**M3/4A16 Assessed Coursework 3** **Darryl Holm**
**Due in class Tuesday December 15, 2009**

Ask in class about clarifying the exact meaning of a question if you're unsure.

## #1 *Exercises in exterior calculus operations*

**Vector notation for differential basis elements:**
*One denotes differential basis elements* $dx^i$ *and* $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$*, for* $i, j, k = 1, 2, 3,$ *in vector notation as*

$$
\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,,\\
d\mathbf{S} &= (dS_1, dS_2, dS_3)\\
&:= (dx^2 \wedge dx^3,\ dx^3 \wedge dx^1,\ dx^1 \wedge dx^2)\,,\\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,,\\
d^{\,3}x &= d\mathrm{Vol} := dx^1 \wedge dx^2 \wedge dx^3\,.
\end{aligned}
$$

**(1a)** ***Vector algebra operations***

**(i)** *Show that contraction with the vector field* $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ *recovers the following familiar operations among vectors*

$$
\begin{aligned}
X \,\lrcorner\, d\mathbf{x} &= \mathbf{X}\,,\\
X \,\lrcorner\, d\mathbf{S} &= \mathbf{X}\times d\mathbf{x}\,,\\
\textit{(or,}\quad X \,\lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k)\\
Y \,\lrcorner\, X \,\lrcorner\, d\mathbf{S} &= \mathbf{X}\times\mathbf{Y}\,,\\
X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,,\\
Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\times\mathbf{Y}\cdot d\mathbf{x} = \epsilon_{ijk}X^iY^jdx^k\,,\\
Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\times\mathbf{Y}\cdot\mathbf{Z}\,.
\end{aligned}
$$

**(ii)** *Show that these are consistent with*

$$
X \,\lrcorner\, (\alpha\wedge\beta) = (X \,\lrcorner\, \alpha)\wedge\beta + (-1)^k\alpha\wedge(X \,\lrcorner\, \beta)\,,
$$

*for a* $k$*-form* $\alpha$*.*

**(iii)** *Use (ii) to compute* $Y \,\lrcorner\, X \,\lrcorner\, (\alpha\wedge\beta)$ *and* $Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, (\alpha\wedge\beta)$*.*

**(1b)** ***Exterior derivative examples in vector notation***
*Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation*

$$
\begin{aligned}
df &= f_{,j}\,dx^j =: \nabla f\cdot d\mathbf{x}\\
0 = d^2 f &= f_{,jk}\,dx^k\wedge dx^j\\
df\wedge dg &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k =: (\nabla f\times\nabla g)\cdot d\mathbf{S}\\
df\wedge dg\wedge dh &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k\wedge h_{,l}\,dx^l =: (\nabla f\cdot\nabla g\times\nabla h)\,d^{\,3}x
\end{aligned}
$$

*Likewise, show that*

$$d(\mathbf{v}\cdot d\mathbf{x}) = (\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S}$$
$$d(\mathbf{A}\cdot d\mathbf{S}) = (\mathrm{div}\,\mathbf{A})\,d^{\,3}x\,.$$

*Verify the compatibility condition* $d^2 = 0$ *for these forms as*

$$0 = d^2 f = d(\nabla f\cdot d\mathbf{x}) = (\mathrm{curl}\,\mathrm{grad}\,f)\cdot d\mathbf{S}\,,$$
$$0 = d^2(\mathbf{v}\cdot d\mathbf{x}) = d\big((\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S}\big) = (\mathrm{div}\,\mathrm{curl}\,\mathbf{v})\,d^{\,3}x\,.$$

*Verify the exterior derivatives of these contraction formulas for* $X = \mathbf{X}\cdot\nabla$

**(i)** $d(X \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) = d(\mathbf{X}\cdot\mathbf{v}) = \nabla(\mathbf{X}\cdot\mathbf{v})\cdot d\mathbf{x}$

**(ii)** $d(X \,\lrcorner\, \boldsymbol{\omega}\cdot d\mathbf{S}) = d(\boldsymbol{\omega}\times\mathbf{X}\cdot d\mathbf{x}) = \mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X})\cdot d\mathbf{S}$

**(iii)** $d(X \,\lrcorner\, f\,d^{\,3}x) = d(f\mathbf{X}\cdot d\mathbf{S}) = \mathrm{div}\,(f\mathbf{X})\,d^{\,3}x$

**(1c)** *Use Cartan's formula,*

$$\pounds_X\alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)$$

*for a* $k-$*form* $\alpha$, $k = 0,1,2,3$ *in* $\mathbb{R}^3$ *to verify the Lie derivative formulas:*

**(i)** $\pounds_X f = X \,\lrcorner\, df = \mathbf{X}\cdot\nabla f$

**(ii)** $\pounds_X\,(\mathbf{v}\cdot d\mathbf{x}) = \big(-\ \mathbf{X}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{X}\cdot\mathbf{v})\big)\cdot d\mathbf{x}$

**(iii)** $\pounds_X(\boldsymbol{\omega}\cdot d\mathbf{S}) = \big(\mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X}) + \mathbf{X}\,\mathrm{div}\,\boldsymbol{\omega}\big)\cdot d\mathbf{S}$
$\qquad = \big(-\,\boldsymbol{\omega}\cdot\nabla\mathbf{X} + \mathbf{X}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\,\mathrm{div}\,\mathbf{X}\big)\cdot d\mathbf{S}$

**(iv)** $\pounds_X(f\,d^{\,3}x) = (\mathrm{div}\,f\mathbf{X})\,d^{\,3}x$

**(v)** *Derive these formulas from the dynamical definition of Lie derivative.*

**(1d)** ***Fourth year students*** *Verify the following Lie derivative identities both by using Cartan's formula and by using the dynamical definition of Lie derivative:*

**(i)** $\pounds_{fX}\alpha = f\pounds_X\alpha + df\wedge(X \,\lrcorner\, \alpha)$

**(ii)** $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$

**(iii)** $\pounds_X(X \,\lrcorner\, \alpha) = X \,\lrcorner\, \pounds_X\alpha$

**(iv)** $\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha)$

**(v)** $\pounds_X(\alpha\wedge\beta) = (\pounds_X\alpha)\wedge\beta + \alpha\wedge\pounds_X\beta$

## #2 *Operations among vector fields*

*The Lie derivative of one vector field by another is called the* ***Jacobi-Lie bracket****, defined as*

$$\pounds_X Y := [X\,,\,Y] := \nabla Y\cdot X - \nabla X\cdot Y = -\,\pounds_Y X$$

*In components, the Jacobi-Lie bracket is*

$$[X\,,\,Y] = \Big[X^k\frac{\partial}{\partial x^k}\,,\,Y^l\frac{\partial}{\partial x^l}\Big] = \bigg(X^k\frac{\partial Y^l}{\partial x^k} - Y^k\frac{\partial X^l}{\partial x^k}\bigg)\frac{\partial}{\partial x^l}$$

*The Jacobi-Lie bracket among vector fields satisfies the Jacobi identity,*

$$[\,X\,,\,[Y\,,\,Z]\,] + [\,Y\,,\,[Z\,,\,X]\,] + [\,Z\,,\,[X\,,\,Y]\,] = 0$$

*Verify the following formulas*

**(2a)** $X \,\lrcorner\, (Y \,\lrcorner\, \alpha) = -\,Y \,\lrcorner\, (X \,\lrcorner\, \alpha)$

**(2b)** $[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X (Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha)$, *for zero-forms (functions) and one-forms.*

**(2c)** $\pounds_{[X\,,Y]}\alpha = \pounds_X \pounds_Y \alpha - \pounds_Y \pounds_X \alpha$, *as a result of* **(b)**. *Use* **2(c)** *to verify the Jacobi identity.*

**(2d)** $\boxed{\textit{Fourth year students}}$
*Derive the formula corresponding to 2(b) for arbitrary k-forms. Hint: Pick a basis of k linearly independent vectors $(v_1, v_2, \dots, v_k)$ and use the chain rule for the Lie derivative of a k-form.*[1]

Problems #1 and #2 are solved in the text. Most of the various parts of these problems were also discussed in class.
$\boxed{\textit{Fourth year students}}$ the formula corresponding to 2(b) for an arbitrary $k-$form $\omega^k$ follows by using the chain rule for the Lie derivative to write,

$$
\begin{aligned}
\pounds_X\Big(\omega^k(v_1, v_2, \dots, v_k)\Big) &= (\pounds_X \omega^k)\big(v_1, v_2, \dots, v_k)\big) + \omega^k(\pounds_X v_1, v_2, \dots, v_k) \\
&\quad + \omega^k(v_1, \pounds_X v_2, \dots, v_k) + \cdots + \omega^k(v_1, , v_2, \dots, \pounds_X v_k)
\end{aligned}
$$

## #3 *A steady Euler fluid flow*

*A steady Euler fluid flow in a rotating frame satisfies*

$$
\pounds_u(\mathbf{v}\cdot d\mathbf{x}) = -\,d(p + \tfrac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v})\,,
$$

*where $\pounds_u$ is Lie derivative with respect to the divergenceless vector field $u = \mathbf{u}\cdot\nabla$, with $\nabla\cdot\mathbf{u} = 0$, and $\mathbf{v} = \mathbf{u} + \mathbf{R}$, with Coriolis parameter* $\mathrm{curl}\,\mathbf{R} = 2\boldsymbol{\Omega}$.

**(3a)** *Write out this Lie-derivative relation in Cartesian coordinates.*

With $\pi := p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}$, the steady flow satisfies

$$
\begin{aligned}
0 &= \pounds_u(\mathbf{v}\cdot d\mathbf{x}) + d\pi \\
&= \Big(\mathbf{u}\cdot\nabla\mathbf{v} + (\nabla\mathbf{u})^T\cdot\mathbf{v} + \nabla\pi\Big)\cdot d\mathbf{x} \\
&= \Big(\underbrace{-\mathbf{u}\times\mathrm{curl}\,\mathbf{v}}_{\text{Lamb vector}} + \underbrace{\nabla\big(p + \tfrac{1}{2}|\mathbf{u}|^2\big)}_{\text{Bernoulli function}}\Big)\cdot d\mathbf{x}
\end{aligned}
$$

upon (1) expanding the Lie derivative and (2) using a vector identity.
A level set of $H$ satisfying $-\mathbf{u}\times\mathrm{curl}\,\mathbf{v} = \nabla H$ is called a $\boxed{\textit{Lamb surface}}$.

[1] R. Palais, Definition of the exterior derivative in terms of the Lie derivative. *Proc. Am. Math. Soc.* 1954; 5: 902-908.

**(3b)** *By taking the exterior derivative, show that this relation implies that the exact two-form*

$$\mathrm{curl}v \lrcorner d^3x = \mathrm{curl}\mathbf{v}\cdot\nabla \lrcorner d^3x = \mathrm{curl}\mathbf{v}\cdot d\mathbf{S} = d(\mathbf{v}\cdot d\mathbf{x}) =: d\Xi\wedge d\Pi$$

*is invariant under the flow of the divergenceless vector field $u$.*

Taking the exterior derivative of $\pounds_u(\mathbf{v}\cdot d\mathbf{x}) + d\pi = 0$ using $d^2 f = 0$ for any continuous function $f$ yields

$$\begin{aligned} 0 &= d\pounds_u(\mathbf{v}\cdot d\mathbf{x}) + d^2(p + \tfrac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}) \\ &= \pounds_u d(\mathbf{v}\cdot d\mathbf{x}) \end{aligned}$$

Hence, by $d^2 = 0$ & commutation of $d$ and $\pounds_u$, the 2-form $d(\mathbf{v}\cdot d\mathbf{x})$ is invariant under the flow of the vector field $u$. Moreover, we have

$$d(\mathbf{v}\cdot d\mathbf{x}) = \mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S} = (\mathrm{curl}\,\mathbf{v}\cdot\nabla)\lrcorner d^3x = \mathrm{curl}v \lrcorner d^3x$$

so the other statements follow, too; the last one by setting $\mathbf{v}\cdot d\mathbf{x} = \Xi d\Pi + d\Theta$.

**(3c)** *Show that Cartan's formula for the Lie derivative in the steady Euler flow condition implies that*

$$u \lrcorner \Big(\mathrm{curl}v \lrcorner d^3x\Big) = dH$$

*and identify the function $H$.*

Cartan's formula for the Lie derivative

$$\pounds_u(\mathbf{v}\cdot d\mathbf{x}) = u \lrcorner d(\mathbf{v}\cdot d\mathbf{x}) + d(u \lrcorner \mathbf{v}\cdot d\mathbf{x})\,,$$

when inserted into the steady Euler flow condition yields

$$\begin{aligned} 0 &= \pounds_u(\mathbf{v}\cdot d\mathbf{x}) + d(p + \tfrac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}) \\ \text{by Cartan's formula} \quad &= u \lrcorner \Big(\mathrm{curl}v \lrcorner d^3x\Big) + d(p + \tfrac{1}{2}|\mathbf{u}|^2)\,. \end{aligned}$$

Hence, $H = -\,(p + \frac{1}{2}|\mathbf{u}|^2)$, up to a constant. That is,

$$u \lrcorner \mathrm{curl}v \lrcorner d^3x = dH = -\,d(p + \tfrac{1}{2}|\mathbf{u}|^2)$$

**(3d)** *Set* $(\mathrm{curl}\mathbf{v}\cdot d\mathbf{S}) = d\Xi\wedge d\Pi$ *and use the results of (3b) and (3c) to write* $\pounds_u\Xi = \mathbf{u}\cdot\nabla\Xi$ *and* $\pounds_u\Pi = \mathbf{u}\cdot\nabla\Xi$ *in terms of the partial derivatives of $H$.*

Since the 2-form curl$v \lrcorner\, d^{\,3}x = d(\mathbf{v}\cdot d\mathbf{x})$ is exact, it may be written as

$$\mathrm{curl}\mathbf{v}\cdot d\mathbf{S} = \mathrm{curl}v \lrcorner\, d^{\,3}x = d\Xi \wedge d\Pi$$

The result of 3(c) then implies

$$\begin{aligned} dH(\Xi,\,\Pi) &= u \lrcorner\,(\mathrm{curl}\mathbf{v}\cdot d\mathbf{S}) \\ &= u \lrcorner\,(d\Xi \wedge d\Pi) \\ &= (\mathbf{u}\cdot\nabla\Xi)\,d\Pi - (\mathbf{u}\cdot\nabla\Pi)\,d\Xi \\ &= \frac{\partial H}{\partial \Pi}\,d\Pi + \frac{\partial H}{\partial \Xi}\,d\Xi\,. \end{aligned}$$

Upon identifying corresponding terms, the steady flow of the fluid velocity $\mathbf{u}$ is found to imply the canonical Hamiltonian equations,

$$\begin{aligned} (\mathbf{u}\cdot\nabla\Xi) = \pounds_u\Xi &= \frac{\partial H}{\partial \Pi}\,, \\ (\mathbf{u}\cdot\nabla\Pi) = \pounds_u\Pi &= -\frac{\partial H}{\partial \Xi}\,. \end{aligned}$$

**(3e)** *What do the results of (3d) mean geometrically? Hint: Is a symplectic form involved?*

The results of 3(d) may be written as

$$\begin{aligned} (\mathbf{u}\cdot\nabla\Xi) &= \big\{\Xi,\,H\big\}\,, \\ (\mathbf{u}\cdot\nabla\Pi) &= \big\{\Pi,\,H\big\}\,, \end{aligned}$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the canonical Poisson bracket for the symplectic form $d\Xi\wedge d\Pi$. This means geometrically that the steady Euler flow is symplectic on level sets of $H(\Xi,\Pi)$. That is, ***Lamb surfaces are symplectic manifolds for the motion of fluids in incompressible steady flows***.

## #4 *Lie derivative formulas for the Lamb vector*

*Euler's equation for the incompressible motion of a fluid in a rotating frame of angular frequency* $\mathbf{\Omega}$ *is expressed in Lie-derivative form as*

$$(\partial_t + \pounds_u)(\mathbf{v}\cdot d\mathbf{x}) = \big(\mathbf{v}_t - \mathbf{u}\times\mathrm{curl}\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x} = -d\pi = -\nabla\pi\cdot d\mathbf{x}\,. \qquad (1)$$

*Here* $\mathbf{v} = \mathbf{u} + \mathbf{R}$ *with Coriolis parameter* $\mathrm{curl}\,\mathbf{R} = 2\mathbf{\Omega}$*, augmented pressure* $\pi := p + u^2/2 - \mathbf{u}\cdot\mathbf{v}$ *and incompressibility condition* $\mathrm{div}\,\mathbf{u} = 0$.

| |
|---|
| Straightforward substitution of definitions. |

**(4a)** *Show that for incompressible flows* $\operatorname{div}\mathbf{u}=0$ *that (1) implies the following equation for pressure,*

$$* \, d * \pounds_u(\mathbf{v}\cdot d\mathbf{x}) = - * d * d\pi = -\triangle\,\pi\,,$$

*which reduces to* $\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u}) = \operatorname{tr}((\nabla\mathbf{u})^T\cdot\nabla\mathbf{u}) = |\nabla\mathbf{u}|^2 = -\triangle p$.

| |
|---|
| Again straightforward substitution of definitions. |

**(4b)** *Show that the spatial exterior derivative of equation (1) yields the vorticity equation,*

$$(\partial_t + \pounds_u)d(\mathbf{v}\cdot d\mathbf{x}) = (\partial_t + \pounds_u)(\boldsymbol{\omega}\cdot d\mathbf{S}) = \Big(\boldsymbol{\omega}_t + \operatorname{curl}(\boldsymbol{\omega}\times\mathbf{u})\Big)\cdot d\mathbf{S} = 0\,,$$

*in which for* $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$ *denotes the* total *vorticity.*

| |
|---|
| Also straightforward. |

***Remark 1 (An Analogy of Fluids with Electricity & Magnetism)***
*We introduce the Roman* d *to denote the space-time differential, as in*

$$\begin{aligned}
\mathrm{d}\Big(\mathbf{v}\cdot d\mathbf{x} - (p+\tfrac{1}{2}u^2)dt\Big) &= -\big(\mathbf{v}_t + \nabla(p+\tfrac{1}{2}u^2)\big)\cdot d\mathbf{x}\wedge dt + \boldsymbol{\omega}\cdot d\mathbf{S} \\
&= \big(\boldsymbol{\omega}\times\mathbf{u}\big)\cdot d\mathbf{x}\wedge dt + \boldsymbol{\omega}\cdot d\mathbf{S}\,,
\end{aligned} \tag{2}$$

*where we have substituted the Euler fluid motion equation (1).*
*This expression is reminiscent of*

$$\begin{aligned}
\mathrm{d}(\mathbf{A}\cdot d\mathbf{x} + A_0 dt) &= -(\mathbf{A}_t - \nabla A_0)\cdot d\mathbf{x}\wedge dt + \operatorname{curl}\mathbf{A}\cdot d\mathbf{S} \\
&= -\mathbf{E}\cdot d\mathbf{x}\wedge dt + \mathbf{B}\cdot d\mathbf{S}\,,
\end{aligned}$$

*for Maxwell fields* $\mathbf{E} = \mathbf{A}_t - \nabla A_0$ *and* $\mathbf{B} = \operatorname{curl}\mathbf{A}$ *in terms of the 4-vector potential* $A_\mu = (\mathbf{A}, A_0)$*, although here we are using the* $\mathbb{R}^4$ *space-time metric, so that* $A_\mu \mathrm{d}x^\mu = \mathbf{A}\cdot d\mathbf{x} + A_0 dt$.

**(4c)** *The E & M gauge transformation is*

$$A_\mu \mathrm{d}x^\mu \to A_\mu \mathrm{d}x^\mu + \mathrm{d}\psi = (\mathbf{A}+\nabla\psi)\cdot d\mathbf{x} + (A_0+\psi_t)dt\,.$$

*What is the corresponding gauge transformation for fluids?*

The analogous gauge transformation is

$$
\begin{aligned}
\mathbf{v}\cdot d\mathbf{x} - (p + \tfrac{1}{2}u^2)dt \quad &\to \quad \mathbf{v}\cdot d\mathbf{x} - (p + \tfrac{1}{2}u^2)dt + \mathrm{d}\psi \\
&= \quad \Big(\mathbf{u} + \nabla\psi + \mathbf{R}\Big)\cdot d\mathbf{x} - \Big(p + \tfrac{1}{2}u^2 - \psi_t\Big)dt
\end{aligned}
$$

This apparently corresponds to a Galilean shift $\mathbf{u} \to \mathbf{u} + \nabla\psi$ and a corresponding change in pressure $p$ in the Bernoulli function $\mathcal{B} := p + \frac{1}{2}u^2$ for the Euler fluid,

$$
p + \tfrac{1}{2}u^2 \to p + \tfrac{1}{2}u^2 + \tfrac{1}{2}|\nabla\psi|^2 + \mathbf{u}\cdot\nabla\psi - \psi_t
$$

**(4d)** *Take another space-time differential of formula (2) and invoke equality of cross derivatives so that $\mathrm{d}^2 = 0$ for the space-time differential, thereby showing that*

$$
\boldsymbol{\omega}_t + \mathrm{curl}(\boldsymbol{\omega}\times\mathbf{u}) = 0 \quad \textit{and} \quad \mathrm{div}\,\boldsymbol{\omega} = 0,
$$

*which recovers Euler's equation for total fluid vorticity.*

Following the directions yields

$$
\begin{aligned}
\mathrm{d}^2\big(\mathbf{v}\cdot d\mathbf{x} - (p + \tfrac{1}{2}u^2)dt\big) \quad &= \quad 0 \\
&= \quad \mathrm{curl}(\boldsymbol{\omega}\times\mathbf{u})\cdot d\mathbf{S}\wedge dt + \boldsymbol{\omega}_t\cdot d\mathbf{S}\wedge dt + \big(\mathrm{div}\,\boldsymbol{\omega}\big)\,dV \\
&= \quad \big(\boldsymbol{\omega}_t + \mathrm{curl}(\boldsymbol{\omega}\times\mathbf{u})\big)\cdot d\mathbf{S}\wedge dt + \big(\mathrm{div}\,\boldsymbol{\omega}\big)\,dV
\end{aligned}
$$

**(4e)** *Show that Euler's formula for the evolution of vorticity implies an equation reminiscent of Faraday's Law for Maxwell's equation*

$$
\frac{d}{dt}\int_S \boldsymbol{\omega}\cdot d\mathbf{S} = -\oint_{\partial S} \boldsymbol{\omega}\times\mathbf{u}\cdot d\mathbf{x}
$$

*by which a time-changing flux of vorticity $\boldsymbol{\omega}$ through a* $\boxed{\textbf{fixed}}$ *surface $S$ generates an opposing "emf" of the Lamb vector*

$$
\boldsymbol{\ell} = \boldsymbol{\omega}\times\mathbf{u}
$$

*on its boundary $\partial S$.*

For a fixed surface we have

$$
\begin{aligned}
\frac{d}{dt}\int_S \boldsymbol{\omega}\cdot d\mathbf{S} \quad &= \quad \frac{d}{dt}\oint_{\partial S}\mathbf{v}\cdot d\mathbf{x} = \oint_{\partial S}\partial_t\mathbf{v}\cdot d\mathbf{x} \\
&= \quad -\oint_{\partial S}\Big(\pounds_u(\mathbf{v}\cdot d\mathbf{x}) + d\pi\Big) = -\oint_{\partial S}\boldsymbol{\omega}\times\mathbf{u}\cdot d\mathbf{x}
\end{aligned}
$$

***Remark 2*** *Thus, the Lamb vector arises naturally as the 1-form* $\boldsymbol{\ell}\cdot d\mathbf{x} = \boldsymbol{\omega}\times\mathbf{u}\cdot d\mathbf{x}$. *This coincidence suggests a superficial analogy with electromagnetism, in which the vorticity* $\boldsymbol{\omega}$ *plays the role of magnetic field* $\mathbf{B}$ *and the Lamb vector* $\boldsymbol{\ell} = \boldsymbol{\omega}\times\mathbf{u}$ *plays the role of the electric field* $\mathbf{E}$. *The analogy is only partial because, unlike the electric and magnetic fields, the vorticity and Lamb vector are* not *independent. Therefore, at some point, the analogy must break down! Even so, as we shall see, the formulas resulting from this analogy of fluid flow with electromagnetism and charge flow may still be interesting from the fluids perspective.*

**(4f)** *Show that the evolution of the Lamb 1-form* $\boldsymbol{\ell}\cdot d\mathbf{x}$ *is given by*

$$(\partial_t + \pounds_u)(\boldsymbol{\ell}\cdot d\mathbf{x}) = -\,\boldsymbol{\omega}\times\big(\boldsymbol{\ell} + \nabla(p + \tfrac{1}{2}u^2)\big)\cdot d\mathbf{x} = -\,\boldsymbol{\omega}\times(\mathbf{u}_t)\cdot d\mathbf{x}$$

Upon using the Euler fluid equations for velocity and total vorticity

$$\mathbf{u}_t = -\big(\boldsymbol{\ell} + \nabla(p + u^2/2)\big) \quad\text{and}\quad \boldsymbol{\omega}_t = -\,\mathrm{curl}\,\boldsymbol{\ell}\,,$$

one finds

$$\begin{aligned}(\partial_t + \pounds_u)(\mathbf{u}\times\boldsymbol{\omega}\cdot d\mathbf{x}) &= \big(\mathbf{u}_t\times\boldsymbol{\omega} + \mathbf{u}\times\boldsymbol{\omega}_t - \mathbf{u}\times\mathrm{curl}(\mathbf{u}\times\boldsymbol{\omega})\big)\cdot d\mathbf{x}\\ &= \mathbf{u}_t\times\boldsymbol{\omega}\cdot d\mathbf{x}\end{aligned}$$

Thus, combining the definition $\mathbf{u}\times\boldsymbol{\omega}\cdot d\mathbf{x} = -\,\boldsymbol{\ell}\cdot d\mathbf{x}$ with Euler's equation for $\mathbf{u}_t$ implies the formula in the statement of the problem.

**(4g)** *The formula in (4f) may be expanded out to pursue its electromagnetic analogy a bit farther.*

$$\begin{aligned}\big(\partial_t + \pounds_u\big)(\boldsymbol{\ell}\cdot d\mathbf{x}) &= \big(\boldsymbol{\ell}_t - \mathbf{u}\times\mathrm{curl}\boldsymbol{\ell} + \nabla(\mathbf{u}\cdot\boldsymbol{\ell})\big)\cdot d\mathbf{x}\\ &= -\,\boldsymbol{\omega}\times\big(\boldsymbol{\ell} + \nabla(p + \tfrac{1}{2}u^2)\big)\cdot d\mathbf{x}\end{aligned}$$

I forgot to set this part as a problem.

***Remark 3*** *If there were a viable analogy with electromagnetism, this equation would represent the electromagnetic analogue of the displacement current in hydrodynamics. The rhs of this equation contains the term* $(\boldsymbol{\ell}\times\boldsymbol{\omega})$, *which is reminiscent of the Poynting vector.*

*To pursue the electromagnetic analogy still farther, one defines the flux of the Lamb vector* $\boldsymbol{\ell} = \boldsymbol{\omega}\times\mathbf{u}$ *as the 2-form,*

$$\boldsymbol{\ell}\cdot d\mathbf{S} = (\boldsymbol{\omega}\times\mathbf{u})\cdot d\mathbf{S} = *d(\mathbf{v}\cdot d\mathbf{x})\wedge\mathbf{u}\cdot d\mathbf{x}$$

*and one computes the formula in the next part of the problem.*

**(4h)** *Show that the flux of the Lamb vector satisfies*

$$(\partial_t + \pounds_u)(\boldsymbol{\ell}\cdot d\mathbf{S}) = -\Big[\mathbf{u}\times(2\mathsf{S}\cdot\boldsymbol{\omega}) + \boldsymbol{\omega}\times\big(2\boldsymbol{\ell} + \nabla(p+\tfrac{1}{2}u^2)\big)\Big]\cdot d\mathbf{S}$$

*where* $2\mathsf{S} := \nabla\mathbf{u} + (\nabla\mathbf{u})^T$ *is twice the strain-rate tensor,* $\mathsf{S}$.

The proof is a direct calculation

$$\begin{aligned}
(\partial_t + \pounds_u)(\boldsymbol{\ell}\cdot d\mathbf{S}) &= (\partial_t + \pounds_u)\Big[(\boldsymbol{\omega}\cdot d\mathbf{x})\wedge(\mathbf{u}\cdot d\mathbf{x})\Big] \\
&= (\boldsymbol{\omega}_t + \mathbf{u}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}_j\nabla u^j)\cdot d\mathbf{x}\wedge(\mathbf{u}\cdot d\mathbf{x}) \\
&\qquad +(\boldsymbol{\omega}\cdot d\mathbf{x})\wedge\Big(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\frac{u^2}{2}\Big)\cdot d\mathbf{x} \\
&= (\boldsymbol{\omega}\cdot\nabla\mathbf{u} + \boldsymbol{\omega}_j\nabla u^j)\cdot d\mathbf{x}\wedge(\mathbf{u}\cdot d\mathbf{x}) \\
&\qquad +(\boldsymbol{\omega}\cdot d\mathbf{x})\wedge\Big(-\nabla p + \mathbf{u}\times 2\boldsymbol{\omega} + \nabla\frac{u^2}{2}\Big)\cdot d\mathbf{x} \\
&= \Big[-\mathbf{u}\times(2\mathsf{S}\cdot\boldsymbol{\omega})\Big]\cdot d\mathbf{S} + \boldsymbol{\omega}\times\Big(-\nabla p + \mathbf{u}\times 2\boldsymbol{\omega} + \nabla\frac{u^2}{2}\Big)\cdot d\mathbf{S}
\end{aligned}$$

as was to be shown.

***Remark 4*** *This formula states that the evolution of the flux of the Lamb vector following a fluid parcel depends partly on the alignment of the total vorticity with the strain-rate tensor and partly on its alignment with the total force. The proof is a direct calculation.*

**(4i)** *Show that the divergence of the Lamb vector satisfies*

$$\partial_t(\mathrm{div}\boldsymbol{\ell}) + \mathrm{div}\Big[\mathbf{u}\,\mathrm{div}\boldsymbol{\ell} + \mathbf{u}\times(2\mathsf{S}\cdot\boldsymbol{\omega}) + \boldsymbol{\omega}\times\big(2\boldsymbol{\ell} + \nabla(p+\tfrac{1}{2}u^2)\big)\Big] = 0\,.$$

Take the spatial differential of the formula in (4h) and commute with the Lie derivative to find

$$\begin{aligned}
(\partial_t + \pounds_u)\Big(\mathrm{div}\boldsymbol{\ell}\,d^3x\Big) &= \big(\partial_t\mathrm{div}\boldsymbol{\ell} + \mathrm{div}(\mathbf{u}\mathrm{div}\boldsymbol{\ell})\big)d^3x \\
&= -\,\mathrm{div}\Big[\mathbf{u}\times(2\mathsf{S}\cdot\boldsymbol{\omega}) + \boldsymbol{\omega}\times\big(2\boldsymbol{\omega}\times\mathbf{u} + \nabla(p+\tfrac{1}{2}u^2)\big)\Big]d^3x
\end{aligned}$$

whose rearrangement yields the desired formula.

***Remark 5*** *To finish the electromagnetic analogy, the second term is the divergence of the current density for the conserved "charge"* $\mathrm{div}\boldsymbol{\ell} = -\,\Delta(p+u^2/2)$. *The equation is potentially interesting as an evolution equation. In turbulence, such as in the exhaust of jet airplane, the jet noise is due to correlations that produce a mean* $\mathrm{div}\boldsymbol{\ell}\neq 0$. *That is, the divergence of the Lamb vector is the source of turbulent jet noise.*