3 M3-4-5A16 Assessed Problems # 3

Exercise 3.1 The dynamical system for the divergence-free motion $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ along the intersection of two orthogonal circular cylinders is given by

$$\dot{x}_1 = x_2 x_3\,, \quad \dot{x}_2 = -\,x_1 x_3\,, \quad \dot{x}_3 = x_1 x_2\,.$$

(a) Write this system in three-dimensional vector $\mathbb{R}^3$-bracket notation as

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where H_1 and H_2 are two conserved functions, whose level sets are circular cylinders oriented, respectively, along the x_3-direction (H_1) and x_1-direction (H_2).

(b) Show that the velocity $\dot{\mathbf{x}} \in T\mathbb{R}^3$ is divergence-free.

(c) Restrict the equations and their $\mathbb{R}^3$ Poisson bracket to a level set of H_2. Show that the Poisson bracket on the parabolic cylinder $H_2 = const$ is symplectic.

(d) Derive the equations of motion on a level set of H_1 and express them in the form of Newton's Law. Do they reduce to something familiar?

□

Exercise 3.2 Consider the divergence-free motion in $\mathbb{R}^3$ along the intersections of a vertically oriented circular cylinder and a sphere off-set by an amount s along the x_2-axis, given respectively by

$$C = \frac{1}{2}(x_1^2 + x_2^2)\,, \quad S = \frac{1}{2}(x_1^2 + (x_2 - s)^2 + x_3^2)$$

(a) Write the corresponding equations of motion in three-dimensional vector $\mathbb{R}^3$-bracket notation as

$$\dot{\mathbf{x}} = \nabla C \times \nabla S\,.$$

(b) Show that this system preserves the level sets of C and S.

(c) Restrict the equations and their $\mathbb{R}^3$ Poisson bracket to a level set of C. Show that the Poisson bracket on the circular cylinder $C = const$ is symplectic.

(d) Derive the equations of motion on a level set of C and express them in the form of Newton's Law. Do they reduce to something familiar?

□

Exercise 3.3 Vector notation for differential basis elements:
One denotes differential basis elements dx^i and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for $i, j, k = 1, 2, 3$, in vector notation as

$$
\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,,\\
d\mathbf{S} &= (dS_1, dS_2, dS_3)\\
&:= (dx^2 \wedge dx^3,\ dx^3 \wedge dx^1,\ dx^1 \wedge dx^2)\,,\\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,,\\
d^{\,3}x &= d\mathrm{Vol} := dx^1 \wedge dx^2 \wedge dx^3\,.
\end{aligned}
$$

(a) ***Vector algebra operations***

(i) Show that contraction with the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ recovers the following familiar operations among vectors

$$
\begin{aligned}
X \,\lrcorner\, d\mathbf{x} &= \mathbf{X}\,,\\
X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,,\\
(\text{or,}\quad X \,\lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k)\\
Y \,\lrcorner\, X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,,\\
X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,,\\
Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y}\cdot d\mathbf{x} = \epsilon_{ijk}X^iY^j dx^k\,,\\
Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y}\cdot \mathbf{Z}\,.
\end{aligned}
$$

(ii) Show that these are consistent with

$$
X \,\lrcorner\, (\alpha \wedge \beta) = (X \,\lrcorner\, \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \,\lrcorner\, \beta)\,,
$$

for a k-form α.

(iii) Use (ii) to compute $Y \,\lrcorner\, X \,\lrcorner\, (\alpha \wedge \beta)$ and $Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, (\alpha \wedge \beta)$.

(b) ***Exterior derivative examples in vector notation***
Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation

$$
\begin{aligned}
df &= f_{,j}\, dx^j =: \nabla f \cdot d\mathbf{x}\\
0 = d^2 f &= f_{,jk}\, dx^k \wedge dx^j\\
df \wedge dg &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k =: (\nabla f \times \nabla g)\cdot d\mathbf{S}\\
df \wedge dg \wedge dh &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \wedge h_{,l}\, dx^l =: (\nabla f \cdot \nabla g \times \nabla h)\, d^{\,3}x
\end{aligned}
$$

Likewise, show that

$$
\begin{aligned}
d(\mathbf{v}\cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S}\\
d(\mathbf{A}\cdot d\mathbf{S}) &= (\mathrm{div}\,\mathbf{A})\, d^{\,3}x\,.
\end{aligned}
$$

Verify the compatibility condition $d^2 = 0$ for these forms as

$$
\begin{aligned}
0 = d^2 f = d(\nabla f \cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathrm{grad}\, f)\cdot d\mathbf{S}\,,\\
0 = d^2(\mathbf{v}\cdot d\mathbf{x}) = d\big((\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S}\big) &= (\mathrm{div}\,\mathrm{curl}\,\mathbf{v})\, d^{\,3}x\,.
\end{aligned}
$$

Verify the exterior derivatives of these contraction formulas for $X = \mathbf{X}\cdot\nabla$

(i) $d(X \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) = d(\mathbf{X}\cdot\mathbf{v}) = \nabla(\mathbf{X}\cdot\mathbf{v})\cdot d\mathbf{x}$

(ii) $d(X \,\lrcorner\, \boldsymbol{\omega}\cdot d\mathbf{S}) = d(\boldsymbol{\omega}\times\mathbf{X}\cdot d\mathbf{x}) = \mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X})\cdot d\mathbf{S}$

(iii) $d(X \intprod f\, d^3x) = d(f\mathbf{X}\cdot d\mathbf{S}) = \operatorname{div}(f\mathbf{X})\, d^3x$

(c) Use Cartan's formula,

$$\pounds_X\alpha = X \intprod d\alpha + d(X \intprod \alpha)$$

for a $k-$form α, $k = 0, 1, 2, 3$ in $\mathbb{R}^3$ to verify the Lie derivative formulas:

(i) $\pounds_X f = X \intprod df = \mathbf{X}\cdot\nabla f$

(ii) $\pounds_X(\mathbf{v}\cdot d\mathbf{x}) = \big(-\ \mathbf{X}\times\operatorname{curl}\mathbf{v} + \nabla(\mathbf{X}\cdot\mathbf{v})\big)\cdot d\mathbf{x}$

(iii) $\pounds_X(\boldsymbol{\omega}\cdot d\mathbf{S}) = \big(\operatorname{curl}(\boldsymbol{\omega}\times\mathbf{X}) + \mathbf{X}\operatorname{div}\boldsymbol{\omega}\big)\cdot d\mathbf{S}$
$= \big(-\boldsymbol{\omega}\cdot\nabla\mathbf{X} + \mathbf{X}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\operatorname{div}\mathbf{X}\big)\cdot d\mathbf{S}$

(iv) $\pounds_X(f\, d^3x) = (\operatorname{div} f\mathbf{X})\, d^3x$

(v) Derive these formulas from the dynamical definition of Lie derivative.

(d) Verify the following Lie derivative identities both by using Cartan's formula and by using the dynamical definition of Lie derivative:

(i) $\pounds_{fX}\alpha = f\pounds_X\alpha + df\wedge(X \intprod \alpha)$

(ii) $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$

(iii) $\pounds_X(X \intprod \alpha) = X \intprod \pounds_X\alpha$

(iv) $\pounds_X(Y \intprod \alpha) = (\pounds_X Y) \intprod \alpha + Y \intprod (\pounds_X\alpha)$

(v) $\pounds_X(\alpha\wedge\beta) = (\pounds_X\alpha)\wedge\beta + \alpha\wedge\pounds_X\beta$

Exercise 3.4 *Operations among vector fields*

The Lie derivative of one vector field by another is called the ***Jacobi-Lie bracket***, defined as

$$\pounds_X Y := [X\,,\,Y] := \nabla Y\cdot X - \nabla X\cdot Y = -\,\pounds_Y X$$

In components, the Jacobi-Lie bracket is

$$[X\,,\,Y] = \Big[X^k\frac{\partial}{\partial x^k}\,,\,Y^l\frac{\partial}{\partial x^l}\Big] = \left(X^k\frac{\partial Y^l}{\partial x^k} - Y^k\frac{\partial X^l}{\partial x^k}\right)\frac{\partial}{\partial x^l}$$

The Jacobi-Lie bracket among vector fields satisfies the Jacobi identity,

$$[\,X\,,\,[Y\,,\,Z]\,] + [\,Y\,,\,[Z\,,\,X]\,] + [\,Z\,,\,[X\,,\,Y]\,] = 0$$

Verify the following formulas

(a) $X \intprod (Y \intprod \alpha) = -\,Y \intprod (X \intprod \alpha)$

(b) $[X\,,\,Y] \intprod \alpha = \pounds_X(Y \intprod \alpha) - Y \intprod (\pounds_X\alpha)$, for zero-forms (functions) and one-forms.

(c) $\pounds_{[X\,,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$, as a result of **(b)**. Use **2(c)** to verify the Jacobi identity.

(d) Verify formula 2(b) for arbitrary $k-$forms using the dynamical definition of the Lie derivative.