1 M3-4-5A16 Assessed Problems # 1

Exercise 1.1 (Poisson brackets for the Hopf map)

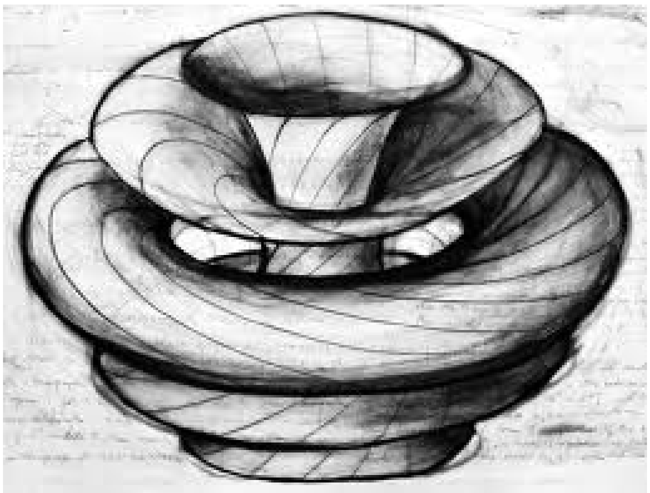

Figure 1: The Hopf map.

In coordinates $(a_1, a_2) \in \mathbb{C}^2$, the Hopf map $\mathbb{C}^2/S^1 \to S^3 \to S^2$ is obtained by transforming to the four quadratic S^1-invariant quantities

$$(a_1, a_2) \to Q_{jk} = a_j a_k^* \,, \quad \text{with} \quad j, k = 1, 2\,.$$

Let the $\mathbb{C}^2$ coordinates be expressed as

$$a_j = q_j + ip_j$$

in terms of canonically conjugate variables satisfying the fundamental Poisson brackets

$$\{q_k, p_m\} = \delta_{km} \quad \text{with} \quad k, m = 1, 2.$$

(A) Compute the Poisson brackets $\{a_j, a_k^*\}$ for $j, k = 1, 2$.

(B) Is the transformation $(q, p) \to (a, a^*)$ canonical? Explain why or why not.

(C) Compute the Poisson brackets among Q_{jk}, with $j, k = 1, 2$.

(D) Make the linear change of variables,

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = 2Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,,$$

and compute the Poisson brackets among (X_0, X_1, X_2, X_3).

(E) Express the Poisson bracket $\{F(\mathbf{X}), H(\mathbf{X})\}$ in vector form among functions F and H of $\mathbf{X} = (X_1, X_2, X_3)$.

(F) Show that the quadratic invariants (X_0, X_1, X_2, X_3) themselves satisfy a quadratic relation.
How is this relevant to the Hopf map?
Is there a momentum map involved?
Hint: What Lie group acts on $\mathbb{C}^2$?

Exercise 1.2 (Motion on a sphere)

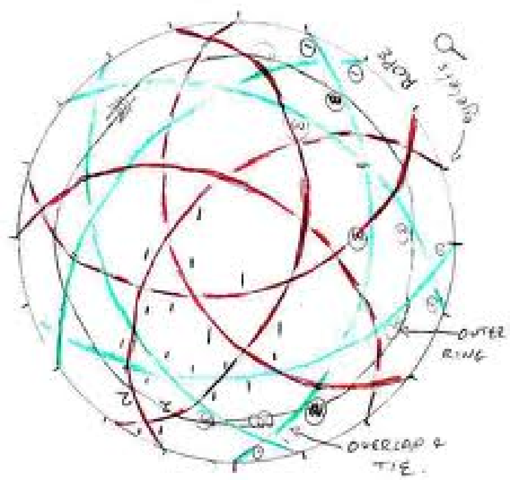

Figure 2: Motion on a sphere.

Motion on a sphere: Part 1, the constraint

Consider Hamilton's principle for the following constrained Lagrangian on $T\mathbb{R}^3$,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - \frac{\mu}{2}\left(1 - \|\mathbf{q}\|^2\right) .$$

Here the quantity μ is called a ***Lagrange multiplier*** and must be determined as part of the solution.

Provide a geometric mechanics description of the dynamical system governed by this Lagrangian. In particular, compute the following for it.

1. Fibre derivative
2. Euler-Lagrange equations
3. Hamiltonian and canonical equations
4. Discussion of solutions

Motion on a sphere: Part 2, the penalty

Provide the same kind of geometric mechanics description of the dynamical system governed by the Lagrangian

$$L_\epsilon(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - \frac{1}{4\epsilon}(1 - \|\mathbf{q}\|^2)^2$$

for a particle with coordinates $\mathbf{q} \in \mathbb{R}^3$ and constants $\epsilon > 0$. For this, let $\gamma_\epsilon(t)$ be the curve in $\mathbb{R}^3$ obtained by solving the Euler-Lagrange equations for L_ϵ with the initial conditions $\mathbf{q}_0 = \gamma_\epsilon(0)$, $\mathbf{v}_0 = \dot{\gamma}_\epsilon(0)$. Show that

$$\lim_{\epsilon \to 0} \gamma_\epsilon(t) = \gamma_0(t)$$

traverses a great circle on the two-sphere S^2, provided that $\mathbf{q}_0$ has unit length and that $\mathbf{q}_0 \cdot \mathbf{v}_0 = 0$.

Exercise 1.3 (The free particle in $\mathbb{H}^2$: #1)

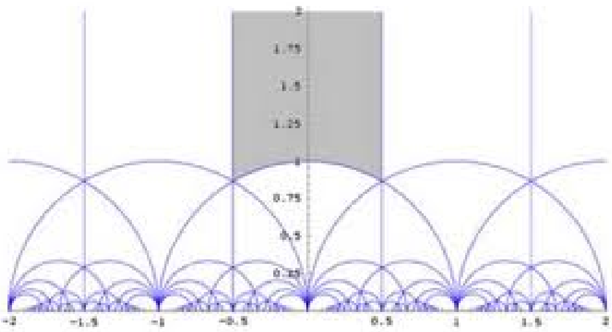

Figure 3: Geodesics on the Lobachevsky half-plane.

In Appendix I of Arnold's book, *Mathematical Methods of Classical Mechanics*, page 303, we read.

> EXAMPLE. We consider the upper half-plane $y > 0$ of the plane of complex numbers $z = x + iy$ with the metric
> $$ds^2 = \frac{dx^2 + dy^2}{y^2}\,.$$
> It is easy to compute that the geodesics of this two-dimensional riemannian manifold are circles and straight lines perpendicular to the x-axis. Linear fractional transformations with real coefficients
> $$z \to \frac{az + b}{cz + d} \tag{1}$$
> are isometric transformations of our manifold ($\mathbb{H}^2$), which is called the *Lobachevsky plane*.[1]

Consider a free particle of mass m moving on $\mathbb{H}^2$. Its Lagrangian is the kinetic energy corresponding to the Lobachevsky metric Namely,

$$L = \frac{m}{2}\left(\frac{\dot{x}^2 + \dot{y}^2}{y^2}\right). \tag{2}$$

(A) (1) Write the fibre derivatives of the Lagrangian (2) and

(2) compute its Euler-Lagrange equations.

These equations represent geodesic motion on $\mathbb{H}^2$.

(3) Evaluate the Christoffel symbols.

(B) (1) List the Lie symmetries of the Lagrangian in (2) and

(2) show that the quantities

$$u = \frac{\dot{x}}{y} \quad \text{and} \quad v = \frac{\dot{y}}{y} \tag{3}$$

are invariant under a subgroup of these symmetry transformations.

(3) Specify the subgroup in terms of the representation (1).

(C) (1) Use the invariant quantities (u, v) in (3) as new variables in Hamilton's principle.

(2) Find the corresponding conserved Noether quantities.

[1]These isometric transformations of $\mathbb{H}^2$ have deep significance in physics. They correspond to the most general Lorentz transformation of space-time.

(D) Transform the Euler-Lagrange equations from x and y to the variables u and v that are invariant under the symmetries of the Lagrangian.

Then:

(1) Show that the resulting system conserves the kinetic energy expressed in these variables.

(2) Discuss its integral curves and critical points in the uv plane.

(3) Show that the u and v equations can be integrated explicitly in terms of sech and tanh.

(E) (1) Legendre transform the system to the Hamiltonian side and

(2) find the Poisson brackets for the variables u and v.

Exercise 1.4 (The free particle in $\mathbb{H}^2$: #2)

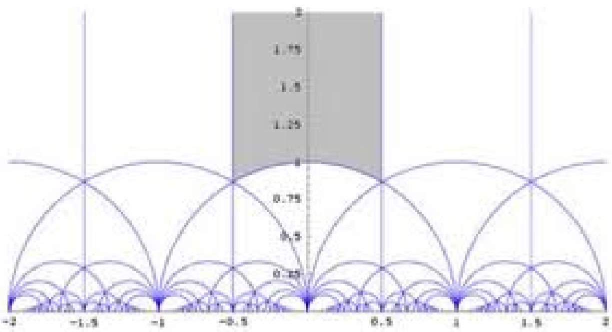

Figure 4: Geodesics on the Lobachevsky half-plane.

Consider the following pair of differential equations for $(u, v) \in \mathbb{R}^2$,

$$\dot{u} = uv\,, \qquad \dot{u} = -\,u^2\,. \tag{4}$$

These equations have discrete symmetries under combined reflection and time reversal, $(u, t) \to (-u, -t)$ and $(v, t) \to (-v, -t)$. (This is called PT symmetry in the (u, v) plane.)

(A) Find 2×2 real matrices L and B for which the system (4) may be written as a Lax pair, namely, as

$$\frac{dL}{dt} = [L, B]\,.$$

Explain what the Lax pair relation means and determine a constant of the motion from it.

(B) Write the system (4) as a double matrix commutator, $\frac{dL}{dt} = [L, [L, N]\,]$. In particular, find N explicitly and explains what this means for the solutions.

(C) Explain why the solution behaviour found in the previous part is consistent with the behaviour predicted by the double bracket relation.

Exercise 1.5 (Nambu Poisson brackets on $\mathbb{R}^3$)

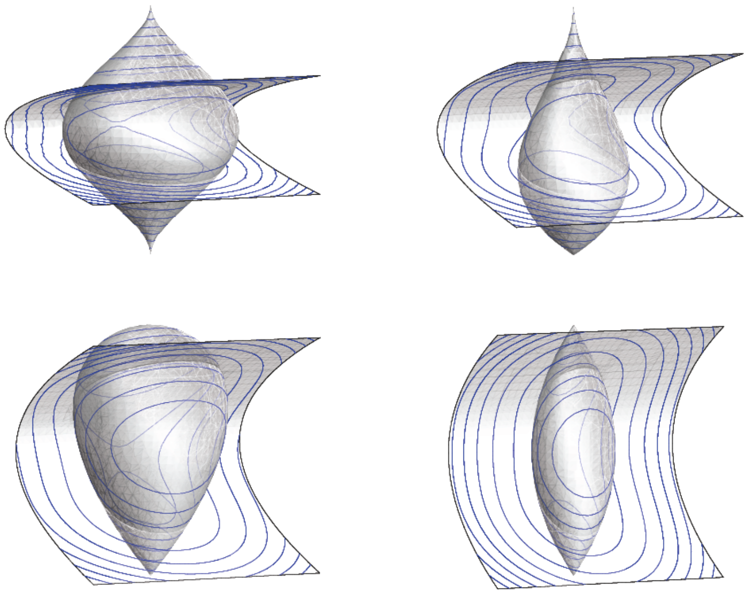

Figure 5: Motion along intersections of surfaces in $\mathbb{R}^3$.

(A) *Show that for smooth functions $c, f, h : \mathbb{R}^3 \to \mathbb{R}$, the $\mathbb{R}^3$-bracket defined by*

$$\{f, h\} = -\nabla c \cdot \nabla f \times \nabla h$$

satisfies the defining properties of a Poisson bracket. Is it also a derivation satisfying the Leibnitz relation for a product of functions on $\mathbb{R}^3$? If so, why?

(B) *How is the $\mathbb{R}^3$-bracket related to the canonical Poisson bracket in the plane?*

(C) *The Casimirs (or distinguished functions, as Lie called them) of a Poisson bracket satisfy*

$$\{c, h\}(\mathbf{x}) = 0\,, \quad \textit{for all} \quad h(\mathbf{x})$$

Part 5 verifies that the $\mathbb{R}^3$-bracket satisfies the defining properties of a Poisson bracket. What are the Casimirs for the $\mathbb{R}^3$ bracket?

(D) *Write the motion equation for the $\mathbb{R}^3$-bracket*

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\}$$

in vector form using gradients and cross products. Show that the corresponding Hamiltonian vector field $X_h = \{\,\cdot\,, h\}$ has zero divergence.

(E) *Show that under the $\mathbb{R}^3$-bracket, the Hamiltonian vector fields $X_f = \{\,\cdot\,, f\}$, $X_h = \{\,\cdot\,, h\}$ satisfy the following anti-homomorphism that relates the commutation of vector fields to the $\mathbb{R}^3$-bracket operation between smooth functions on $\mathbb{R}^3$,*

$$[X_f, X_h] = -X_{\{f,h\}}.$$

Hint: commutation of divergenceless vector fields does satisfy the Jacobi identity.

(F) *Show that the motion equation for the $\mathbb{R}^3$-bracket is invariant under a certain linear combination of the functions c and h. Interpret this invariance geometrically.*