3 M3-4-5 A16 Assessed Problems # 3 Due 2pm 15 Dec 2011

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 3.1 ***A cyclically symmetric problem with three oscillators on*** $\mathbb{C}^3$

Consider the (1,2,3) cyclically symmetric dynamical system,

$$\frac{da_1^*}{dt} = a_2a_3\,, \quad \frac{da_2^*}{dt} = a_3a_1\,, \quad \frac{da_3^*}{dt} = a_1a_2\,,$$

where $a_1, a_2, a_3 \in \mathbb{C}^3$ and a_k^* denotes the complex conjugate of a_k.

Problem statement:

(a) Show that this system is Hamiltonian for the canonical Poisson bracket $\{a_j\,,\,a_k^*\} = -2i\delta_{jk}$.

Note: a dynamical system $\dot{x} = F(x)$ is called Hamiltonian, if it can be expressed as $\dot{x} = \{x, H\}$ for a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ and Hamiltonian $H(x)$. Here you are given the Poisson bracket and the phase space, $T^*\mathbb{C}^3$. Just find the Hamiltonian.

Answer

For the canonical Poisson bracket $\{a_j^*\,,\,a_k\} = 2i\delta_{jk}$

$$\frac{da_k^*}{dt} = \{a_k^*, H\} = 2i\frac{\partial H}{\partial a_k} \quad \text{yields} \quad \frac{da_1^*}{dt} = a_2a_3, \quad \text{and cyclic permutations,}$$

for the Hamiltonian $H : \mathbb{C}^3 \to \mathbb{R}$ given by

$$H = \mathrm{Im}(a_1a_2a_3) = \frac{1}{2i}(a_1a_2a_3 - a_1^*a_2^*a_3^*) = \frac{1}{2i}\left(\prod_{k=1}^3 a_k - \prod_{k=1}^3 a_k^*\right).$$

▲

(b) Find two other constants of motion that generate S^1 symmetries of the Hamiltonian.

Answer

Two other constants of motion for this system are given by

$$I_2 = |a_1|^2 - |a_2|^2, \quad I_3 = |a_1|^2 - |a_3|^2\,.$$

These two quantities are conserved, because they generate the following two S^1 symmetries of the Hamiltonian in the previous part,

$$I_2 : (a_1, a_2, a_3) \to \left(e^{-2i\phi_2}a_1,\, e^{2i\phi_2}a_2,\, a_3\right),$$
$$I_3 : (a_1, a_2, a_3) \to \left(e^{-2i\phi_3}a_1,\, a_2,\, e^{2i\phi_3}a_3\right).$$

▲

(c) Is this dynamical system completely integrable? That is, are there enough symmetries and constants of motion in involution to reduce it to a Hamiltonian system on the phase plane, and reconstruct the 'angles' canonically conjugate to the constants of motion?

(i) Begin by showing that the following transformation of variables is canonical,

$$a_1 = ze^{-i(\phi_2+\phi_3)}\,, \qquad a_2 = |a_2|e^{i\phi_2}\,, \qquad a_3 = |a_3|e^{i\phi_3}\,, \quad \text{with} \quad z = |z|e^{i\zeta} \in \mathbb{C}.$$

(ii) Show that the Hamiltonian may be written solely in terms of z, z^*, I_2, I_3.

(iii) Are there any limitations to the range of variables? If so, what are they?

(iv) Draw a figure showing the level sets of H in the (z_1, z_2) phase plane with $z = z_1 + iz_2$.

(v) Write a closed equation in z that provides the dynamics of both the amplitude and phase of $z = |z|e^{i\zeta}$. Show that solving it would determine the magnitudes, $|a_2|$ and $|a_3|$.

(vi) Extra credit: Write the phase equations and solve for the phases ζ, ϕ_2 and ϕ_3.

Answer

This 3 degree-of-freedom canonically Hamiltonian system has 2 additional constants of motion which will turn out to be in involution, that is, $\{I_2, I_3\} = 0$. Therefore, the system is completely integrable, i.e., it is reducible to action-angle form. It's enough to reduce it to a phase plane and identify the action-angle variables. The reduction follows the same procedure in Chapter 5 of GM, Part 1, for the 3-wave equations and uses a remarkable canonical transformation.

(i) The transformation of variables

$$a_1 = ze^{-i(\phi_2+\phi_3)}\,, \qquad a_2 = |a_2|e^{i\phi_2}\,, \qquad a_3 = |a_3|e^{i\phi_3}\,, \quad \text{with} \quad z = |z|e^{i\zeta} \in \mathbb{C},$$

is canonical, since it verifies the relation

$$da_1 \wedge da_1^* + da_2 \wedge da_2^* + da_3 \wedge da_3^* = dz \wedge dz^* + i(dI_2 \wedge d\phi_2 + dI_3 \wedge d\phi_3)\,,$$

with canonical 'action-angle variables' satisfying $\{I_k,\, \phi_k\} = 1$.
The proof of this relation uses polar coordinates, e.g., $a_k = r_k e^{i\phi_k}$, $|a_1|^2 = r_1^2 = |z|^2$, and $|a_k|^2 = r_1^2 - I_k$ for $k = 2, 3$.

(ii) In these complex canonical variables, the Hamiltonian is given by

$$H(z, z^*) = \mathrm{Im}(z)\,\sqrt{|z|^2 - I_2}\sqrt{|z|^2 - I_3} = |z|\sin(\zeta)\,\sqrt{|z|^2 - I_2}\sqrt{|z|^2 - I_3}\,,$$

which is independent of the angles ϕ_2 and ϕ_3, thereby reconfirming that their canonically conjugate actions I_2 and I_3 are constants of the motion in involution, that is, $\{I_2, I_3\} = 0$.

(iii) For the Hamiltonian to remain real-valued, we must have $|z|^2 > I_2$ and $|z|^2 > I_3$. Since the conserved quantities I_2 and I_3 are not sign definite, these inequality conditions should not pose much of a limitation.

(iv) Figure 1 shows the level sets of H in the phase plane, courtesy of M. Ben Slama, J. Kirsten and A. Lucas.

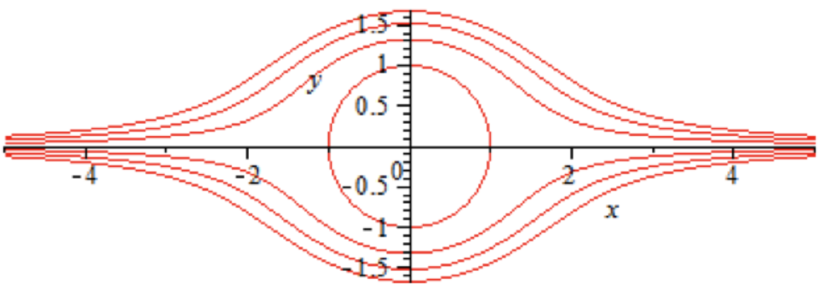

Figure 1: Phase portrait for J = 0 and H = 0 (circle with radius $N/2 = 1$) and $H = 1, 2, 3$.

(v) The Poisson bracket is $\{z, z^*\} = -2i$ and the canonical equations reduce to

$$\dot{z} = \{z, H\} = -2i\frac{\partial H}{\partial z^*} = \sqrt{|z|^2 - I_2}\sqrt{|z|^2 - I_3} - \left(\frac{1}{|z|^2 - I_2} + \frac{1}{|z|^2 - I_3}\right) Hiz\,.$$

This provides the dynamics of both the amplitude and phase of $z = |z|e^{i\zeta}$ as an integrable Hamiltonian system in a planar phase space.

(vi) This was expected, because the original system has enough constants of motion in involution to be completely integrable. In this case, polar coordinates would be summoned. In particular, we have

$$\frac{d|z|}{dt} = \frac{z + z^*}{2|z|}\sqrt{|z|^2 - I_2}\sqrt{|z|^2 - I_3} = \cos(\zeta)\sqrt{|z|^2 - I_2}\sqrt{|z|^2 - I_3}$$

Writing $\cos(\zeta)$ in terms of H, $|z|^2$, I_2 and I_3, and setting $|z|^2 = Q$ yields

$$\frac{1}{2}\frac{dQ}{dt} = \sqrt{Q(Q - I_2)(Q - I_3) - H^2}$$

which yields $Q(t)$ as an elliptic integral. This solution for $Q(t)$ determines the magnitudes, $|a_2| = \sqrt{Q - I_2}$ and $|a_3| = \sqrt{Q - I_3}$.

(vii) After the solution for $Q(t)$ is known, the phases, or angle variables, ζ, ϕ_2 and ϕ_3 may be reconstructed by quadratures from their dynamical equations

$$\dot{\zeta} = \frac{\partial H}{\partial Q}\,, \quad \dot{\phi}_k = \frac{\partial H}{\partial I_k}$$

with

$$H = \sin(\zeta)\sqrt{Q(Q - I_2)(Q - I_3)}$$

▲

(d) Will the analysis here generalise to n degrees of freedom? Is the corresponding system on $\mathbb{C}^n$ completely integrable? Write this system explicitly and justify your answer.

Answer

The (1,2,...,n) cyclically symmetric problem with n oscillators on $\mathbb{C}^n$ is represented by the dynamical system

$$\frac{da_k^*}{dt} = \prod_{j\neq k}^{n} a_j\,, \quad j, k = 1, \dots, n,$$

and everything goes through as before, with natural generalisations.
For example, the Hamiltonian $H : \mathbb{C}^n \to \mathbb{R}$ will be given by

$$H = \mathrm{Im}\left(\prod_{k=1}^{n} a_k\right) = \frac{1}{2i}\left(\prod_{k=1}^{n} a_k - \prod_{k=1}^{n} a_k^*\right).$$

This Hamiltonian admits S^1 symmetries generated canonically by the conserved quantities

$$I_k = |a_1|^2 - |a_k|^2\,, \quad k = 2, 3, \dots, n.$$

These $n - 1$ constants of motion are enough to solve the problem via the canonical transformation of variables

$$a_1 = ze^{-i\sum_{k=2}^{n}\phi_k}, \quad \text{with} \quad z = |z|e^{i\zeta} \in \mathbb{C}, \quad \text{and} \quad a_k = |a_k|e^{i\phi_k},$$

which verifies

$$\sum_{k=1}^{n} da_k \wedge da_k^* = dz \wedge dz^* + i\sum_{k=2}^{n} dI_k \wedge d\phi_k\,,$$

so the motion again reduces to the (z, z^*) phase plane, with $n-1$ action-angle variables. The Hamiltonian in these variables becomes

$$H(z, z^*, \{I_2 \dots, I_k\}) = \mathrm{Im}(z)\sqrt{\prod_{k=2}^{n}(|z|^2 - I_k)} = \sin(\zeta)\sqrt{|z|^2\prod_{k=2}^{n}(|z|^2 - I_k)}\,,$$

and the solution proceeds as before, with, e.g.,

$$\dot{z} = \{z, H\} = -2i\frac{\partial H}{\partial z^*} = \sqrt{\prod_{k=2}^{n}(|z|^2 - I_k)} - Hiz\sum_{k=2}^{n}\frac{1}{|z|^2 - I_k}\,.$$

For example, one finds

$$\frac{d|z|}{dt} = \cos(\zeta)\sqrt{\prod_{k=2}^{n}(|z|^2 - I_k)}\,,$$

which yields, upon setting $|z|^2 = Q$,

$$\frac{1}{2}\frac{dQ}{dt} = \sqrt{Q\prod_{k=2}^{n}(Q - I_k) - H^2}$$

so that $Q(t)$ is found as a hyper-elliptic integral.

▲

Exercise 3.2 ***Spherical pendulum in a constant vertical magnetic field***

Explain the effects that an external constant vertical magnetic field $B_0\hat{\mathbf{e}}_3$ can have on a spherical pendulum with unit charge on its mass.

Draw your conclusions analytically by taking the following steps.

Problem statement:

Begin with the Lagrangian $L(\mathbf{x},\dot{\mathbf{x}}) : T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L(\mathbf{x},\dot{\mathbf{x}}) = \tfrac{1}{2}|\dot{\mathbf{x}}|^2 + B_0\hat{\mathbf{e}}_3\cdot(\mathbf{x}\times\dot{\mathbf{x}}) - g\hat{\mathbf{e}}_3\cdot\mathbf{x} - \tfrac{1}{2}\mu(1-|\mathbf{x}|^2),$$

in which the Lagrange multiplier μ constrains the motion to remain on the sphere S^2 by enforcing $(1-|\mathbf{x}|^2)=0$ when it is varied in Hamilton's principle.

(a) Derive the constrained Euler–Lagrange equation from Hamilton's principle for this Lagrangian.

Answer

$$0 = \delta S = \delta\int_a^b L(\mathbf{x},\dot{\mathbf{x}})dt = \int_a^b \big(\underbrace{\dot{\mathbf{x}} + B_0\hat{\mathbf{e}}_3\times\mathbf{x}}_{\mathbf{y}}\big)\cdot\delta\dot{\mathbf{x}} + \big(\dot{\mathbf{x}}\times B_0\hat{\mathbf{e}}_3 - g\hat{\mathbf{e}}_3\cdot\mathbf{x} - \mu\mathbf{x}\big)\cdot\delta\mathbf{x}\,dt$$

$$= \int_a^b \big(-\ddot{\mathbf{x}} + \dot{\mathbf{x}}\times 2B_0\hat{\mathbf{e}}_3 - g\hat{\mathbf{e}}_3\cdot\mathbf{x} - \mu\mathbf{x}\big)\cdot\delta\mathbf{x}\,dt + \big[\mathbf{y}\cdot\delta\mathbf{x}\big]_a^b$$

The corresponding Euler–Lagrange equation is

$$\ddot{\mathbf{x}} = -g\hat{\mathbf{e}}_3 + \dot{\mathbf{x}}\times 2B_0\hat{\mathbf{e}}_3 + \mu\mathbf{x}\,.$$

This equation preserves both of the TS^2 relations $1-|\mathbf{x}|^2=0$ and $\mathbf{x}\cdot\dot{\mathbf{x}}=0$, provided the Lagrange multiplier is given by

$$\mu = \Big(g\hat{\mathbf{e}}_3 - \dot{\mathbf{x}}\times 2B_0\hat{\mathbf{e}}_3\Big)\cdot\mathbf{x} - |\dot{\mathbf{x}}|^2\,.$$

▲

(b) Use the S^1 symmetry of the Lagrangian under rotations about the vertical to obtain its Noether conservation laws.

Answer

Under rotations about the vertical $\delta\mathbf{x} = \hat{\mathbf{e}}_3\times\mathbf{x}$ and the Noether conserved quantity is the vertical component of angular momentum

$$\mathbf{J}_3 = \mathbf{y}\cdot\hat{\mathbf{e}}_3\times\mathbf{x} = \hat{\mathbf{e}}_3\cdot\mathbf{x}\times\mathbf{y}\,,\quad\text{with}\quad \mathbf{y} = \dot{\mathbf{x}} + B_0\hat{\mathbf{e}}_3\times\mathbf{x} = \frac{\partial L}{\partial\dot{\mathbf{x}}}$$

▲

(c) Transform to the Hamiltonian side and write the Hamiltonian in terms of S^1-invariant quantities that reduce to those used in class for the spherical pendulum in the absence of a magnetic field.

Answer

The S^1 invariants take the same form as before,

$$\begin{array}{llllll} \sigma_1 & = x_3 & \sigma_3 & = y_1^2+y_2^2+y_3^2 & \sigma_5 & = x_1y_1+x_2y_2 \\ \sigma_2 & = y_3 & \sigma_4 & = x_1^2+x_2^2 & \sigma_6 & = x_1y_2-x_2y_1 \end{array}\,.$$

However, this time, the canonical momentum includes the magnetic term,

$$\mathbf{y} = \dot{\mathbf{x}} + B_0\hat{\mathbf{e}}_3\times\mathbf{x}\,.$$

▲

(d) Write the Hamiltonian equations of motion for the S^1 invariants in $\mathbb{R}^3$-bracket form and reduce the dynamics to a phase plane.

Answer

$$
\begin{aligned}
H &= \mathbf{y}\cdot\dot{\mathbf{x}} - L(\mathbf{x},\dot{\mathbf{x}}) \\
&= \frac{1}{2}|\mathbf{y} - B_0\hat{\mathbf{e}}_3\times\mathbf{x}|^2 + g\hat{\mathbf{e}}_3\cdot\mathbf{x} + \tfrac{1}{2}\mu(1-|\mathbf{x}|^2) \\
&= \tfrac{1}{2}|\mathbf{y}|^2 - B_0\hat{\mathbf{e}}_3\cdot\mathbf{x}\times\mathbf{y} + \tfrac{1}{2}B_0^2(x_2^2+x_2^2) + g\hat{\mathbf{e}}_3\cdot\mathbf{x} + \tfrac{1}{2}\mu(1-|\mathbf{x}|^2) \\
&= \tfrac{1}{2}\sigma_3 - B_0\sigma_6 + \tfrac{B_0^2}{2}\sigma_4 + g\sigma_1 + \mu(\underbrace{1-\sigma_4-\sigma_1^2}_{\text{vanishes}})
\end{aligned}
$$

$$
H = \tfrac{1}{2}\sigma_3 - B_0\sigma_6 + \tfrac{B_0^2}{2}(1-\sigma_1^2) + g\sigma_1
$$

$$
C = \sigma_3(1-\sigma_1^2) - \sigma_2^2 - \sigma_6^2 = 0
$$

$$
\dot{\boldsymbol{\sigma}} = \nabla C\times\nabla H = \det\begin{bmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ -2\sigma_1\sigma_3 & -2\sigma_2 & 1-\sigma_1^2 \\ g - B_0^2\sigma_1 & 0 & \frac{1}{2} \end{bmatrix}.
$$

Thus,

$$
\begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3 \end{bmatrix} = \begin{bmatrix} -\sigma_2 \\ \sigma_1\sigma_3 + (1-\sigma_1^2)(g - B_0^2\sigma_1) \\ 2\sigma_2(g - B_0^2\sigma_1) \end{bmatrix}.
$$

One may eliminate σ_3 in favour of the Hamiltonian H by writing

$$
\sigma_3 = 2(H + B_0\sigma_6 - g\sigma_1) - B_0^2(1-\sigma_1^2)
$$

Then a short calculation gives

$$
\ddot{\sigma}_1 = -\frac{d}{d\sigma_1}\Big(\tfrac{B_0^2}{2}\sigma_1^4 - g\sigma_1^3 + (H + B_0\sigma_6 - B_0^2)\sigma_1^2 + g\sigma_1\Big).
$$

This is Newtonian motion in a quartic potential, whose equilibria in general are two centres with a saddle between them. This is something like a Duffing equation, but the potential is not symmetric under reflections $\sigma_1\to-\sigma_1$.

▲

(e) Draw its phase portraits and discuss the types of motion that are available to this system, as the magnitude B_0 of its external magnetic field is varied.

Answer

The phase plane portrait of the orbit is a figure of eight in general, which degenerates to the fish as the magnetic field strength B_0 is decreased to zero.

Figure 2 shows the potentials and phase portraits for $B_0 = 0.0, 0.3, 0.9, 1.7$ (left to right), also courtesy of M. Ben Slama, J. Kirsten and A. Lucas.

▲

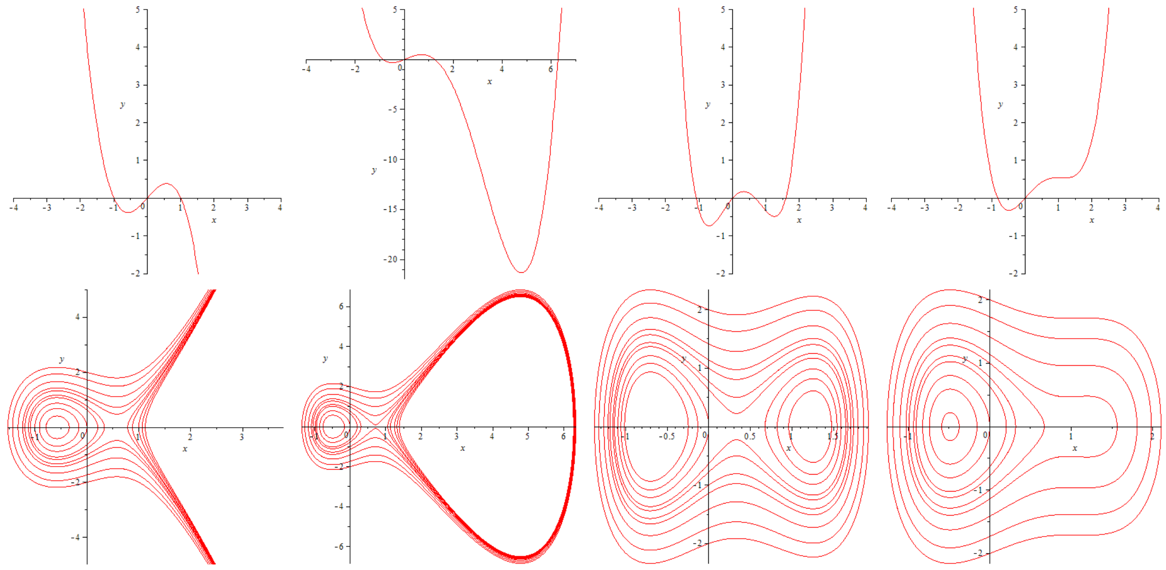

Figure 2: Potentials and phase portraits for $B_0 = 0.0, 0.3, 0.9, 1.7$ (left to right)

Exercise 3.3 ***Complex Maxwell-Bloch equations***

Begin by reading Section 6.1 of GM Part 1, which derives the complex Maxwell-Bloch (CMB) equations

$$\dot{x} = y, \quad \dot{y} = xz, \quad \dot{z} = \frac{1}{2}\left(x^* y + x y^*\right), \quad \text{for } (x, y, z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R},$$

by averaging the Lagrangian for the Maxwell-Schrödinger equations. Section 6.1 of GM Part 1 points out that the 5D CMB system conserves three quantities,

$$H = \frac{i}{2}\left(x^* y - x y^*\right), \quad K = z + \frac{1}{2}|x|^2 \quad \text{and} \quad C = |y|^2 + z^2 .$$

This problem is about understanding the relationships among these conserved quantities.

Let's first introduce notation that will conveniently make all factors real in the subsequent calculations.

$$U_0 = \begin{bmatrix} w_0 & q_0 \\ r_0 & -w_0 \end{bmatrix} = \begin{bmatrix} -4z & -4iy \\ 4iy^* & 4z \end{bmatrix} \qquad U_1 = \begin{bmatrix} w_1 & q_1 \\ r_1 & -w_1 \end{bmatrix} = \begin{bmatrix} Q & 2ix \\ 2ix^* & -Q \end{bmatrix}$$

where the variable Q is a space-holder for the 6^{th} dimension (!) that will later be set to zero.

Problem statement:

(a) Show that the CMB equations arise from the Lax pair (L, M) given by the 2×2 matrix commutator relation

$$\frac{dL}{dt} = [M, L]\,, \quad \text{with} \quad L = A\lambda^2 + U_1\lambda + U_0 \quad \text{and} \quad M = A\lambda + U_1$$

where A is a 2×2 constant matrix to be determined.

Answer

The Lax pair calculation produces

$$\dot{U}_1 = [U_1, U_0] \quad \text{and} \quad \dot{U}_0 = [A, U_0].$$

Explicitly, this is

$$\dot{U}_1 = \begin{bmatrix} \dot{w}_1 & \dot{q}_1 \\ \dot{r}_1 & -\dot{w}_1 \end{bmatrix} = \begin{bmatrix} q_1 r_0 - r_1 q_0 & 2(w_1 q_0 - w_0 q_1) \\ 2(w_0 r_1 - w_1 r_0) & -(q_1 r_0 - r_1 q_0) \end{bmatrix}$$

and

$$\dot{U}_0 = \begin{bmatrix} \dot{w}_1 & \dot{q}_1 \\ \dot{r}_1 & -\dot{w}_1 \end{bmatrix} = \begin{bmatrix} 0 & 2q_0 \\ -2r_0 & 0 \end{bmatrix}, \quad \text{when} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These recover the CMB equations when one sets $w_1 = Q = 0$, which $\dot{w}_1 = 0$ allows.

▲

(b) Explain why the conservation laws for this system h_1, h_2, h_3, w_1 are obtained from $\mathrm{tr}L^2$, the trace of the square of the Lax matrix and how they are related to ones we already know, H, K, C.

Answer

One computes directly

$$\frac{1}{2}\frac{d}{dt}\mathrm{tr}L^2 = \mathrm{tr}(L[M,L]) = \mathrm{tr}(LML) - \mathrm{tr}(LLM) = 0$$

▲

(c) Find these conservation laws.

Answer

$$\mathrm{tr}L^2 = \lambda^4 + 2\lambda^3 w_1 + \lambda^2 h_1 + \lambda h_2 + h_3$$

with the Hamiltonians

$$\begin{aligned} h_1 &= 2w_0 + q_1 r_1 + w_1^2 = -8K + w_1^2 \\ h_2 &= q_0 r_1 + q_1 r_0 + 2w_1 w_0 = -16iH + 2w_1 w_0 \\ h_3 &= q_0 r_0 + w_0^2 = 16C \end{aligned}$$

This calculation recovers all of the constants of motion we had previously known for this problem, plus showing that the additional place-holder variable w_1 is conserved, so that it may be set to zero after the calculations with it are finished. ▲

(d) Compute the three 6×6 Hamiltonian matrix operators of the forms

$$D_0 = -\begin{bmatrix} J_1 & J_2 \\ J_2 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} J_0 & 0 \\ 0 & -J_2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & J_0 \\ J_0 & J_1 \end{bmatrix}$$

by identifying the 3×3 matrices J_0, J_1, J_2, that recover the CMB equations in vector form $U = (q_0, r_0, w_0, q_1, r_1, w_1)^T$ via the tri-Hamiltonian ladder relations,

$$\dot{U} = D_0 \nabla h_3 = D_1 \nabla h_2 = D_2 \nabla h_1 \,.$$

That is, obtain the CMB equations in this vector tri-Hamiltonian ladder form from the Hamiltonians h_1, h_2, h_3 above, after setting $w_1 = 0$, by determining the submatrices J_0, J_1, J_2 that produce the Hamiltonian matrices D_0, D_1, D_2.

Answer

$$J_0 = \begin{bmatrix} 0 & -2w_0 & q_0 \\ 2w_0 & 0 & -r_0 \\ -q_0 & r_0 & 0 \end{bmatrix}, \qquad J_1 = \begin{bmatrix} 0 & -2w_1 & q_1 \\ 2w_1 & 0 & -r_1 \\ -q_1 & r_1 & 0 \end{bmatrix}, \qquad J_0 = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

▲

(e) Find the Casimir functions for the the tri-Hamiltonian operators, D_0, D_1, D_2.

Answer

The Casimir for one Hamiltonian matrix is the Hamiltonian for another, since

$$0 = D_0 \nabla h_1 = D_2 \nabla h_2 = D_1 \nabla h_3 = D_2 \nabla h_3 \,.$$

In particular, the unitarity condition $L = const$ is the Hamiltonian h_3 for D_1 and is the Casimir for both of the other Hamiltonian matrices.

Also, combining these results with the triHamiltonian equations above, leads to the following *ladder relations*,

$$\begin{aligned} \nabla h_2 &= (D_1^{-1} D_2) \nabla h_1 \,, \qquad & (D_1^{-1} D_2) \nabla h_2 = 0 \,, \\ \nabla h_3 &= (D_0^{-1} D_1) \nabla h_2 \,, \qquad & (D_0^{-1} D_1) \nabla h_3 = 0 \,. \end{aligned}$$

▲

(f) Does this system possess enough constants of motion to be completely solved analytically? Prove it.

Answer

The Lax pair and the independence of 4 Poisson-commuting conserved quantities (the Hamiltonians h_1, h_2, h_3) and the quantity w_1 bodes well for the complete integrability of the CMB system. The proof is easily shown by elementary methods, as follows.

The CMB equations

$$\dot{x} = y, \quad \dot{y} = xz, \quad \dot{z} = \frac{1}{2}\left(x^* y + x y^*\right), \quad \text{for } (x, y, z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R},$$

conserve three quantities,

$$H = \frac{i}{2}\left(x^* y - x y^*\right), \quad K = z + \frac{1}{2}|x|^2 \quad \text{and} \quad C = |y|^2 + z^2 \,.$$

On a level set of K we have

$$z = K - \frac{1}{2}|x|^2 \,, \qquad C = |y|^2 + \left(K - \frac{1}{2}|x|^2\right)^2$$

and

$$\dot{x} = y = \frac{\partial C}{\partial y^*} \,, \qquad \dot{y} = x(K - \frac{1}{2}|x|^2) = -\frac{\partial C}{\partial x^*} \,, \qquad \dot{K} = 0 \,.$$

These are *canonical* equations with Hamiltonian C and Poisson bracket $\{x, y^*\} = 1$. Under this bracket the infinitesimal transformations of x, y generated by H are

$$x'(\phi) = \{x, H\} = \frac{\partial H}{\partial y^*} = -\frac{i}{2} x \,, \qquad y'(\phi) = \{y, H\} = -\frac{\partial H}{\partial x^*} = -\frac{i}{2} y \,.$$

The finite transformations are phase shifts $x(\phi) = e^{i\phi/2} x(0)$ and $y(\phi) = e^{i\phi/2} y(0)$. This S^1 symmetry suggests we should either transform to S^1 invariants, or choose polar coordinates.

(i) If we transform to S^1 invariants we will find an $\mathbb{R}^3$ bracket for say

$$R_1 = |x|^2, \quad R_2 = |y|^2, \quad R_3 + iH = 2x^* y \,,$$

in which $H^2 = R_1 R_2 - R_3^2$ and $C = R_2 + (K - R_1/2)^2$. Level sets of H^2 are hyperboloids of revolution and level sets of C are parabolic cylinders. The motion takes place in $\mathbb{R}^3$ in the usual way along the intersections of these quadratic surfaces. Thus, the CMB system is completely integrable.

(ii) The motion may also be visualised in complex polar coordinates, in which we write

$$x = Qe^{i\theta}, \text{ so that } \dot{x} = y \text{ implies } y = (\dot{Q} + iQ\dot{\theta})e^{i\theta}.$$

In these coordinates, level sets of H^2 and C are expressed as

$$H = \mathrm{Im}(xy^*) = -\,Q^2\dot{\theta} \quad \text{and} \quad C/2 = C' := \frac{1}{2}\dot{Q}^2 + \frac{H^2}{2Q^2} + \frac{1}{2}\left(K - \frac{Q^2}{2}\right)^2 .$$

We then have the phase plane relation

$$\frac{1}{2}\dot{Q}^2 = C' - \frac{H^2}{2Q^2} + \frac{1}{2}\left(K - \frac{Q^2}{2}\right)^2$$

which yields a quadrature for $Q(t)$. We also have

$$\dot{\theta} = -\,H/Q^2\,, \quad \dot{H} = 0\,,$$

which is a second quadrature for $\theta(t)$, given H and $Q(t)$ obtained from the first quadrature. Thus, having been reduced to quadratures for $Q(t)$ and $\theta(t)$ on a level set of Z, the CMB system is completely integrable.

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