M3/4A16 Assessed Coursework 2 **Darryl Holm**
Due in class Tuesday November 24, 2009

[#1] ***Reduced Kepler problem***

Newton's equation for the ***reduced Kepler problem*** *for planetary motion is*

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = 0\,, \tag{1}$$

in which μ is a constant and $r = |\mathbf{r}|$ with $\mathbf{r} \in \mathbb{R}^3$.

Scale invariance of this equation under the changes $R \to s^2 R$ and $T \to s^3 T$ in the units of space R and time T for any constant (s) means that it admits families of solutions whose space and time scales are related by $T^2/R^3 = const.$ This is ***Kepler's Third Law****.*

[A] *Show that Newton's equation (1) conserves the quantities,*

$$\begin{aligned} E &= \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{r} \quad \text{(energy)}\,, \\ \mathbf{L} &= \mathbf{r} \times \dot{\mathbf{r}} \quad \text{(specific angular momentum)}\,. \end{aligned}$$

Since, $\mathbf{r} \cdot \mathbf{L} = 0$, the planetary motion in $\mathbb{R}^3$ takes place in a plane to which vector $\mathbf{L}$ is perpendicular. This is the orbital plane.

[B] ***Use conservation of the magnitude $L = |\mathbf{L}|$ to show that the orbit sweeps out equal areas in equal times. This is Kepler's Second Law. Express the period in terms of $L = |\mathbf{L}|$ and the spatial orbital parameters.***

[C] *The unit vectors for polar coordinates in the orbital plane are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. Show that these vectors satisfy*

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \quad \textit{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\,\dot{\theta}\,\hat{\mathbf{r}}\,, \quad \textit{where} \quad \dot{\theta} = \frac{L}{r^2}\,.$$

Show that Newton's equation (1) also conserves the following two vector quantities,

$$\begin{aligned} \mathbf{K} &= \dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}} \quad \text{(Hamilton's vector)}, \\ \mathbf{J} &= \dot{\mathbf{r}} \times \mathbf{L} - \mu \mathbf{r}/r \quad \text{(Laplace-Runge-Lenz vector, or LRL vector)}\,, \end{aligned}$$

which both lie in the orbital plane, since $\mathbf{J} \cdot \mathbf{L} = 0 = \mathbf{K} \cdot \mathbf{L}$. Hint: How are these two vectors related? Their constancy means that certain attributes of the orbit, particularly, its orientation, are fixed in the orbital plane.

[D] *From their definitions, show that these conserved quantities are related by*

$$L^2 + \frac{J^2}{(-2E)} = \frac{\mu^2}{(-2E)} \quad \textit{and} \quad \mathbf{J} \cdot \mathbf{K} \times \mathbf{L} = K^2 L^2 = J^2\,, \tag{2}$$

where $J^2 := |\mathbf{J}|^2$, etc. and $-2E > 0$ for bounded orbits.

[E] *Choose the conserved LRL vector $\mathbf{J}$ in the orbital plane to point along the reference line for the measurement of the polar angle θ, say from the center of the orbit (Sun) to the perihelion (point of nearest approach, at mid-summer's day), so that*

$$\mathbf{r} \cdot \mathbf{J} = rJ\cos\theta = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{L} - \mu\,\mathbf{r}/r)\,.$$

Use this relation to write the Kepler orbit $r(\theta)$ in plane polar coordinates, as

$$r(\theta) = \frac{L^2}{\mu + J\cos\theta} = \frac{l_\perp}{1 + e\cos\theta}\,,$$

with eccentricity $e = J/\mu$ and semi latus rectum $l_\perp = L^2/\mu$. The expression $r(\theta)$ for the Kepler orbit is the formula for a conic section. This is Kepler's First Law. How is the value of the eccentricity associated to the types of orbits?

[F] ***Show that the period of the orbit is given by***

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{\mu} = \frac{\mu^2}{(-2E)^3}$$

The relation $T^2/a^3 = constant$ is Kepler's Third Law. The constant is Newton's constant.

[G] *Make a table of the canonical Poisson brackets*

$$\{F,\,H\} := \frac{\partial F}{\partial \mathbf{q}}\cdot\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}}\cdot\frac{\partial H}{\partial \mathbf{q}}\,, \tag{3}$$

among $(q_k, p_k, (\mathbf{q}\cdot\mathbf{p}), |\mathbf{q}|^2, |\mathbf{p}|^2, q_k/|\mathbf{q}|)$.

$\{\cdot\,,\,\cdot\}$	q_l	p_l	$\mathbf{p}\cdot\mathbf{q}$	$\|\mathbf{p}\|^2$	$\|\mathbf{q}\|^2$	$q_l/\|\mathbf{q}\|$
q_k	0					
p_k		0				
$\mathbf{p}\cdot\mathbf{q}$			0			
$\|\mathbf{p}\|^2$				0		
$\|\mathbf{q}\|^2$					0	
$q_k/\|\mathbf{q}\|$						0

(Which of these arise in computing $\{J_k, J_l\}$*?)*

[H] *Write the Poisson brackets among the set of quadratic combinations*

$$X_1 = |\mathbf{q}|^2 \ge 0\,,\quad X_2 = |\mathbf{p}|^2 \ge 0\,,\quad X_3 = \mathbf{p}\cdot\mathbf{q}\,, \tag{4}$$

as an $\mathbb{R}^3$*-bracket and identify its Casimir function.*

Write the dynamics on $\mathbf{X} = (X_1, X_2, X_3) \in \mathbb{R}^3$ *for the Hamiltonian of Kepler's problem* $H = \frac{1}{2}X_2 - \mu/\sqrt{X_1}$. *Any ideas about how to solve the resulting system?*

[I] *List all the canonical Poisson brackets among* $(\mathbf{q},\mathbf{p})$ *with* $(H, \mathbf{L}, \mathbf{J})$ *given in these canonical variables by*

$$\begin{aligned} H &= \tfrac{1}{2}|\mathbf{p}|^2 - \frac{\mu}{|\mathbf{q}|}\,,\\ \mathbf{L} &= \mathbf{q}\times\mathbf{p}\,,\\ \mathbf{J} &= \mathbf{p}\times(\mathbf{q}\times\mathbf{p}) - \mu\,\mathbf{q}/|\mathbf{q}|\\ &= -\,(\mathbf{q}\cdot\mathbf{p})\,\mathbf{p} - 2H\,\mathbf{q}\,. \end{aligned}$$

As a check, show that

$$\tfrac{1}{2}\big\{|\mathbf{q}|^2, \mathbf{J}\big\} = \mathbf{q}\times\mathbf{L}\,, \quad \tfrac{1}{2}\big\{|\mathbf{p}|^2, \mathbf{J}\big\} = -\mu\,\mathbf{q}\times\mathbf{L}/|\mathbf{q}|^3\,, \quad \big\{(\mathbf{q}\cdot\mathbf{p}), \mathbf{J}\big\} = \mathbf{p}\times\mathbf{L}\,,$$

as well as $\frac{1}{2}\big\{|\mathbf{q}|^2, J^2\big\} = L^2(\mathbf{q}\cdot\mathbf{p})$.

[J] ***Fourth year, MSc and MSci students***

List all the Poisson brackets amongst the components of the vectors $\mathbf{L}$ *and* $\mathbf{J}$. *Show that they close amongst themselves. You may wish to recall that* $\{\mathbf{q}, \mathbf{L}\cdot\boldsymbol{\xi}\} = \boldsymbol{\xi}\times\mathbf{q}$ *and* $\{\mathbf{p}, \mathbf{L}\cdot\boldsymbol{\xi}\} = \boldsymbol{\xi}\times\mathbf{p}$ *for any constant vector* $\boldsymbol{\xi}\in\mathbb{R}^3$. *Show that*

$$\{\mathbf{J},\, J^2\} = -4H\mathbf{J}\times\mathbf{L} = 4HL^2\mathbf{K} = -2H\{L^2,\, \mathbf{J}\}\,.$$

[K] ***Fourth year, MSc and MSci students***

For bounded orbits, in which $-2H > 0$, *set* $\mathbf{J}/\sqrt{-2H} = \mathbf{A}$

$$\mathbf{M} = (\mathbf{L}+\mathbf{A})/2 \quad \textit{and} \quad \mathbf{N} = (\mathbf{L}-\mathbf{A})/2\,,$$

then compute all the Poisson brackets amongst the components of the vectors $\mathbf{M}$ *and* $\mathbf{N}$, *evaluated on a level set of* H *and using the conservation properties of Newton's equation, that* $\{J_i, H\} = 0 = \{L_i, H\}$ *for the Hamiltonian* H *of Kepler's problem.*

To complete the transformation, solve for the Hamiltonian H *in the new variables* $\mathbf{M}$ *and* $\mathbf{N}$.

Answer. [#1] ***Reduced Kepler problem***

[A] $\dot{E} = 0$ and $\dot{\mathbf{L}} = 0$ both follow easily by direct verification using Newton's equation for Kepler's problem.

[B] **Solution:** ***We use*** $L = r^2\dot\theta$ ***in computing the area swept out during time*** t_1 ***to time*** t_2***, as***

$$A = \int_{\theta(t_1)}^{\theta(t_2)} \tfrac{1}{2}r\cdot r d\theta = \int_{t_1}^{t_2} \tfrac{1}{2}r\cdot(r\dot\theta)dt = \int_{t_1}^{t_2} \tfrac{1}{2}Ldt = \tfrac{1}{2}L(t_2-t_1)$$

So the area swept out is linear in the duration $t_2 - t_1$***. This is Kepler's second law. For an elliptic orbit with semi-axes*** a ***and*** b***, the area (***πab***) and period (***T***) are related by***

$$\pi ab = \tfrac{1}{2}LT$$

So the period of the orbit satisfies

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^2b^2}{L^2} = \frac{a^4(1-e^2)}{L^2}$$

with eccentricity $e \geq 0$ ***defined by*** $b^2/a^2 = (1-e^2)$***.***

[C] The required relations for time derivatives of

$$\hat{\mathbf{r}} = (\cos\theta, \sin\theta) \quad \text{and} \quad \hat{\boldsymbol{\theta}} = (-\sin\theta, \cos\theta)$$

follow from their definitions.

Conservation of Hamilton's vector. Conservation of $\mathbf{K}$ follows from Newton's equation and its conservation of $\mathbf{L}$, with some additional geometry. By using the relations,

$$\frac{\dot{\theta}}{L} = \frac{1}{r^2}\,, \quad \frac{dL}{dt} = 0 \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\,\dot{\theta}\,\hat{\mathbf{r}}\,,$$

Newton's equation of motion (1) for the Kepler problem may be rewritten equivalently as

$$0 = \ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = \ddot{\mathbf{r}} + \frac{\mu}{L}\dot{\theta}\,\hat{\mathbf{r}} = \frac{d}{dt}\Big(\dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big)\,.$$

This equation implies conservation of the following vector in the plane of motion

$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}} \quad (\textbf{\textit{Hamilton's vector}})\,.$$

The vector $\mathbf{J}$ in the plane is given by the cross-product of the two conserved vectors $\mathbf{K}$ and $\mathbf{L}$,

$$\mathbf{J} = \mathbf{K}\times\mathbf{L} = \dot{\mathbf{r}}\times\mathbf{L} - \mu\hat{\mathbf{r}} \quad (\textbf{\textit{Laplace-Runge-Lenz vector}})\,,$$

so it is also conserved. The vectors $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{L}$ are mutually orthogonal, with $\mathbf{L}$ normal to the orbital plane.

[D] From their definitions, these conserved quantities are related by

$$K^2 = 2E + \frac{\mu^2}{L^2} = \frac{J^2}{L^2}\,.$$

Hence, $J^2 = 2E\,L^2 + \mu^2$.

[E] Choose the conserved Laplace-Runge-Lenz vector $\mathbf{J}$ in the plane of the orbit as the reference line for the measurement of the polar angle θ. The scalar product of $\mathbf{r}$ and $\mathbf{J}$ then yields an elegant result for the Kepler orbit in plane polar coordinates. We are given

$$\mathbf{r}\cdot\mathbf{J} = rJ\cos\theta = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L} - \mu\,\mathbf{r}/r) = L^2 - \mu r\,,$$

which implies and expression for $r(\theta)$

$$r(\theta) = \frac{L^2}{\mu + J\cos\theta} = \frac{l_\perp}{1+e\cos\theta} \quad \text{with eccentricity} \quad e := J/\mu\,.$$

As expected, the orbit is a conic section. ***The eccentricity e takes values $0<e<1$ for an ellipse, $e=1$ for a parabola and $e>1$ for a hyperbola. For bounded periodic orbits (for which $-2E>0$) the formula for $r(\theta)$ describes an ellipse in polar coordinates that are centred at one of***

the two foci. This verifies Kepler's First Law. The eccentricity of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

$$e^2 = J^2/\mu^2 = 1 - b^2/a^2$$

and its half-width at a focus (the semi latus rectum) is given by

$$l_\perp = L^2/\mu = (1 - e^2)a = b^2/a$$

Hence, $J^2 = 2E\,L^2 + \mu^2$ ***implies*** $(1 - J^2/\mu^2) = -2E(L^2/\mu^2)$***, so that***

$$1 = -2E(a/\mu) \quad \textit{or} \quad -2E = \frac{\mu}{a}$$

for elliptical orbits. (The sign changes to $+2E = \mu/a$ ***for hyperbolic orbits.)*** The eccentricity vanishes ($e = 0$) for a circle and correspondingly $K = 0$ implies that $\dot{\mathbf{r}} = \mu\,\hat{\boldsymbol{\theta}}/L$.

[F] Solution: ***From previous parts of the problem, we know that***

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^4(1 - e^2)}{L^2}$$

with $b^2/a^2 = (1 - e^2)$ ***for elliptic orbits and***

$$L^2 = \frac{\mu^2 - J^2}{-2E} = \frac{\mu^2(1 - e^2)}{-2E} = \mu(1 - e^2)a$$

after using $e^2 := J^2/\mu^2$ ***and*** $-2E = \mu/a$ ***for elliptic orbits. Therefore,***

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{\mu} = \frac{\mu^2}{(-2E)^3}$$

The first equation is Kepler's Third Law. The second equation relates the period of the orbit to its energy.

[G] Let $\Pi_{kl} = \delta_{kl} - q_k q_l/|\mathbf{q}|^2$ be the projection operator so that $\Pi_{kl} q_l = 0$. The Poisson brackets among $(q_k, p_k, (\mathbf{q}\cdot\mathbf{p}), |\mathbf{q}|^2, |\mathbf{p}|^2, q_k/|\mathbf{q}|)$ are given in the following table. ***(Note semidirect product form!)***

$\{\cdot\,,\,\cdot\}$	q_l	p_l	$\mathbf{p}\cdot\mathbf{q}$	$\lvert\mathbf{p}\rvert^2$	$\lvert\mathbf{q}\rvert^2$	$q_l/\lvert\mathbf{q}\rvert$
q_k	0	δ_{kl}	q_k	$2p_k$	0	0
p_k	$-\delta_{kl}$	0	$-p_k$	0	$-2q_k$	$-\Pi_{kl}/\lvert\mathbf{q}\rvert$
$\mathbf{p}\cdot\mathbf{q}$	$-q_l$	p_l	0	$2\lvert\mathbf{p}\rvert^2$	$-2\lvert\mathbf{q}\rvert^2$	0
$\lvert\mathbf{p}\rvert^2$	$-2p_l$	0	$-2\lvert\mathbf{p}\rvert^2$	0	$-4\mathbf{p}\cdot\mathbf{q}$	$-2p_k\Pi_{kl}/\lvert\mathbf{q}\rvert$
$\lvert\mathbf{q}\rvert^2$	0	$2q_l$	$2\lvert\mathbf{q}\rvert^2$	$4\mathbf{p}\cdot\mathbf{q}$	0	0
$q_k/\lvert\mathbf{q}\rvert$	0	$\Pi_{kl}/\lvert\mathbf{q}\rvert$	0	$2p_l\Pi_{kl}/\lvert\mathbf{q}\rvert$	0	0

$\{\cdot\,,\,\cdot\}$	c	q_l	p_l	$\mathbf{p}\cdot\mathbf{q}$	$\lvert\mathbf{p}\rvert^2$	$\lvert\mathbf{q}\rvert^2$
c	0	0	0	0	0	0
q_k	0	0	δ_{kl}	q_k	$2p_k$	0
p_k	0	$-\delta_{kl}$	0	$-p_k$	0	$-2q_k$
$\mathbf{p}\cdot\mathbf{q}$	0	$-q_l$	p_l	0	$2\lvert\mathbf{p}\rvert^2$	$-2\lvert\mathbf{q}\rvert^2$
$\lvert\mathbf{p}\rvert^2$	0	$-2p_l$	0	$-2\lvert\mathbf{p}\rvert^2$	0	$-4\mathbf{p}\cdot\mathbf{q}$
$\lvert\mathbf{q}\rvert^2$	0	0	$2q_l$	$2\lvert\mathbf{q}\rvert^2$	$4\mathbf{p}\cdot\mathbf{q}$	0

All of these Poisson brackets arise in computing $\{J_k, J_l\}$.

[H] The Poisson brackets among the X's in (4) produce the vector field

$$\dot{\mathbf{X}} = \{\mathbf{X}, H\} = \nabla S^2 \times \nabla H \quad \text{with Casimir} \quad S^2 = |\mathbf{q}\times\mathbf{p}|^2 = X_1X_2 - X_3^2$$

One may also make the table of these Poisson brackets

$$\{X_i,\, X_j\} = \begin{array}{|c|ccc|} \hline \{\cdot\,,\,\cdot\} & X_1 & X_2 & X_3 \\ \hline X_1 & 0 & 4X_3 & 2X_1 \\ X_2 & -4X_3 & 0 & -2X_2 \\ X_3 & -2X_1 & 2X_2 & 0 \\ \hline \end{array}$$

For the Hamiltonian $H = \frac{1}{2}X_2 - \mu/X_1^{-1/2}$ of Kepler's problem, this yields

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{1}} & \hat{\mathbf{2}} & \hat{\mathbf{3}} \\ X_2 & X_1 & -2X_3 \\ \frac{\mu}{2}X_1^{-3/2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} X_3 \\ -\mu X_3 X_1^{-3/2} \\ \frac{1}{2}X_2 - \frac{\mu}{2}X_1^{-1/2} \end{bmatrix}$$

These variable do not seem to be optimal for Kepler's problem.

[I] The Poisson brackets of $(\mathbf{q},\mathbf{p})$ with $(H,\mathbf{L},\mathbf{J})$ given in canonical variables are:

$$\begin{aligned}
\{\,\mathbf{q}\,,\,H\} &= \frac{\partial H}{\partial \mathbf{p}} = \mathbf{p}\,, \\
\{\,\mathbf{p}\,,\,H\} &= -\frac{\partial H}{\partial \mathbf{q}} = -\mu\,\frac{\mathbf{q}}{|\mathbf{q}|^3}\,, \\
\{\,q_k\,,\,L_l\} &= \frac{\partial L_l}{\partial p_k} = \epsilon_{klm}q_m \\
\{\,p_k\,,\,L_l\} &= -\frac{\partial L_l}{\partial q_k} = \epsilon_{klm}p_m\,, \\
\{\,q_k\,,\,J_l\} &= \frac{\partial J_l}{\partial p_k} = p_kq_l - \delta_{kl}(\mathbf{q}\cdot\mathbf{p}) + (p_kq_l - q_kp_l) \\
&= p_kq_l - \delta_{kl}(\mathbf{q}\cdot\mathbf{p}) - \epsilon_{klm}L_m\,, \\
\{\,p_k\,,\,J_l\} &= -\frac{\partial J_l}{\partial q_k} = \big(p_kp_l - \delta_{kl}|\mathbf{p}|^2\big) - \frac{\mu}{|\mathbf{q}|^3}\big(q_kq_l - \delta_{kl}|\mathbf{q}|^2\big)\,, \\
\text{with}\quad J_l &= q_l\,|\mathbf{p}|^2 - q_l\,\frac{\mu}{|\mathbf{q}|} - p_l\,(\mathbf{q}\cdot\mathbf{p}) \quad \text{and} \quad (p_kq_l - q_kp_l) = -\,\epsilon_{klm}L_m\,.
\end{aligned}$$

[J] Poisson brackets amongst the components of the vectors $\mathbf{L}$ and $\mathbf{J}$ are, as follows. (Note that sign of last term; $-2H > 0$ for bounded orbits.)

$$\begin{aligned}\{L_i,\, L_j\} &= \epsilon_{ijk}L_k\,,\\ \{L_i,\, J_j\} &= \epsilon_{ijk}J_k\,,\\ \{J_i,\, J_j\} &= -\,2H\epsilon_{ijk}L_k\,.\end{aligned}$$

Importantly, this means that

$$\{J_i,\, J^2\} = -\,4H\epsilon_{ijk}J_jL_k = -\,4H(\mathbf{J}\times\mathbf{L})_i = -\,2H\{L^2,\, J_i\}\,.$$

Because $\{J_i, H\} = 0$ we have the required check of this formula, that

$$\big\{J_i, \big(J^2 - 2H\,L^2\big)\big\} = \big\{J_i, \mu^2\big\} = 0\,.$$

- Poisson brackets with the LRL vector $\mathbf{J}$ affect the shape of the ellipse, *both* in its eccentricity (J^2) and its latus rectum (L^2), while preserving energy.
- Constancy of the LRL vector $\mathbf{J}$ and angular momentum vector $\mathbf{L}$ implies that the shape and orientation of the planar orbit remain constant.

[K] For bounded orbits, $-2H > 0$ and Poisson brackets amongst the components of the vectors obtained by setting $\mathbf{J}/\sqrt{-2H} = \mathbf{A}$

$$\mathbf{M} = (\mathbf{L}+\mathbf{A})/2 \quad \text{and} \quad \mathbf{N} = (\mathbf{L}-\mathbf{A})/2\,,$$

evaluated on a level set of H are:

$$\begin{aligned}\{M_i,\, M_j\} &= \epsilon_{ijk}M_k\,,\\ \{\,N_i,\, N_j\,\} &= \epsilon_{ijk}N_k\,,\\ \{M_i,\, N_j\} &= 0\,.\end{aligned}$$

Note that they split into two separate versions of the angular momentum Poisson bracket.

For these bounded orbits, in which $-2H > 0$ equation (2) may be rewritten as the spherical condition

$$L^2 + A^2 = \frac{1}{4}M^2 = \frac{1}{4}N^2 = \frac{\mu^2}{(-2H)} > 0\,, \tag{5}$$

(on recalling that $\mathbf{L}\cdot\mathbf{A} = 0$) and then (2) may be solved for the Hamiltonian as

$$\frac{-2H}{4\mu^2} = \frac{1}{M^2} = \frac{1}{M_1^2+M_2^2+M_3^2} = \frac{1}{N^2} = \frac{1}{N_1^2+N_2^2+N_3^2}\,. \tag{6}$$

So for these variables the Hamiltonian is just the reciprocal of the Casimir and no dynamics occurs at all. Let's look for a more interesting Hamiltonian in these variables.

▲

[#2] *Hamiltonian reduction by stages*

[A] *Write Hamilton's equations using the Poisson brackets*

$$\{M_i,\, M_j\} = \epsilon_{ijk} M_k\,, \quad \{\, N_i,\, N_j\,\} = \epsilon_{ijk} N_k\,, \quad \{M_i,\, N_j\} = 0\,,$$

among the components of the $\mathbb{R}^3$ *vectors* $\mathbf{M}$ *and* $\mathbf{N}$ *in the previous exercise.*

[B] *Compute the equations of motion and identify the functionally independent conserved quantities for the following two Hamiltonians*

$$H_1 = \hat{\mathbf{z}} \cdot (\mathbf{M} \times \mathbf{N}) \quad \textit{and} \quad H_2 = \mathbf{M} \cdot \mathbf{N}\,. \tag{7}$$

[C] *Determine whether these Hamiltonians have sufficiently many symmetries and associated conservation laws to be completely integrable (i.e., reducible to Hamilton's canonical equations for a single degree of freedom) and explain why.*

[D] *Transform the Hamiltonians in (7) from Cartesian components of the vectors* $(\mathbf{M}, \mathbf{N}) \in \mathbb{R}^3 \times \mathbb{R}^3$ *into spherical coordinates* $(\theta, \phi) \in S^2$ *and* $(\bar{\theta}, \bar{\phi}) \in S^2$*, respectively.*

[E] ***Fourth year, MSc and MSci students***

Use the S^1 *symmetries and their associated conservation laws to reduce the dynamics in* $\mathbb{R}^3 \times \mathbb{R}^3$ *to canonical Hamiltonian equations first on* $S^2 \times S^2$ *and then on* S^2 *by a two-stage sequence of canonical transformations.*

Answer.* [#2] *Hamiltonian reduction by stages

[A] The Hamiltonian equations for this system are

$$\dot{\mathbf{M}} = \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} \quad \text{and} \quad \dot{\mathbf{N}} = \mathbf{N} \times \frac{\partial H}{\partial \mathbf{N}}$$

[B] The system with Hamiltonian H_1 conserves $|\mathbf{M}|^2$, $|\mathbf{N}|^2$ and $L_3 = M_3 + N_3$. The first two conservation laws reduce the problem to $S^2 \times S^2$ and the last one provides a further $SO(2)$ symmetry under simultaneous rotation of each of the spheres about its vertical 3-axis. As we shall see, this symmetry and its conservation law are enough to reduce $S^2 \times S^2$ to S^2 and thereby make the system completely integrable.

[C] The system with Hamiltonian H_2 conserves $|\mathbf{M}|^2$, $|\mathbf{N}|^2$ and all the components of $\mathbf{L} = \mathbf{M} + \mathbf{N}$. These conserved quantities are not all functionally independent, since

$$|\mathbf{L}|^2 = |\mathbf{M} + \mathbf{N}|^2 = |\mathbf{M}|^2 + |\mathbf{N}|^2 + 2\,\mathbf{M} \cdot \mathbf{N}\,.$$

However, enough symmetry still remains for these equations to be integrated by employing $L_3 = M_3 + N_3$ and its associated $SO(2)$ symmetry for simultaneous rotation of each of the spheres about its vertical 3-axis. This symmetry reduces its $S^2 \times S^2$ phase space to S^2 and thereby allows it to be integrated as before.

[D] The vectors $\mathbf{M}$ and $\mathbf{N}$ may be written in spherical coordinates (θ, ϕ) and $(\bar\theta, \bar\phi)$, respectively, as

$$\begin{aligned} \mathbf{M} &= (M_1, M_2, M_3)^T \\ &= M(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T\,, \\ \mathbf{N} &= (N_1, N_2, N_3)^T \\ &= N(\sin\bar\theta\cos\bar\phi, \sin\bar\theta\sin\bar\phi, \cos\bar\theta)^T \end{aligned}$$

In terms of these variables we may write

$$H_1 = M_1N_2 - M_2N_1 \text{ and } H_2 = M_1N_1 + M_2N_2 + M_3N_3\,.$$

[E] Reduction $S^2 \times S^2 \to S^2$ may be accomplished by a canonical transformation using conservation of $|\mathbf{M}|^2$, $|\mathbf{N}|^2$ and $L_3 = M_3 + N_3$. The symplectic form on $S^2 \times S^2$ is given in spherical coordinates by

$$\omega = M^2 d\cos\theta \wedge d\phi + N^2 d\cos\bar\theta \wedge d\bar\phi\,. \tag{8}$$

We transform to weighted sum and difference variables by

$$\begin{aligned} \sqrt{2}\lambda &= M_3 + N_3 = M\cos\theta + N\cos\bar\theta\,, \qquad \sqrt{2}\alpha = M\phi + N\bar\phi\,, \\ \sqrt{2}\kappa &= M_3 - N_3 = M\cos\theta - N\cos\bar\theta\,, \qquad \sqrt{2}\beta = M\phi - N\bar\phi\,. \end{aligned}$$

This transformation is canonical and yields the new symplectic form,

$$\omega = d\kappa \wedge d\beta + d\lambda \wedge d\alpha\,. \tag{9}$$

Expressing the Hamiltonians H_1 and H_2 in terms of these new canonical variables reduces the problem to the (κ, β) phase plane, with motion parameterised by the 3rd component of total angular momentum λ and independent of its canonically conjugate angle, α. In each case Hamilton's canonical equations separate into reduced dynamics on S^2, plus reconstruction of the phase, $\alpha \in S^1$,

$$\underbrace{\dot\kappa = -\frac{\partial H}{\partial \beta}\,, \quad \dot\beta = \frac{\partial H}{\partial \kappa}\,,}_{\text{Reduced dynamics on } S^2} \qquad \underbrace{\dot\lambda = -\frac{\partial H}{\partial \alpha} = 0\,, \quad \dot\alpha = \frac{\partial H}{\partial \lambda}}_{\text{Reconstruction of the phase, } \alpha}$$

▲

Cultural background for $so(4)$

Exercise: Flow on $O(4)$***.***

[A] *Show that the flow of an orthogonal matrix in four dimensions satisfying* $O^TO = \mathrm{Id}_{4\times4}$ *may be represented by* $O(t) = e^{\widehat{\Psi}t}O(0)$*, where* $\widehat{\Psi} = O^{-1}\dot{O}(t)$ *is a* 4×4 *skew-symmetric matrix. (The matrix* $\widehat{\Psi} = O^{-1}\dot{O}(t)$ *is the angular velocity of rotation for rotations* $O(t) \in SO(4)$*.)*

Answer. The time derivative of $O(t)^T O(t) = \mathrm{Id}_{4\times 4}$ yields $\widehat{\Psi}^T = -\widehat{\Psi}$. Any 4×4 skew-symmetric matrix may be represented as a linear combination of 4×4 basis matrices with three-dimensional vector coefficients $\Omega,\ \Lambda \in \mathbb{R}^3$ in the form

$$\widehat{\Psi} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 & -\Lambda_1 \\ \Omega_3 & 0 & -\Omega_1 & -\Lambda_2 \\ -\Omega_2 & \Omega_1 & 0 & -\Lambda_3 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 & 0 \end{pmatrix} = \Omega \cdot \widehat{J} + \Lambda \cdot \widehat{K} = \Omega_a \widehat{J}_a + \Lambda_b \widehat{K}_b \,.$$

This is the formula for the angular velocity of rotation in four dimensions.

▲

[B] *Write a basis for the 4×4 skew-symmetric matrices. Hint: add a row and column to the 3×3 basis.*

★

Answer. The 4×4 basis set $\widehat{J} = (J_1,\ J_2,\ J_3)^T$ and $\widehat{K} = (K_1,\ K_2,\ K_3)^T$ consists of the following six linearly independent 4×4 skew-symmetric matrices, $\widehat{J}_a,\ \widehat{K}_b$ with $a, b = 1, 2, 3$,

$$\widehat{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{K}_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\widehat{J}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{K}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\widehat{J}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{K}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrices $\widehat{J}_a$ with $a = 1, 2, 3$, embed the basis for 3×3 skew-symmetric matrices into the 4×4 matrices by adding a row and column of zeros. The skew matrices $\widehat{K}_a$ with $a = 1, 2, 3$, then extend the 3×3 basis to 4×4. ▲

Commutation relations. *The skew matrix basis $\widehat{J}_a,\ \widehat{K}_b$ with $a, b = 1, 2, 3$, satisfies the commu-*

tation relations,

$$
\begin{aligned}
\big[\,\widehat{J}_a,\,\widehat{J}_b\,\big] &= \widehat{J}_a\widehat{J}_b - \widehat{J}_b\widehat{J}_a = \epsilon_{abc}\widehat{J}_c\,,\\
\big[\,\widehat{J}_a,\,\widehat{K}_b\,\big] &= \widehat{J}_a\widehat{K}_b - \widehat{K}_b\widehat{J}_a = \epsilon_{abc}\widehat{K}_c\,,\\
\big[\,\widehat{K}_a,\,\widehat{K}_b\,\big] &= \widehat{K}_a\widehat{K}_b - \widehat{K}_b\widehat{K}_a = \epsilon_{abc}\widehat{J}_c\,.
\end{aligned}
$$

These commutation relations may be verified by a series of direct calculations, as $[\,\widehat{J}_1,\,\widehat{J}_2\,] = \widehat{J}_3$, *etc.*

Hat map for 4×4 ***skew matrices.*** *The map above for the* 4×4 *skew matrix* $\widehat{\Psi}$ *may be written as*

$$
\widehat{\Psi} = \Omega\cdot\widehat{J} + \Lambda\cdot\widehat{K} = \Omega_a\widehat{J}_a + \Lambda_b\widehat{K}_b\,,\quad \textit{sum on}\quad a,b=1,2,3\,.
$$

This map provides the 4×4 *version of the* ***hat map****, written now as* $(\,\cdot\,)\widehat{}\;:\;\mathbb{R}^3\times\mathbb{R}^3\mapsto so(4)$. *Here* $so(4)$ *is the Lie algebra of the* 4×4 *special orthogonal matrices, which consists of the* 4×4 *skew matrices represented in the six-dimensional basis of* $\widehat{J}$*'s and* $\widehat{K}$*'s.*

Commutator as intertwined vector product. *The commutator of* 4×4 *skew matrices corresponds to an intertwined vector product, as follows. For any vectors* $\Omega,\ \Lambda,\ \omega,\ \lambda\in\mathbb{R}^3$, *one has*

$$
\begin{aligned}
&\Big[\,\Omega\cdot\widehat{J} + \Lambda\cdot\widehat{K}\,,\,\omega\cdot\widehat{J} + \lambda\cdot\widehat{K}\,\Big]\\
&\quad= \Big(\Omega\times\omega + \Lambda\times\lambda\Big)\cdot\widehat{J} + \Big(\Omega\times\lambda - \Lambda\times\omega\Big)\cdot\widehat{K}\,.
\end{aligned}
$$

Matrix pairing as inner product of vectors.
Likewise, the ***matrix pairing*** $\langle\,A,B\,\rangle = \mathrm{tr}(A^TB)$ *is related to the vector dot-product pairing in* $\mathbb{R}^3$ *by*

$$
\Big\langle\,\Omega\cdot\widehat{J} + \Lambda\cdot\widehat{K}\,,\,\omega\cdot\widehat{J} + \lambda\cdot\widehat{K}\,\Big\rangle = \Omega\cdot\omega + \Lambda\cdot\lambda\,.
$$

That is,

$$
\Big\langle\,\widehat{J}_a\,,\,\widehat{J}_b\,\Big\rangle = \delta_{ab} = \Big\langle\,\widehat{K}_a\,,\,\widehat{K}_b\,\Big\rangle \quad\textit{and}\quad \Big\langle\,\widehat{J}_a\,,\,\widehat{K}_b\,\Big\rangle = 0\,.
$$

Hamiltonian form on $so(4)^*$***.***
As for the Kepler problem, the equations of motion on $so(4)^*$ *may be expressed in Hamiltonian form as*

$$
\frac{d}{dt}\begin{bmatrix}\Pi\\ \Xi\end{bmatrix} = \begin{bmatrix}\Pi\times & \Xi\times\\ \Xi\times & \Pi\times\end{bmatrix}\begin{bmatrix}\delta h/\delta\Pi\\ \delta h/\delta\Xi\end{bmatrix}. \tag{10}
$$

The corresponding Lie-Poisson bracket is given by

$$
\{f,\,h\} = -\,\Pi\cdot\left(\frac{\delta f}{\delta\Pi}\times\frac{\delta h}{\delta\Pi} + \frac{\delta f}{\delta\Xi}\times\frac{\delta h}{\delta\Xi}\right) - \,\Xi\cdot\left(\frac{\delta f}{\delta\Pi}\times\frac{\delta h}{\delta\Xi} - \frac{\delta h}{\delta\Pi}\times\frac{\delta f}{\delta\Xi}\right).
$$

The Hamiltonian matrix (10) has two null eigenvectors for the variational derivatives of $C_1 = |\Pi|^2 + |\Xi|^2$ *and* $C_2 = \Pi\cdot\Xi$. *The functions* C_1, C_2 *are the Casimirs of the* $so(4)$ *Lie-Poisson bracket. That is* $\{C_1,\,H\} = 0 = \{C_2,\,H\}$ *for every Hamiltonian* $H(\Pi,\,\Xi)$.

Manakov integrability

Show that the rigid body motion on $SO(4)$ *governed by the two Hamiltonians in equations (7) both satisfy Manakov's criteria for complete integrability and write the matrices* A *and* B *in their integrable deformations.*

[#3] *Modulation equations*

The real 3-wave modulation equations on $\mathbb{R}^3$ are

$$\dot{X}_1 = -X_2X_3\,, \quad \dot{X}_2 = -X_3X_1\,, \quad \dot{X}_3 = +X_1X_2\,.$$

[A] *Write these equations using an $\mathbb{R}^3$ bracket of the form,*

$$\dot{\mathbf{X}} = \nabla C \times \nabla H\,,$$

where level sets of C and H are each circular cylinders.

Characterise the equilibrium points geometrically in terms of the gradients of C and H. How many are there? Which are stable?

[B] *Choose cylindrical polar coordinates along the axis of the circular cylinder that represents the level set of C and restrict the $\mathbb{R}^3$ Poisson bracket to that level set.*

[C] *Write the equations of motion on that level set. Do they reduce to something familiar?*

[D] ***Fourth year, MSc and MSci students***

For any closed orbits on the level set of C, write formulas for its geometric and dynamic phases.

Answer. [#3] Modulation equations

The real 3-wave modulation equations on $\mathbb{R}^3$ are

$$\dot{X}_1 = -X_2X_3\,, \quad \dot{X}_2 = -X_3X_1\,, \quad \dot{X}_3 = +X_1X_2\,.$$

[A] These equations may be written in the $\mathbb{R}^3$ bracket form as,

$$\dot{\mathbf{X}} = \nabla C \times \nabla H = \nabla \frac{1}{2}(X_1^2 + X_3^2) \times \nabla \frac{1}{2}(X_2^2 + X_3^2)\,,$$

where level sets of C and H are circular cylinders, oriented along the X_2 and X_1 axes, respectively.

Characterise the equilibrium points geometrically in terms of the gradients of C and H. How many are there? Which are stable?

Equilibria occur at points where the cross product of gradients $\nabla C \times \nabla H$ vanishes. In the orthogonal intersection of two circular cylinders as above, this may occur at points where the circular cylinders are tangent, and at points where the axis of one cylinder punctures normally through the surface of the other. The elliptic cylinders are tangent at one $\mathbb{Z}_2$-symmetric pair of points along the X_3 axis, and the elliptic cylinders have normal axial punctures at two other $\mathbb{Z}_2$-symmetric pairs of points along the X_1 and X_2 axes. There is a total of 6 equilibrium points. 4 are stable and 2 are unstable.

[B] Cylindrical polar coordinates are chosen along the axis of the circular cylinder level set of C by writing $(X_1, X_3) = (-r\cos\theta,\, r\sin\theta)$. Thus,

$$X_1^2 + X_3^2 = r^2(\cos^2\theta + \sin^2\theta) = 2C$$

Thus,

$$d^3X = -dX_2 \wedge dX_1 \wedge dX_3 = dX_2 \wedge d\frac{r^2}{2} \wedge d\theta = dC \wedge d\theta \wedge dX_2$$

Restricting the $\mathbb{R}^3$ Poisson bracket to a level set of C yields

$$\{F,\, H\}d^3X = dC \wedge \{F,\, H\}_C d\theta \wedge dX_2$$

where on a level set of C,

$$\{F,\, H\}_C = \frac{\partial F}{\partial \theta}\frac{\partial H}{\partial X_2} - \frac{\partial H}{\partial \theta}\frac{\partial F}{\partial X_2} \quad \text{so that} \quad \{\theta,\, X_2\}_C = 1\,.$$

[C] The Hamiltonian H on a level set of C is given by

$$H = \frac{1}{2}X_2^2 + 2C\sin^2\theta\,.$$

The equations of motion on a level set of C are given by

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial X_2} = X_2 \qquad \frac{dX_2}{dt} = -\,\frac{\partial H}{\partial \theta} = -\,2C\sin\theta\cos\theta$$

These reduce to the pendulum equation,

$$\frac{d^2\theta}{dt^2} = -\,C\sin 2\theta\,.$$

[D] ***Fourth year, MSc and MSci students***

The geometric phase for any closed orbit on the level set of C is the integral

$$\Delta\phi_{geom} = \frac{1}{C}\int_A d\theta \wedge dX_2 = -\frac{1}{C}\oint_{\partial A} X_2 d\theta\,,$$

by Stokes theorem. Here A is the area enclosed by the solution orbit ∂A on a level set of C. Then

$$\begin{aligned}\Delta\phi_{geom} &= -\frac{1}{C}\oint_{\partial A} X_2\,\dot\theta(t)\,dt = -\frac{1}{C}\oint_{\partial A} X_2\frac{\partial H}{\partial X_2}\,dt \\ &= = -\frac{1}{C}\oint_{\partial A} 2H - 4C\sin^2\theta\,dt = -\frac{2T}{C}\Big(H - 2C\Big\langle V\Big\rangle\Big)\,,\end{aligned}$$

where

$$\Big\langle V\Big\rangle = \frac{1}{T}\oint_{\partial A} 2C\sin^2\theta(t)\,dt$$

is the average of the potential energy over the orbit.

The dynamic phase is given by the formula,

$$
\begin{aligned}
\Delta\phi_{dyn} &= \frac{1}{C}\oint_{\partial A}\Big(X_2\dot\theta + C\dot\phi\Big)dt \\
&= \frac{1}{C}\oint_{\partial A}\Big(X_2\frac{\partial H}{\partial X_2} + C\frac{\partial H}{\partial C}\Big)dt \\
&= \frac{1}{C}\oint_{\partial A} X_2^2 + 2C\sin^2\theta\,dt \\
&= \frac{1}{C}\oint_{\partial A} 2H - 2C\sin^2\theta\,dt \\
&= \frac{2T}{C}\Big(H - C\Big\langle V\Big\rangle\Big)
\end{aligned}
$$

where ϕ is the angle conjugate to C and T is the period of the orbit around which the integration is performed. Thus, the total phase change around the orbit is

$$\Delta\phi_{tot} = \Delta\phi_{dyn} + \Delta\phi_{geom} = 2T\Big\langle V\Big\rangle\,.$$

▲

[#4] ***2D coupled oscillators***

Consider the 2D oscillator Hamiltonian $H:\mathbb{C}^2\to\mathbb{R}$*, with complex 2-vector* $\mathbf{a}=(a_1,a_2)\in\mathbb{C}^2$ *and constant frequencies* ω_j*,*

$$H = \frac{1}{2}\sum_{j=1}^{2}\omega_j|a_j|^2 = \frac{1}{4}(\omega_1+\omega_2)\big(|a_1|^2+|a_2|^2\big) + \frac{1}{4}(\omega_1-\omega_2)\big(|a_1|^2-|a_2|^2\big)\,.$$

[A] *Compute its canonical Hamiltonian dynamics with*

$$\{a_j, a_k^*\} = -2i\delta_{jk}.$$

Explain why this is the sum of a $1:1$ *resonant oscillator and a* $1:-1$ *oscillator.*

[B] *Find the transformations generated by* X, Y, Z, R *on* a_1, a_2*, where*

$$
\begin{aligned}
R &= |a_1|^2 + |a_2|^2\,, \\
Z &= |a_1|^2 - |a_2|^2\,, \\
X - iY &= 2a_1a_2^*\,,
\end{aligned}
$$

are the S^1 *invariants of the* $1:1$ *resonance. Express these infinitesimal transformations as matrix operations and identify their corresponding finite transformations.*

[C] *For the starting Hamiltonian,*

$$
\begin{aligned}
H &= \frac{\omega_1}{2}(R+Z) + \frac{\omega_2}{2}(R-Z) \\
&= \frac{1}{2}(\omega_1+\omega_2)R + \frac{1}{2}(\omega_1-\omega_2)Z\,,
\end{aligned}
$$

write the equations $\dot X, \dot Y, \dot Z, \dot R$ *for the* S^1 *invariants* X, Y, Z, R*.*

Write these equations in vector form, with $\mathbf{X}=(X,Y,Z)^T$*, and describe this motion in terms of level sets of the Poincaré sphere and the Hamiltonian* H*.*

[D] $\boxed{\textbf{\textit{Fourth year, MSc and MSci students}}}$

Re-do this problem for the same Hamiltonian, but using the $1:-1$ *invariants*

$$\begin{aligned} S &= |a_1|^2 - |a_2|^2, \\ Y_1 &= |a_1|^2 + |a_2|^2, \\ Y_2 + iY_3 &= 2a_1 a_2\,, \end{aligned}$$

for which level sets of S^2 *satisfy*

$$Y_2^2 + Y_3^2 = 4|a_1|^2|a_2|^2 = Y_1^2 - S^2\,,$$

as hyperboloids of revolution around the Y_1*-axis.*

Express the starting Hamiltonian as

$$H \;=\; \frac{1}{2}(\omega_1 + \omega_2)Y_1 + \frac{1}{2}(\omega_1 - \omega_2)S\,,$$

in terms of the $1:-1$ *invariants and write the equations of motion in vector form, with* $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$.

Describe this motion in terms of level sets of the Hamiltonian H *and the hyperboloids along the* Y_1*-axis parameterised by* S.

[E] $\boxed{\textbf{\textit{Fourth year, MSc and MSci students}}}$

The Hamiltonian $\mathbb{C}^2 \to \mathbb{R}$ *for a certain* $2:1$ *resonance is given by*

$$H = \tfrac{1}{2}|a_1|^2 - |a_2|^2 + \tfrac{1}{2}\mathrm{Im}(a_1^{*2}a_2)\,,$$

in terms of canonical variables $(a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2$ *whose Poisson bracket relation is*

$$\{a_j, a_k^*\} = -2i\delta_{jk}, \quad \textit{for} \quad j,k = 1,2.$$

is invariant under the $2:1$ *resonance* S^1 *transformation*

$$a_1 \to e^{i\phi} \quad \textit{and} \quad a_2 \to e^{2i\phi}.$$

- a *Write the motion equations in terms of the canonical variables* $(a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2$
- b *Introduce the orbit map* $\mathbb{C}^2 \to \mathbb{R}^4$

$$\pi:\; (a_1, a_1^*, a_2, a_2^*) \to \{X, Y, Z, R)\} \tag{11}$$

  and transform the Hamiltonian H *on* $\mathbb{C}^2$ *to new variables* $X, Y, Z, R \in \mathbb{R}^4$ *given by*

$$\begin{aligned} R &= \tfrac{1}{2}|a_1|^2 + |a_2|^2\,, \\ Z &= \tfrac{1}{2}|a_1|^2 - |a_2|^2\,, \\ X + iY &= 2a_1^{*2}a_2\,, \end{aligned}$$

  that are invariant under the $2:1$ *resonance* S^1 *transformation.*
- c *Show that these variables are functionally dependent, because they satisfy a cubic algebraic relation* $C(X, Y, Z, R) = 0$.

d *Use the orbit map* $\mathbb{C}^2 \to \mathbb{R}^4$ *to make a table of Poisson brackets among the four quadratic* $2:1$ *resonance* S^1*-invariant variables* $X, Y, Z, R \in \mathbb{R}^4$.

e *Show that both* R *and the cubic algebraic relation* $C(X, Y, Z, R) = 0$ *are Casimirs for these Poisson brackets.*

f *Write the Hamiltonian, Poisson bracket and equations of motion in terms of the variables* $\mathbf{X} = (X, Y, Z)^T \in \mathbb{R}^3$.

g *Describe this motion in terms of level sets of the Hamiltonian* H *and the orbit manifold for the 2:1 resonance, given by* $C(X, Y, Z, R) = 0$.

h *Restrict the dynamics to a level set of the Hamiltonian and show that it reduces there to the equation of motion for a point particle in a cubic potential. Explain its geometrical meaning.*

i *Compute the geometric and dynamic phases for any closed orbit on a level set of* H.

Answer. [#4] 2D coupled oscillators

The 2D oscillator Hamiltonian $H : \mathbb{C}^2 \to \mathbb{R}$, with complex 2-vector $\mathbf{a} = (a_1, a_2) \in \mathbb{C}^2$ and constant frequencies ω_j,

$$H = \frac{1}{2}\sum_{j=1}^{2} \omega_j |a_j|^2 = \frac{1}{4}(\omega_1 + \omega_2)\big(|a_1|^2 + |a_2|^2\big) + \frac{1}{4}(\omega_1 - \omega_2)\big(|a_1|^2 - |a_2|^2\big) .$$

[A] The canonical Hamiltonian dynamics with $\{a_j, a_k^*\} = -2i\delta_{jk}$ is

$$H = \frac{1}{2}\sum_{j=1}^{2} \omega_j |a_j|^2 = \frac{1}{4}(\omega_1 + \omega_2)\underbrace{\big(|a_1|^2 + |a_2|^2\big)}_{1\,:\,1 \text{ resonance}} + \frac{1}{4}(\omega_1 - \omega_2)\underbrace{\big(|a_1|^2 - |a_2|^2\big)}_{1\,:\,-1 \text{ resonance}} .$$

This is the linear combination of Hamiltonians for a $1:1$ resonant oscillator and a $1:-1$ oscillator.

[B] The infinitesimal transformations generated by X, Y, Z, R on a_1, a_2, are

$$\begin{aligned}
X_R a_j &= \{a_j, R\} = -\,2i\,\frac{\partial R}{\partial a_j^*} = -\,2i\,a_j\,, \\
X_Z a_1 &= \{a_1, Z\} = -\,2i\,a_1\,, \qquad X_Z a_2 = \{a_2, Z\} = 2i\,a_2\,, \\
X_X a_1 &= \{a_1, X\} = -\,2i\,a_2\,, \qquad X_X a_2 = \{a_2, X\} = -\,2i\,a_1\,, \\
X_Y a_1 &= \{a_1, Y\} = -2a_2\,, \qquad X_Y a_2 = \{a_2, Y\} = 2a_1\,.
\end{aligned}$$

These infinitesimal transformations may be expressed as matrix operations,

$$\begin{aligned}
X_Z \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= -2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{or} \quad X_Z\mathbf{a} = -\,2i\,\sigma_3\mathbf{a}\,, \\
X_X \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= -2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{or} \quad X_X\mathbf{a} = -\,2i\,\sigma_1\mathbf{a}\,, \\
X_Y \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= -2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{or} \quad X_Y\mathbf{a} = -\,2i\,\sigma_2\mathbf{a}\,.
\end{aligned}$$

From these expressions, one recognises that the finite transformations, or flows, of the Hamiltonian vector fields for (X, Y, Z) are rotations about the (X, Y, Z) axes, respectively.

[C] For the Hamiltonian,

$$
\begin{aligned}
H &= \frac{\omega_1}{2}(R+Z) + \frac{\omega_2}{2}(R-Z) \\
&= \frac{1}{2}(\omega_1+\omega_2)R + \frac{1}{2}(\omega_1-\omega_2)Z\,,
\end{aligned}
\tag{12}
$$

the equations $\dot{X}, \dot{Y}, \dot{Z}, \dot{R}$ for the S^1 invariants X, Y, Z, R of the $1:1$ resonance may be written as

$$\dot{F} = \{F, H\}$$

which produces

$$\dot{R} = 0 = \dot{Z}\,, \quad \dot{X} = \frac{1}{2}(\omega_1-\omega_2)Y \quad \text{and} \quad \dot{Y} = -\frac{1}{2}(\omega_1-\omega_2)X\,.$$

In vector form, with $\mathbf{X} = (X, Y, Z)^T$, this is

$$\dot{\mathbf{X}} = \frac{1}{2}(\omega_1-\omega_2)\mathbf{X}\times\widehat{\mathbf{Z}}\,,$$

where $\widehat{\mathbf{Z}}$ is the unit vector in the Z-direction $(\cos\theta = 0)$. This motion is uniform rotation in the positive direction along a latitude of the Poincaré sphere $R = const$.

This azimuthal rotation on a latitude at fixed polar angle on the sphere occurs along the intersections of level sets of the Poincaré sphere $R = const$ and the planes $Z = const$, which are level sets of the Hamiltonian for a fixed value of R.

[D] ***Fourth year, MSc and MSci students***

The following are quadratic $1:-1$ invariants: $|a_1|^2, |a_2|^2, a_1a_2, a_1^*a_2^*$.
Then the following linear combinations of these are also invariant

$$
\begin{aligned}
S &= |a_1|^2 - |a_2|^2, \\
Y_1 &= |a_1|^2 + |a_2|^2, \\
Y_2 + iY_3 &= 2a_1a_2\,.
\end{aligned}
\tag{13}
$$

Thus,

$$Y_2^2 + Y_3^2 = 4|a_1|^2|a_2|^2 = Y_1^2 - S^2\,,$$

and the level sets of the orbital manifold are the hyperboloids of revolution around the Y_1-axis parameterised by S. That is,

$$S^2 = Y_1^2 - Y_2^2 - Y_3^2\,. \tag{14}$$

We remark that:
$S = |a_1|^2 - |a_2|^2 = const$ is an ***hyperboloid*** in both $\mathbb{C}^2$ and $\mathbb{R}^3$.
$Y_1 = |a_1|^2 + |a_2|^2 = const$ is a ***sphere*** $S^3 \in \mathbb{C}^2$, and it is a ***plane*** in $\mathbb{R}^3$.

One may write the starting Hamiltonian as,

$$H = \frac{1}{2}(\omega_1 + \omega_2)Y_1 + \frac{1}{2}(\omega_1 - \omega_2)S\,,$$

in terms of the $1 : -1$ invariants and thereby write the equations of motion in vector form, with $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$.
For the $1 : -1$ resonance Hamiltonian $H = Y_1$, the evolution of S, Y_2, Y_3 is described by

$$\dot{\mathbf{Y}} = \nabla S^2 \times \widehat{\mathbf{Y}}_\mathbf{1} = 2\widehat{\mathbf{Y}}_\mathbf{1} \times \mathbf{Y} \tag{15}$$

$$\begin{aligned} \dot{S} &= \{S, H\} = 0, \\ \dot{Y}_1 &= \{Y_1, H\} = \{Y_1, Y_1\} = 0, \\ \dot{Y}_2 &= \{Y_2, H\} = \{Y_2, Y_1\} = -2Y_3, \\ \dot{Y}_3 &= \{Y_3, H\} = \{Y_3, Y_1\} = 2Y_2\,. \end{aligned} \tag{16}$$

Thus, Y_2 and Y_3 rotate clockwise around the Y_1-axis in a plane at $Y_1 = const$.

[E] ***Fourth year, MSc and MSci students***

The Hamiltonian $\mathbb{C}^2 \to \mathbb{R}$ for a certain $2 : 1$ resonance is given by

$$H = \tfrac{1}{2}|a_1|^2 - |a_2|^2 + \tfrac{1}{2}\mathrm{Im}(a_1^{*2}a_2)\,, \tag{17}$$

a Write the motion equations in terms of canonical variables, by using the bracket relation, $\{a_j, a_k^*\} = -2i\delta_{jk}$.

$$\begin{aligned} \dot{a}_1 &= \{a_1, H\} = -2i\frac{\partial H}{\partial a_1^*} = -ia_1 + a_1^* a_2\,, \\ \dot{a}_2 &= \{a_2, H\} = -2i\frac{\partial H}{\partial a_2^*} = 2ia_2 + \frac{1}{2}a_1^2\,. \end{aligned}$$

b Transform the motion equations to the $2 : 1$ invariants (X, Y, Z, R) given by,

$$\begin{aligned} R &= \tfrac{1}{2}|a_1|^2 + |a_2|^2\,, \\ Z &= \tfrac{1}{2}|a_1|^2 - |a_2|^2\,, \\ X + iY &= 2a_1^{*2}a_2\,. \end{aligned}$$

- In these variables, the Hamiltonian (17) is given by

$$H = Y + Z \quad \Longleftarrow \boxed{\text{Should be } H = \tfrac{1}{4}Y + H}.$$

- The variables (X, Y, Z, R) satisfy the functional relation

$$\begin{aligned} |X + iY|^2 &= X^2 + Y^2 \\ &= 4\,|a_1|^4\,|a_2|^2 \\ &= 2\,(R + Z)^2\,(R - Z)\,, \end{aligned}$$

which defines the orbit manifold for 2 : 1 resonance,

$$C(X,\,Y,\,Z,\,R) := X^2+Y^2-2\,(R+Z)^2\,(R-Z)=0\,. \tag{18}$$

This is a cubic of revolution defined along an interval of the Z-axis.

c Write the motion equations in vector form, with $\mathbf{X}=(X,Y,Z)^T$, and describe this motion in terms of level sets of the Hamiltonian H and the orbit manifold for the 2:1 resonance, given by

$$\dot{\mathbf{X}} = \nabla C \times \nabla H$$

$$\begin{aligned}\begin{bmatrix}\dot X\\ \dot Y\\ \dot Z\end{bmatrix} &= \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\\ 2X & 2Y & -2(R+Z)(R-3Z)\\ 0 & 1 & 1\end{bmatrix}\\ &= 2\begin{bmatrix}Y+(R+Z)(R-3Z)\\ -X\\ X\end{bmatrix}.\end{aligned}$$

The motion takes place in $\mathbb{R}^3$ along the intersections of constant level sets of the plane $H=Y+Z$ and the surface $C(X,\,Y,\,Z,\,R)=0$, where $R=const$.

d Write the solution through the origin analytically and explain its geometrical meaning.

The solution through the origin has $R=0$, so that

$$\begin{bmatrix}\dot X\\ \dot Y\\ \dot Z\end{bmatrix} = 2\begin{bmatrix}Y-3Z^2\\ -X\\ X\end{bmatrix} = 2\begin{bmatrix}H-Z-3Z^2\\ -X\\ X\end{bmatrix}.$$

Hence,

$$\tfrac{1}{2}\ddot Z = 2H-2Z-6Z^2$$

which has a first integral (total energy)

$$\tfrac{1}{4}\dot Z^2 + Z^2 + 2Z^3 - HZ = const$$

which is then solved the usual way. (This approach works just as well when $R\neq 0$.)

Explain why the left and right S^1 reductions form a dual pair.

e

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