $\mathbf{u}$.

Proof. The dynamical definition of the Lie derivative in (27) yields the following for the time rate of change of this circulation integral:

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})} \mathbf{v}\cdot d\mathbf{x} &= \oint_{c(\mathbf{u})} \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x}) \\
&= \oint_{c(\mathbf{u})} \left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x^j}u^j + v_j\frac{\partial u^j}{\partial \mathbf{x}}\right)\cdot d\mathbf{x} \\
&= - \oint_{c(\mathbf{u})} \nabla\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right)\cdot d\mathbf{x} \\
&= - \oint_{c(\mathbf{u})} d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right) = 0\,.
\end{aligned}
\tag{36}
$$

The last step in the proof follows, because the integral of an exact differential around a closed loop vanishes. □

The exterior derivative of the Euler fluid equation in the form (29) yields Stokes' theorem, after using the commutativity of the exterior and Lie derivatives $[d,\ \pounds_{\mathbf{u}}] = 0$,

$$
\begin{aligned}
d\pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= d\big(-\mathbf{u}\times \mathrm{curl}\mathbf{v}\cdot d\mathbf{x} + d(\mathbf{u}\cdot\mathbf{v})\big) \\
&= \pounds_{\mathbf{u}}(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) \\
&= -\,\mathrm{curl}\big(\mathbf{u}\times \mathrm{curl}\,\mathbf{v}\big)\cdot d\mathbf{S} \\
&= \big[\mathbf{u}\cdot\nabla \mathrm{curl}\,\mathbf{v} + \mathrm{curl}\,\mathbf{v}(\mathrm{div}\,\mathbf{u}) - (\mathrm{curl}\,\mathbf{v})\cdot\nabla\mathbf{u}\big]\cdot d\mathbf{S}\,, \\
(\text{by } \mathrm{div}\,\mathbf{u}=0)\ \ &= \big[\mathbf{u}\cdot\nabla \mathrm{curl}\,\mathbf{v} - (\mathrm{curl}\,\mathbf{v})\cdot\nabla\mathbf{u}\big]\cdot d\mathbf{S} \\
&=: [\,u,\ \mathrm{curl}\,v\,]\cdot d\mathbf{S}\,,
\end{aligned}
\tag{37}
$$

where $[\,u,\ \mathrm{curl}\,v\,]$ denotes (minus) the ***Jacobi–Lie bracket*** of the vector fields u and $\mathrm{curl}\,v$.

This calculation proves the following.

Theorem 20. *Euler's fluid equations (24) imply that*

$$
\frac{\partial\omega}{\partial t} = -\,[\,u,\ \omega\,]
\tag{38}
$$

where $[\,u,\ \omega\,]$ *denotes the Jacobi–Lie bracket of the divergenceless vector fields* u *and* $\omega := \mathrm{curl}\,v$.

The exterior derivative of Euler's equation in its geometric form (29) is equivalent to the curl of its vector form (24). That is,

$$d\left(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x})=\left(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\right)(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S})=0\,. \tag{39}$$

Hence from the calculation in (37) and the Helmholtz vorticity equation (39) we have

$$\left(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\right)(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S})=\Big(\partial_t\boldsymbol{\omega}-\mathrm{curl}(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S}=0\,, \tag{40}$$

in which one denotes $\boldsymbol{\omega}:=\mathrm{curl}\,\mathbf{v}$. This Lie-derivative version of the Helmholtz vorticity equation may be used to prove the following form of Stokes' theorem for the Euler equations in a rotating frame.

Theorem 21. ***[Kelvin/Stokes' theorem for vorticity of a rotating fluid]***

$$\begin{aligned}\frac{d}{dt}\oint_{c(\mathbf{u})}\mathbf{v}\cdot d\mathbf{x}&=\frac{d}{dt}\iint_{S(\mathbf{u})}\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}\\&=\iint_{S(\mathbf{u})}\left(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\right)(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S})\\&=\iint_{S(\mathbf{u})}\Big(\partial_t\boldsymbol{\omega}-\mathrm{curl}\,(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S}=0\,,\end{aligned} \tag{41}$$

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid.

3.5 The conserved helicity of ideal incompressible flows

Definition 22 (Helicity)**.** *The helicity $\Lambda[\operatorname{curl}\mathbf{v}]$ of a divergence-free vector field* $\operatorname{curl} v$ *that is tangent to the boundary ∂D of a simply connected domain $D \in \mathbb{R}^3$ is defined as*

$$\Lambda[\operatorname{curl}\mathbf{v}] = \int_D \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\, d^{\,3}x\,, \tag{42}$$

where $\mathbf{v}$ is a divergence-free vector-potential for the field $\operatorname{curl}\mathbf{v}$*.*

Remark 23.
The helicity is unchanged by adding a gradient to the vector $\mathbf{v}$. Thus, $\mathbf{v}$ is not unique and $\operatorname{div}\mathbf{v} = 0$ is not a restriction for simply connected domains in $\mathbb{R}^3$, provided $\operatorname{curl}\mathbf{v}$ is tangent to the boundary ∂D.

The helicity of a vector field $\operatorname{curl} v$ measures the total linking of its field lines, or their relative winding. (For details and mathematical history, see [ArKh1998].) The idea of helicity goes back to Helmholtz and Kelvin in the 19th century. The principal feature of this concept for fluid dynamics is embodied in the following theorem.

Theorem 24 (***Euler flows preserve helicity***)**.** *When homogeneous or periodic boundary conditions are imposed, Euler's equations for an ideal incompressible fluid flow in a rotating frame with Coriolis parameter* $\operatorname{curl}\mathbf{R} = 2\boldsymbol{\Omega}$ *preserves the helicity*

$$\Lambda[\operatorname{curl}\mathbf{v}] = \int_D \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\, d^{\,3}x\,, \tag{43}$$

with $\mathbf{v} = \mathbf{u}+\mathbf{R}$*, for which* $\mathbf{u}$ *is the divergenceless fluid velocity* $(\operatorname{div}\mathbf{u} = 0)$ *and* $\operatorname{curl}\mathbf{v} = \operatorname{curl}\mathbf{u} + 2\boldsymbol{\Omega}$ *is the total vorticity.*

Proof. Rewrite the geometric form of the Euler equations (29) for rotating incompressible flow with unit mass density in terms of the circulation one-form $v := \mathbf{v}\cdot d\mathbf{x}$ as

$$(\partial_t + \pounds_u)v = -\,d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right) =: -\,d\,\varpi\,, \tag{44}$$

and $\pounds_u\, d^{\,3}x = 0$. Here, ϖ is an augmented pressure variable,

$$\varpi := p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\,. \tag{45}$$

The fluid velocity vector field is denoted as $u = \mathbf{u}\cdot\nabla$ with $\operatorname{div}\mathbf{u} = 0$. Then the ***helicity density***, defined as

$$v\wedge dv = \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\, d^{\,3}x = \lambda\, d^{\,3}x\,, \quad \text{with} \quad \lambda = \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\,, \tag{46}$$

obeys the dynamics it inherits from the Euler equations,

$$\big(\partial_t + \pounds_u\big)(v \wedge dv) = -d\varpi \wedge dv - v \wedge d^2\varpi = -d(\varpi\, dv)\,, \tag{47}$$

after using $d^2\varpi = 0$ and $d^2v = 0$. In vector form, this result may be expressed as a conservation law,

$$\big(\partial_t\lambda + \operatorname{div}\lambda\mathbf{u}\big)d^{\,3}x = -\operatorname{div}(\varpi\operatorname{curl}\mathbf{v})\,d^{\,3}x\,. \tag{48}$$

Consequently, the time derivative of the integrated helicity in a domain D obeys

$$\begin{aligned}\frac{d}{dt}\Lambda[\operatorname{curl}\mathbf{v}] &= \int_D \partial_t\lambda\,d^{\,3}x = -\int_D \operatorname{div}(\lambda\mathbf{u} + \varpi\operatorname{curl}\mathbf{v})\,d^{\,3}x \\ &= -\oint_{\partial D}(\lambda\mathbf{u} + \varpi\operatorname{curl}\mathbf{v})\cdot d\mathbf{S}\,,\end{aligned} \tag{49}$$

which vanishes when homogeneous, or periodic, or even Neumann boundary conditions are imposed on the values of $\mathbf{u}$ and $\operatorname{curl}\mathbf{v}$ at the boundary ∂D. □

Remark 25.

This result means the ***helicity integral***

$$\Lambda[\operatorname{curl}\mathbf{v}] = \int_D \lambda\,d^{\,3}x$$

is conserved in periodic domains, or in all of $\mathbb{R}^3$ with vanishing boundary conditions at spatial infinity. However, if either the velocity or total vorticity at the boundary possesses a nonzero normal component, then the boundary is a source of helicity (that is, it causes winding of field lines of curl $\mathbf{v}$). For a fixed impervious boundary, the normal component of velocity does vanish, but no such condition is imposed on the total vorticity by the physics of fluid flow. Thus, we have the following.

Corollary 26. *A flux of total vorticity* curl $\mathbf{v}$ *into the domain is a source of helicity.*

Exercise. Use Cartan's formula in (27) to compute $\pounds_u(v \wedge dv)$ in Equation (47).

★

Theorem 27 (Diffeomorphisms preserve helicity)**.** *The helicity $\Lambda[\xi]$ of any divergenceless vector field ξ is preserved under the action on ξ of any volume-preserving diffeomorphism of the manifold M [ArKh1998].*

Remark 28 (Helicity is a topological invariant)**.**
The helicity $\Lambda[\xi]$ is a topological invariant, not a dynamical invariant, because its invariance is independent of which diffeomorphism acts on ξ. This means the invariance of helicity is independent of which Hamiltonian flow produces the diffeomorphism. This is the hallmark of a Casimir function. Although it is defined above with the help of a metric, every volume-preserving diffeomorphism carries a divergenceless vector field ξ into another such field with the same helicity. However, independently of any metric properties, the action of diffeomorphisms does not create or destroy linkages of the characteristic curves of divergenceless vector fields.

3.6 Ertel theorem for potential vorticity

Euler–Boussinesq equations The Euler–Boussinesq equations for the incompressible motion of an ideal flow of a stratified fluid and velocity $\mathbf{u}$ satisfying $\operatorname{div}\mathbf{u} = 0$ in a rotating frame with Coriolis parameter $\operatorname{curl}\mathbf{R} = 2\boldsymbol{\Omega}$ are given by

$$\underbrace{\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{\text{acceleration}} = \underbrace{-\,gb\nabla z}_{\text{buoyancy}} + \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}} \tag{50}$$

where $-g\nabla z$ is the constant downward acceleration of gravity and b is the buoyancy, a scalar function of space and time which satisfies the ***advection relation***,

$$\partial_t b + \mathbf{u}\cdot\nabla b = 0\,. \tag{51}$$

As for Euler's equations without buoyancy, requiring preservation of the divergence-free (volume-preserving) constraint $\nabla\cdot\mathbf{u} = 0$ results in a Poisson equation for pressure p,

$$-\,\Delta\left(p + \frac{1}{2}|\mathbf{u}|^2\right) = \operatorname{div}(-\,\mathbf{u}\times\operatorname{curl}\mathbf{v}) + g\partial_z b\,, \tag{52}$$

which satisfies a Neumann boundary condition because the velocity $\mathbf{u}$ must be tangent to the boundary. where we denote

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{R}\,, \quad \boldsymbol{\omega} = \operatorname{curl}\mathbf{v} = \operatorname{curl}\mathbf{u} + 2\boldsymbol{\Omega}\,, \tag{53}$$

The Newton's law form of the Euler–Boussinesq equations (50) may be rearranged as

$$\partial_t \mathbf{v} - \mathbf{u} \times \operatorname{curl} \mathbf{v} + gb\nabla z + \nabla \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) = 0\,, \qquad (54)$$

where $\mathbf{v} \equiv \mathbf{u} + \mathbf{R}$ and $\nabla \cdot \mathbf{u} = 0$.

Exercise. Prove that the Euler–Boussinesq equations in (50) emerge as Euler-Poincaré equations from Hamilton's variational principle for the following action integral.

$$0 = \delta S = \delta \int_0^T \frac{1}{2} D|\mathbf{u}|^2 + D\mathbf{u} \cdot \mathbf{R} - Dbz - p(D-1)\, d^3x\, dt$$

★

Theorem 29. ***[The Kelvin/Stokes' theorem for a stratified, rotating fluid]***

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})} \mathbf{v}\cdot d\mathbf{x} &= \frac{d}{dt}\iint_{S(\mathbf{u})} \operatorname{curl}\mathbf{v}\cdot d\mathbf{S} \\
&= \iint_{S(\mathbf{u})} \Big(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\Big)(\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}) \\
&= \iint_{S(\mathbf{u})} \Big(\partial_t\,\boldsymbol{\omega} - \operatorname{curl}\,(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S} \\
&= \iint_{S(\mathbf{u})} \Big(-\; g\nabla b \times \nabla z\Big)\cdot d\mathbf{S}\,,
\end{aligned}
\tag{55}
$$

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid. Thus, non-alignment of the gradient of buoyancy ∇b with the vertical ∇z creates circulation. Compare this result with equation (41) in the absence of stratification.

Geometrically, equation (54) may be written as

$$
\big(\partial_t + \pounds_u\big)v + gbdz + d\varpi = 0\,, \tag{56}
$$

where ϖ is defined in (45). In addition, the buoyancy satisfies

$$
\big(\partial_t + \pounds_u\big)b = 0\,, \quad \text{with} \quad \pounds_u\, d^{\,3}x = 0\,. \tag{57}
$$

The fluid velocity vector field is denoted as $u = \mathbf{u} \cdot \nabla$ and the circulation one-form as $v = \mathbf{v} \cdot d\mathbf{x}$. The exterior derivatives of the two equations in (56) are written as

$$\big(\partial_t + \pounds_u\big) dv = -g db \wedge dz \quad \text{and} \quad \big(\partial_t + \pounds_u\big) db = 0\,. \tag{58}$$

Consequently, one finds from the product rule for Lie derivatives that

$$\big(\partial_t + \pounds_u\big)(dv \wedge db) = 0 \quad \text{or} \quad \partial_t\, q + \mathbf{u} \cdot \nabla q = 0\,, \tag{59}$$

in which the quantity

$$q = \nabla b \cdot \operatorname{curl} \mathbf{v}\,, \tag{60}$$

is called ***potential vorticity*** and is abbreviated as PV. The potential vorticity is an important diagnostic for many processes in geophysical fluid dynamics. Conservation of PV on fluid parcels is called ***Ertel's theorem***.

Remark 30 (***Ertel's theorem for the vorticity vector field).***
Writing the vorticity vector field $\omega = \boldsymbol{\omega} \cdot \nabla$, we have

$$\big(\partial_t + \pounds_u\big)\omega = \partial_t \omega + [u,\, \omega] = g \nabla z \times \nabla b \cdot \nabla\,.$$

Thus, conservation of the potential vorticity may also be proved by the product rule, as

$$\big(\partial_t + \pounds_u\big) q = \big(\partial_t + \pounds_u\big)(\boldsymbol{\omega} \cdot \nabla b) = \big(\partial_t + \pounds_u\big)(\omega b) = \Big(\big(\partial_t + \pounds_u\big)\omega\Big) b + \omega\big(\partial_t + \pounds_u\big) b = 0\,.$$

Remark 31 (***Material derivative formulation).***
Denoting

$$\frac{D}{Dt} = \partial_t + \pounds_u \quad \text{and} \quad \omega = \boldsymbol{\omega}\cdot\nabla$$

provides an intuitive expression of the Ertel theorem (59) that helps understand it in terms of the time derivative $\frac{D}{Dt}$ following the flow of the fluid particles. Namely, it suggests writing in vector form

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla) = g\nabla z\times\nabla b\cdot\nabla \quad \text{and} \quad \frac{Db}{Dt} = 0\,,$$

so that the product rule for derivatives yields conservation of PV on fluid parcels, as

$$\frac{Dq}{Dt} = \frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla b) = \Big(\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla)\Big)b + (\boldsymbol{\omega}\cdot\nabla)\frac{Db}{Dt} = g\nabla z\times\nabla b\cdot\nabla b + (\boldsymbol{\omega}\cdot\nabla)\frac{Db}{Dt} = 0\,.$$

Remark 32 (***The conserved quantities associated with Ertel's theorem).***
The constancy of the scalar quantities b and q on fluid parcels implies conservation of the spatially integrated quantity,

$$C_\Phi = \int_D \Phi(b,q)\,d^{\,3}x\,, \tag{61}$$

for any smooth function Φ for which the integral exists.

Proof.

$$
\begin{aligned}
\frac{d}{dt}C_\Phi &= \int_D \Phi_b\partial_t b + \Phi_q\partial_t q \, d^{\,3}x = -\int_D \Phi_b \mathbf{u}\cdot\nabla b + \Phi_q \mathbf{u}\cdot\nabla q \, d^{\,3}x \\
&= -\int_D \mathbf{u}\cdot\nabla\Phi(b,q)\, d^{\,3}x = -\int_D \nabla\cdot\big(\mathbf{u}\,\Phi(b,q)\big)\, d^{\,3}x = -\oint_{\partial D} \Phi(b,q)\,\mathbf{u}\cdot\hat{\mathbf{n}}\, d\,S = 0\,,
\end{aligned}
$$

when the normal component of the velocity $\mathbf{u}\cdot\hat{\mathbf{n}}$ vanishes at the boundary ∂D.

□

Remark 33 (***Energy conservation).***
In addition to C_Φ, the Euler–Boussinesq fluid equations (54) also conserve the total energy

$$
E = \int_D \frac{1}{2}|\mathbf{u}|^2 + bz \; d^{\,3}x\,, \tag{62}
$$

which is the sum of the kinetic and potential energies.

We do not develop the Hamiltonian formulation of the three-dimensional stratified rotating fluid equations studied here. However, one may imagine that the conserved quantity C_Φ with the arbitrary function Φ would play an important role. For more explanation in the framework of Geometric Mechanics, see [Ho2011GM] and references therein.

3.7 Rotating shallow water (RSW) equations

Consider dynamics of rotating shallow water (RSW) on a two dimensional domain with horizontal planar coordinates $\mathbf{x} = (x, y)$. This RSW motion is governed by the following nondimensional equations for variables depending on $(\mathbf{x}, t)$ comprising the horizontal fluid velocity vector $\mathbf{u} = (u, v)$ and the total depth η,

$$\epsilon \frac{d}{dt}\mathbf{u} + f(\mathbf{x})\hat{\mathbf{z}} \times \mathbf{u} + \nabla h = 0\,, \qquad \frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}) = 0\,, \tag{63}$$

with notation

$$\frac{d}{dt} := \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \quad \text{and} \quad h := \left(\frac{\eta - B}{\epsilon \mathcal{F}} \right),$$

where $\epsilon \ll 1$ and $\mathcal{F} = O(1)$ are nondimensional constants. These equations include spatially variable Coriolis parameter $f(\mathbf{x})\hat{\mathbf{z}} = \mathrm{curl}\mathbf{R}(\mathbf{x})$ and mean depth $B = B(\mathbf{x})$.

Exercise.

(i) Show that the RSW equations in (63) follow as Euler-Poincaré equations

$$\big(\partial_t + \pounds_u\big)\frac{1}{\eta}\frac{\delta l}{\delta u} = \frac{1}{\eta}\frac{\delta l}{\delta \eta} \diamond \eta \quad \text{and} \quad \big(\partial_t + \pounds_u\big)\big(\eta\, d^2x\big) = 0\,,$$

from Hamilton's variational principle for the following action integral.

$$0 = \delta S \text{ with } S = \int_0^T l(\mathbf{u}, \eta) dt \text{ and } l(\mathbf{u}, \eta) = \int \frac{\epsilon}{2}\eta|\mathbf{u}|^2 + \eta \mathbf{u} \cdot \mathbf{R}(\mathbf{x}) - \frac{(\eta - B(\mathbf{x}))^2}{2\epsilon \mathcal{F}} \, d^2x$$

in which $\eta(\mathbf{x}, t)\, d^2x$ is an advected quantity. Recall that $\diamond : V^* \times V \to \mathfrak{X}^*$ is defined by $\langle \frac{\delta \ell}{\delta a} \diamond a \,,\, v \rangle := \langle \frac{\delta \ell}{\delta a} \,,\, -\pounds_v a \rangle$ for vector field $v \in \mathfrak{X}$ and L^2 pairing $\langle \,\cdot\,,\,\cdot\, \rangle$.

(ii) Use the Euler-Poincaré equations to show that the RSW equations satisfy Kelvin's circulation theorem

$$\frac{d}{dt} \oint_{c_t} \mathbf{v} \cdot d\mathbf{x} = 0 \,,$$

with $\mathbf{v} = \epsilon \mathbf{u} + \mathbf{R}(\mathbf{x})$.

(iii) Use the Euler-Poincaré equations to show that the RSW equations satisfy

$$\big(\partial_t + \pounds_u\big) d\big(\mathbf{v} \cdot d\mathbf{x}\big) = 0 \,,$$

with $\mathbf{v} = \epsilon \mathbf{u} + \mathbf{R}(\mathbf{x})$.

(iv) Show that $d(\mathbf{v} \cdot d\mathbf{x}) = \omega \, d^2x$, with $\omega := \hat{\mathbf{z}} \cdot \mathrm{curl}\mathbf{v}$.

(v) Use $(\partial_t + \pounds_u)(\omega \, d^2x) = 0$ obtained in the previous two parts to derive conservation of potential vorticity on fluid particles.

★

Answer.

1. The Euler-Poincaré equations are

$$\big(\partial_t + \pounds_u\big)\frac{1}{\eta}\frac{\delta l}{\delta u} = \frac{1}{\eta}\frac{\delta l}{\delta \eta}\diamond \eta \quad \text{and} \quad \big(\partial_t + \pounds_u\big)\big(\eta\, d^2x\big) = 0\,,$$

where $\eta^{-1}\frac{\delta l}{\delta u} = (\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x}))\cdot d\mathbf{x} =: \mathbf{v}\cdot d\mathbf{x}$ and $\eta^{-1}\frac{\delta l}{\delta\eta}\diamond\eta = d(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h)$. Thus,

$$\big(\partial_t + \pounds_u\big)(\mathbf{v}\cdot d\mathbf{x}) = d\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big)$$

with $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$.

2. Integrating the previous equation around a loop moving with the fluid produces

$$\frac{d}{dt}\oint_{c_t}\mathbf{v}\cdot d\mathbf{x} = \oint_{c_t} d\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big) = 0\,,$$

with $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$.

3. The differential of the Euler-Poincaré equation yields with $\omega := \hat{\mathbf{z}}\cdot\mathrm{curl}\mathbf{v}$

$$\big(\partial_t + \pounds_u\big)(\omega d^2x) = \big(\partial_t + \pounds_u\big)d(\mathbf{v}\cdot d\mathbf{x}) = d^2\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big) = 0$$

upon commuting the differential d with the Lie derivative and using $d^2 = 0$.

4. By direct computation,

$$
\begin{aligned}
d(\mathbf{v}\cdot d\mathbf{x}) &= v_{i,j}dx^j\wedge dx^i = v_{1,2}dx^2\wedge dx^1 + v_{2,1}dx^1\wedge dx^2\\
&= (v_{2,1}-v_{1,2})\,d^2x = \hat{\mathbf{z}}\cdot \mathrm{curl}\mathbf{v}\,d^2x = \omega\,d^2x
\end{aligned}
$$

5. We have $(\partial_t+\pounds_u)(\omega d^2x)$ and $(\partial_t+\pounds_u)(\eta d^2x)$. Therefore, by the product rule for the evolutionary operator $(\partial_t+\pounds_u)$ we have

$$
0 = (\partial_t+\pounds_u)\Big(\frac{\omega}{\eta}(\eta d^2x)\Big) = \Big((\partial_t+\pounds_u)\frac{\omega}{\eta}\Big)(\eta d^2x) + \frac{\omega}{\eta}(\partial_t+\pounds_u)(\eta d^2x)
$$

Since the second term vanishes via the continuity equation, $(\partial_t+\pounds_u)(\eta d^2x)$, the first term yields

$$
0 = (\partial_t+\pounds_u)\frac{\omega}{\eta} = \left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\frac{\omega}{\eta}\,.\quad \text{Hence,}\quad \frac{dq}{dt}=0\,,\quad \text{with}\quad q:=\omega/\eta\,.
$$

This is conservation of potential vorticity on fluid particles.

▲

References

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