1 M3-4-5 A16 Assessed Problems # 1 Due 2pm 16 Nov 2011

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 1.1

(a) *Consider the zero locus map in the space of* 3×3 *real matrices,*

$$S_{\mathsf{K}} = \{O \in GL(3,\mathbb{R}) |\, O^T \mathsf{K} O - \mathsf{K} = 0\}.$$

Explain why S_{K} *is a manifold for* $\mathsf{K} = \mathsf{K}^T \in GL(3,\mathbb{R})$. *Hint: Is it a submersion?*

(b) *Assuming that* S_{K} *is a manifold, prove that it is a matrix Lie group.*

(c) *Write the defining relation for the tangent space to* S_{K} *at its identity,* $T_e S_{\mathsf{K}}$. *This relation defines the matrix Lie algebra* $\mathfrak{s}_{\mathsf{K}}$.

(d) *Show that for any pair of matrices* $\widehat{X},\, \widehat{Y} \in T_e S_{\mathsf{K}}$, *the matrix commutator* $[\widehat{X},\, \widehat{Y}] \equiv \widehat{X}\,\widehat{Y} - \widehat{Y}\,\widehat{X}$ *is in* $T_e S_{\mathsf{K}}$.

(e) *Suppose the* 3×3 *matrices* $\widehat{X} \in T_e S_{\mathsf{K}}$ *and* $\mathsf{K} = \mathsf{K}^T$ *satisfy*

$$\widehat{X}^T \mathsf{K} + \mathsf{K} \widehat{X} = 0\,.$$

Show that $\exp(\widehat{X}^T t) \mathsf{K} \exp(\widehat{X} t) = \mathsf{K}$ *for all* t.

Explain what this result means. Is it surprising? Why, or why not?

(f) *Define the following* **hat map** *from basis vectors* $(\widehat{e}_1, \widehat{e}_2, \widehat{e}_3) \in \mathfrak{s}_{\mathsf{K}}$ *to basis vectors* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \in \mathbb{R}^3$,

$$\widehat{\ }: \mathfrak{s}_{\mathsf{K}} \to \mathbb{R}^3 \quad \text{is defined by} \quad (\mathsf{K}\widehat{\mathbf{e}})_{ij} = -\,\epsilon_{ijk} \mathsf{K}^{kl} \mathbf{e}_l = -\,c_{ij}{}^l \mathbf{e}_l = -\,[\mathbf{e}_i,\, \mathbf{e}_j]_{\mathsf{K}}\,.$$

Under what conditions on K *is the hat map* $(\widehat{\cdot})$ *a linear isomorphism? (This is easy.)*

(g) *For any vectors* $\mathbf{x} = x^i \mathbf{e}_i$, $\mathbf{y} = y^j \mathbf{e}_j \in \mathbb{R}^3$ *with components* x^j, y^k, *where* $j,k = 1,2,3$, *show that the Lie algebra structure* $[\mathbf{e}_i,\, \mathbf{e}_j]_{\mathsf{K}}$ *is represented on* $\mathbb{R}^3$ *by the vector product*

$$[\mathbf{x}, \mathbf{y}]_{\mathsf{K}} = \mathsf{K}(\mathbf{x} \times \mathbf{y})\,. \tag{1}$$

(h) *Compute the Euler-Poincaré equation on* $\mathbb{R}^3$ *for a Lagrangian* $\ell : \mathfrak{s}_K \to \mathbb{R}$ *by using the hat map representation of* $\mathfrak{s}_K$ *on* $\mathbb{R}^3$.

(i) *Legendre transform to the Hamiltonian side and compute the corresponding Lie-Poisson bracket* $\{F(\mathbf{x}), H(\mathbf{x})\}_{\mathsf{K}}$ *for smooth real functions* $F, H : \mathbf{x} \in \mathbb{R}^3 \to \mathbb{R}$.

(j) *Rewrite this Lie-Poisson bracket as a triple scalar product of gradients of smooth real functions on* $\mathbb{R}^3$ *and find its Casimir(s)* $C : \{C(\mathbf{x}), H(\mathbf{x})\}_{\mathsf{K}} = 0$, *for all* H.

(k) *Compute the corresponding equations of motion for the Hamiltonian* $H = \frac{1}{2}\|\mathbf{x}\|^2$. *How are the resulting equations related to Euler's equations for rigid body motion?*

Exercise 1.2

(a) ***Gauge invariance*** *Show that the Euler-Lagrange equations are unchanged under*

$$L(\mathbf{q}(t),\dot{\mathbf{q}}(t)) \to L' = L + \frac{d}{dt}\gamma(\mathbf{q}(t),\dot{\mathbf{q}}(t))\,,$$

for any function $\gamma : \mathbb{R}^{6N} = \{(\mathbf{q},\dot{\mathbf{q}}) \mid \mathbf{q},\dot{\mathbf{q}} \in \mathbb{R}^{3N}\} \to \mathbb{R}$.

(b) ***Generalized coordinate theorem*** *Show that the Euler-Lagrange equations are* ***unchanged in form*** *under any smooth invertible mapping* $f : \{\mathbf{q} \mapsto \mathbf{s}\}$. *That is, with*

$$L(\mathbf{q}(t),\dot{\mathbf{q}}(t)) = \tilde{L}(\mathbf{s}(t),\dot{\mathbf{s}}(t))\,,$$

show that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \quad \Longleftrightarrow \quad \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\mathbf{s}}}\right) - \frac{\partial \tilde{L}}{\partial \mathbf{s}} = 0\,.$$

(c) *How do the Euler-Lagrange equations transform under* $\mathbf{q}(t) = \mathbf{r}(t) + \mathbf{s}(t)$, *when* $\mathbf{r}(t)$ *and* $\mathbf{s}(t)$ *are independent of each other?*

(d) *State and prove Noether's theorem that each continuous symmetry of Hamilton's principle imples a conservation law for the corresponding Euler-Lagrange equations on the tangent space* TM *of a smooth manifold* M.

(e) *Show that* ***conservation of energy results from Noether's theorem*** *if, in Hamilton's principle, the variations of* $L(\mathbf{q}(t),\dot{\mathbf{q}}(t))$ *are chosen as*

$$\delta\mathbf{q}(t) = \frac{d}{ds}\bigg|_{s=0}\mathbf{q}(t,s)\,,$$

corresponding to symmetry of the Lagrangian under reparametrizations of time along the given curve $q(t) \to q(\tau(t,s))$.

Exercise 1.3 (Example Lagrangians)

Write the Euler-Lagrange equations, then apply the Legendre transformation to determine the Hamiltonian and Hamilton's canonical equations for the following Lagrangians.

Determine which of them are hyperregular. (A Lagrangian is hyperregular if its fibre derivative is invertible, so that the velocity may be expressed in terms of the position and canonical momentum.)

(a) *The kinetic energy Lagrangian* $K(q,\dot{q}) = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$ *with* $i,j - 1,2,\dots,N$

(b) $L(\dot{\mathbf{q}}) = -\Big(1 - \dot{\mathbf{q}}\cdot\dot{\mathbf{q}}\Big)^{1/2}$

(c) *The Lagrangian for free motion of a particle of unit mass in a moving frame of velocity* $\mathbf{R}(\mathbf{q},t)$ *is obtained by setting*

$$L(\dot{\mathbf{q}},\mathbf{q},t) = \frac{1}{2}\|\dot{\mathbf{q}} + \mathbf{R}(\mathbf{q},t)\|^2\,.$$

For example, a frame rotating with time-dependent frequency $\Omega(t)$ *about the vertical axis* $\hat{\mathbf{z}}$ *is obtained by choosing* $\mathbf{R}(\mathbf{q},t) = \mathbf{q}\times\Omega(t)\hat{\mathbf{z}}$.

(d) *The Lagrangian for a charged particle of mass m in a magnetic field $\mathbf{B} = \text{curl}\mathbf{A}$ is*

$$L(q, \dot{q}) = \frac{m}{2}\dot{\mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{e}{c}\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q}),$$

for constants m, e, c and prescribed function $\mathbf{A}(\mathbf{q})$.

How do the Euler-Lagrange equations for this Lagrangian differ from those of the previous part for free motion in a moving frame of velocity $\frac{e}{mc}\mathbf{A}(\mathbf{q})$?

(e) *Let Q be the manifold $\mathbb{R}^3 \times S^1$ with variables $(\mathbf{q}, \theta)$. Introduce the Lagrangian $L : TQ \simeq T\mathbb{R}^3 \times TS^1 \mapsto \mathbb{R}$ as*

$$L(\mathbf{q}, \theta, \dot{\mathbf{q}}, \dot{\theta}) = \frac{m}{2}\|\dot{\mathbf{q}}\|^2 + \frac{e}{2c}\Big(\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q}) + \dot{\theta}\Big)^2 .$$

The Lagrangian L is positive definite in $(\dot{\mathbf{q}}, \dot{\theta})$; so it may be regarded as the kinetic energy of a metric. Interpret the motion as geodesic.

How do the Euler-Lagrange equations for this Lagrangian differ from those of the previous part for a charged particle with mass moving in a magnetic field?

Remember in each part to apply the Legendre transformation to determine the Hamiltonian and Hamilton's canonical equations for all of these Lagrangians.

(f) *Consider the Lagrangian*

$$L_\epsilon(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - g\mathbf{e}_z \cdot \mathbf{q} - \frac{1}{4\epsilon}(1 - \|\mathbf{q}\|^2)^2 + \frac{\pi}{\epsilon}\left(\mathbf{q} \cdot \dot{\mathbf{q}}\right)$$

for a particle with coordinates $\mathbf{q} \in \mathbb{R}^3$, constants g, ϵ and vertical unit vector $\mathbf{e}_z$. Let $\gamma_\epsilon(t)$ be the curve in $\mathbb{R}^3$ obtained by solving the Euler-Lagrange equations for L_ϵ with the initial conditions $\mathbf{q}_0 = \gamma_\epsilon(0)$, $\dot{\mathbf{q}}_0 = \dot{\gamma}_\epsilon(0)$.

Show in either spherical coordinates or stereoscopic coordinates that

(i) *In the limit*

$$\lim_{g \to 0, \epsilon \to 0} \gamma_\epsilon(t)$$

the motion is along is a great circle on the two-sphere S^2, provided that the initial conditions satisfy $\|\mathbf{q}_0\|^2 = 1$ and $\mathbf{q}_0 \cdot \dot{\mathbf{q}}_0 = 0$.

(ii) *For constant $g > 0$ the limit*

$$\lim_{\epsilon \to 0} \gamma_\epsilon(t)$$

recovers the dynamics of a spherical pendulum.

(g) *How does the motion in the previous part differ from that obtained via Hamilton's principle for the following Lagrangian?*

$$L_\epsilon(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - g\mathbf{e}_z \cdot \mathbf{q} - \mu(1 - \|\mathbf{q}\|^2)$$

*where μ is called a **Lagrange multiplier** and must be determined as part of the solution.*

Exercise 1.4 (Poisson brackets)

(a) Show that the canonical Poisson bracket is bilinear, skew symmetric, satisfies the Jacobi identity and acts as a derivation on products of functions in phase space.

(b) Given two constants of motion, what does the Jacobi identity imply about additional constants of motion associated with their Poisson bracket?

(c) Compute the Poisson brackets among the $\mathbb{R}^3$-valued functions of $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^3$

$$J_i = \epsilon_{ijk} p_j q_k \quad \textit{or} \quad \mathbf{J} = \mathbf{p} \times \mathbf{q} \quad \textit{in vector notation.}$$

(d) Answer the following questions about these Poisson brackets.

(i) Do the Poisson brackets $\{J_l, J_m\}$ close among themselves?

(ii) Write the Poisson bracket $\{F(\mathbf{J}), H(\mathbf{J})\}$ for the restriction to functions of $\mathbf{J} = (J_1, J_2, J_3)$.

(iii) Write in vector notation the equation $\dot{\mathbf{J}} = \{\mathbf{J}, H(\mathbf{J})\}$ for any Hamiltonian function $H(\mathbf{J})$.

(iv) Compute the dynamical equation for the Hamiltonian function

$$H(\mathbf{J}) = J^{\xi} = \boldsymbol{\xi} \cdot \mathbf{J}$$

for any vector $\boldsymbol{\xi} \in \mathbb{R}^3$. Interpret the solutions for this flow geometrically.

Exercise 1.5 (Nambu Poisson brackets on $\mathbb{R}^3$)

(a) Show that for smooth functions $c, f, h : \mathbb{R}^3 \to \mathbb{R}$, the $\mathbb{R}^3$-bracket defined by

$$\{f, h\} = -\nabla c \cdot \nabla f \times \nabla h$$

satisfies the defining properties of a Poisson bracket. Is it also a derivation satisfying the Leibnitz relation for a product of functions on $\mathbb{R}^3$? If so, why?

(b) How is the $\mathbb{R}^3$-bracket related to the canonical Poisson bracket in the plane? Hint: restrict $\mathbb{R}^3$-bracket to a level set of c.

(c) The Casimirs (or distinguished functions, as Lie called them) of a Poisson bracket satisfy

$$\{c, h\}(\mathbf{x}) = 0\,, \quad \textit{for all} \quad h(\mathbf{x})$$

Part (a) verifies that the $\mathbb{R}^3$-bracket satisfies the defining properties of a Poisson bracket. What are the Casimirs for the $\mathbb{R}^3$ bracket?

(d) Write the motion equation for the $\mathbb{R}^3$-bracket

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\}$$

in vector form using gradients and cross products. Show that the corresponding Hamiltonian vector field $X_h = \{\,\cdot\,, h\}$ has zero divergence.

(e) *Show that under the* $\mathbb{R}^3$*-bracket, the Hamiltonian vector fields* $X_f = \{\,\cdot\,, f\}$, $X_h = \{\,\cdot\,, h\}$ *satisfy the following anti-homomorphism that relates the commutation of vector fields to the* $\mathbb{R}^3$*-bracket operation between smooth functions on* $\mathbb{R}^3$,

$$[X_f, X_h] = -X_{\{f,h\}}.$$

Hint: the commutator of divergenceless vector fields does satisfy the Jacobi identity.

(f) *Show that the motion equation for the* $\mathbb{R}^3$*-bracket is invariant under a certain linear combination of the functions* c *and* h*. Interpret this invariance geometrically.*

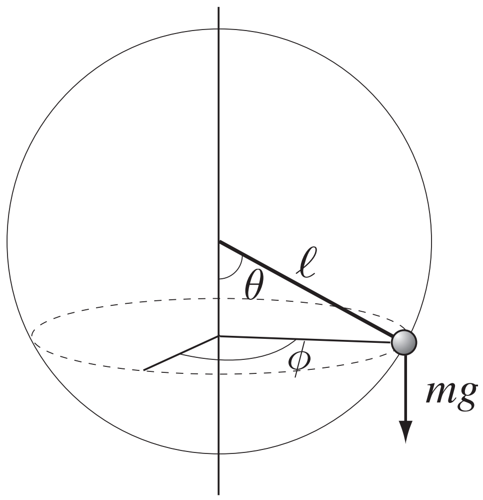

Figure 1: Spherical pendulum: $x_1 = \ell \sin\theta \cos\phi$, $x_2 = \ell \sin\theta \sin\theta$, $x_3 = -\,\ell \cos\theta$.

Exercise 1.6 (Spherical pendulum)

A spherical pendulum of length ℓ *swings from a fixed point of support under the constant downward force of gravity* mg.

Use spherical coordinates with azimuthal angle $0 \le \phi < 2\pi$ *and polar angle* $0 \le \theta < \pi$ *measured from the* ***downward*** *vertical defined in terms of Cartesian coordinates by (note minus sign in z)*

(a) *Find its equations of motion according to the approaches of*

  (i) *Newton,*

  (ii) *Lagrange and*

  (iii) *Hamilton.*

(b) *Transform the Hamiltonian equations to quadratic variables that are invariant under rotations about the vertical.*

(c) *Find a Nambu bracket on* $\mathbb{R}^3$ *in these quadratic invariant variables.*

(d) *Reduce the equations of motion to a level set of the Hamiltonian and classify their solutions.*

Exercise 1.7 (The Hopf map)

In coordinates $(a_1, a_2) \in \mathbb{C}^2$*, the Hopf map* $\mathbb{C}^2/S^1 \to S^3 \to S^2$ *is obtained by transforming to the four quadratic* S^1*-invariant quantities*

$$(a_1, a_2) \to Q_{jk} = a_j a_k^* , \quad \textit{with} \quad j, k = 1, 2 .$$

Let the $\mathbb{C}^2$ *coordinates be expressed as*

$$a_j = q_j + ip_j$$

in terms of canonically conjugate variables satisfying the fundamental Poisson brackets

$$\{q_k, p_m\} = \delta_{km} \quad \textit{with} \quad k, m = 1, 2.$$

(a) *Compute the Poisson brackets* $\{a_j, a_k^*\}$ *for* $j, k = 1, 2$.

(b) *Is the transformation* $(q, p) \to (a, a^*)$ *canonical? Explain why or why not.*

(c) *Compute the Poisson brackets among* Q_{jk}*, with* $j, k = 1, 2$.

(d) *Make the linear change of variables,*

$$X_0 = Q_{11} + Q_{22} , \quad X_1 + iX_2 = 2Q_{12} , \quad X_3 = Q_{11} - Q_{22} ,$$

and compute the Poisson brackets among (X_0, X_1, X_2, X_3).

(e) *Express the Poisson bracket* $\{F(\mathbf{X}), H(\mathbf{X})\}$ *in vector form among functions* F *and* H *of* $\mathbf{X} = (X_1, X_2, X_3)$.

(f) *Show that the quadratic invariants* (X_0, X_1, X_2, X_3) *themselves satisfy a quadratic relation.*

How is this relevant to the Hopf map?