2 M345 PA16 Assessed Coursework #2 Spring Term 2020

Exercise 2.1 (Classical) Let $y(t) = f(x(t))$ for $x, y \in \mathbb{R}^n$ and f a fixed invertible and differentiable coordinate transformation. Define the transformed Lagrangian in Hamilton's principle as

$$\widetilde{L}(y(t), \dot{y}(t)) = L\Big(f(x(t)), \frac{d}{dt}f(x(t))\Big)$$

(a) Use Hamilton's principle to derive the Euler-Lagrange equation for $\widetilde{L}(y(t), \dot{y}(t))$.

(b) Use Hamilton's principle to derive the Euler-Lagrange equation for $L\Big(f(x(t)), \frac{d}{dt}f(x(t))\Big)$.

(c) Show that the Euler-Lagrange equation for $\widetilde{L}(y(t), \dot{y}(t))$ implies the Euler-Lagrange equation for $L\Big(f(x(t)), \frac{d}{dt}f(x(t))\Big)$. For instance, you could show that δy being arbitrary implies the same for δx, by using the chain rule property of the variational derivative.

(d) Rewrite Hamilton's principle for $\widetilde{L}(y(t), \dot{y}(t))$ by Legendre transforming the Lagrangian $\widetilde{L}(y(t), \dot{y}(t))$ to its corresponding phase-space Lagrangian. Then take variations and derive the symplectic Poisson bracket for the canonical Hamiltonian formulation.

Exercise 2.2 (Geometric) Define the transformed Lagrangian as $\widetilde{L}(y(t), \dot{y}(t)) = L\Big(g_t y_0, \dot{g}_t y_0\Big)$, where g_t is a time-dependent curve in the $n \times n$ matrix Lie group G, with $g_0 = Id$.

Assume that the Lagrangian L is invariant under linear transformations defined by the left action of the matrix Lie group G on $\mathbb{R}^n$. This means that

$$L\Big(g_t y_0, \dot{g}_t y_0\Big) = L\Big(y_0, g_t^{-1}\dot{g}_t y_0\Big) =: \ell(g_t^{-1}\dot{g}_t)\,,$$

for every initial y_0. Define $\xi := g_t^{-1}\dot{g}_t$ with $\delta\xi = \delta(g_t^{-1}\dot{g}_t)$ where $\xi \in \mathfrak{g}$ is an element of the left matrix Lie algebra of the matrix Lie group G.

(a) Derive the Euler-Poincaré equation resulting from Hamilton's principle with action integral

$$S = \int_{t_1}^{t_2} \ell(\xi) + \Big\langle \mu\,,\, g_t^{-1}\dot{g}_t - \xi \Big\rangle\, dt\,,$$

where the brackets $\langle\,,\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, e.g., $\langle \mu\,,\,\xi \rangle := \frac{1}{2}\mathrm{tr}(\mu^T\xi)$ denote the Frobenius pairing of $n \times n$ matrices $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$ and in this pairing μ^T is the transpose matrix. In particular,

$$\delta\ell(\xi) = \Big\langle \frac{\delta \ell}{\delta\xi}\,,\,\delta\xi \Big\rangle = \frac{1}{2}\mathrm{tr}\Big(\frac{\delta\ell}{\delta\xi}^T\delta\xi\Big).$$

Hint: To compute the variation of $\xi = g_t^{-1}\dot{g}_t$, begin by defining $\nu := g_t^{-1}\delta g_t$ and proving that

$$\delta\xi = \frac{d}{dt}\nu + \xi\nu - \nu\xi =: \frac{d}{dt}\nu + [\xi, \nu] =: \frac{d}{dt}\nu + \mathrm{ad}_\xi \nu\,,$$

where matrix multiplication is denoted by concatenation.

(b) Derive the Euler-Poincaré equation resulting from Hamilton's principle with the phase-space Lagrangian in the action integral

$$S = \int_{t_1}^{t_2} \Big\langle \mu\,,\, g_t^{-1}\dot{g}_t \Big\rangle - h(\mu)\, dt\,.$$

(c) From the result of Hamilton's principle with the phase-space Lagrangian, derive the Lie–Poisson bracket for the Hamiltonian formulation on $n \times n$ matrices $\mu \in \mathfrak{g}^*$ as a linear functional of the matrix Lie algebra commutator.

(d) Explain why the Lie–Poisson bracket derived in the previous part satisfies the Jacobi condition. Hint: Commutation $[\xi, \nu] := \xi\nu - \nu\xi$ in the matrix Lie algebra $\mathfrak{g}$ does satisfy the Jacobi condition.

Exercise 2.3 Show that the following operations on vector fields and differential forms are natural under pull-back. Namely,

(a) wedge product: Prove $\phi^*(\alpha\wedge\beta)=\alpha\wedge\phi^*\beta$ for $\alpha\in\Lambda^k(M)$ and $\alpha\in\Lambda^l(M)$, $k+l\le\dim M$.

(b) vector field commutator: Prove $c_t^*\big[Q,\,R\big]=\big[c_t^*Q,\,c_t^*R\big]$ in which the transformation between the vector fields is $R=c^*Q$.

For this proof, it may be useful to review the meaning of the notation $(Qh)\circ c=R(h\circ c)$ by redoing the calculus manipulation done in class, which was as a direct change of variables

$$Q=f^i(q)\frac{\partial}{\partial q^i}\quad\mapsto\quad R=g^j(r)\frac{\partial}{\partial r^j}\quad\text{with}\quad g^j(r)\frac{\partial c^i}{\partial r^j}=f^i(c(r))\quad\text{or}\quad g=c_r^{-1}f\circ c\,,$$

obtained by substituting $q=c(r)$ into the operation of the vector field Q on a function h, as

$$(Qh)\circ c=c^*(Qh)=\frac{\partial h(c(r))}{\partial r^j}\Big(\Big[\frac{\partial c}{\partial r}\Big]^{-1}\Big)^j_i f^i(c(r))=g_j(r)\frac{\partial h((c(r))}{\partial r^j}=R(h\circ c)\,.$$

(c) differential: Prove $\phi^*(d\alpha)=d(\phi^*\alpha)$ for $\alpha\in\Lambda(M)$ and $k<\dim M$. What happens if $k=\dim M$?

(d) contraction of a vector field with a differential form:

With $\phi^*X(m)=X(\phi(m))$ for a point $m\in M$, as in the calculation above for transforming coordinates in vector fields, prove that

$$\phi^*X\,\lrcorner\,\phi^*\alpha(X_1,X_2,\dots.)=\phi^*(X\,\lrcorner\,\alpha)(X_2,X_3,\dots)\,.$$

(e) Lie derivative: Prove that, given the geometric definition of the Lie derivative,

$$\begin{aligned}\frac{d}{dt}(\phi_t^*\alpha)=\phi_t^*(\mathcal{L}_X\alpha)&=\phi_t^*(X\,\lrcorner\,d\alpha+d(X\,\lrcorner\,\alpha))\\&=\phi_t^*X\,\lrcorner\,d\alpha+X\,\lrcorner\,d(\phi_t^*\alpha)+d(\phi_t^*X\,\lrcorner\,\alpha+X\,\lrcorner\,\phi_t^*\alpha)\end{aligned}$$

where the vector field X is tangent to the flow of ϕ_t at the identity $t=0$.

Exercise 2.4

1. The Lie derivative of one vector field by another is called the ***Jacobi-Lie bracket***, defined as

$$\pounds_XY:=[X\,,\,Y]:=(X\cdot\nabla)Y-(Y\cdot\nabla)X=-\,\pounds_YX$$

Verify the Jacobi identity for the Jacobi-Lie bracket using streamlined notation

$$[X,\,Y]=X(Y)-Y(X)\,,$$

and invoking bilinearity of the bracket.

2. Verify the following Lie derivative identities by choosing a convenient definition of Lie derivative:

   (i) $\pounds_{fX}\alpha=f\pounds_X\alpha+df\wedge(X\,\lrcorner\,\alpha)$
   (ii) $\pounds_Xd\alpha=d\big(\pounds_X\alpha\big)$
   (iii) $\pounds_X(X\,\lrcorner\,\alpha)=X\,\lrcorner\,\pounds_X\alpha$
   (iv) $\pounds_X(Y\,\lrcorner\,\alpha)=(\pounds_XY)\,\lrcorner\,\alpha+Y\,\lrcorner\,(\pounds_X\alpha)$
   (v) $\pounds_X(\alpha\wedge\beta)=(\pounds_X\alpha)\wedge\beta+\alpha\wedge\pounds_X\beta$

3. Verify $[X\,,\,Y]\,\lrcorner\,\alpha=\pounds_X(Y\,\lrcorner\,\alpha)-Y\,\lrcorner\,(\pounds_X\alpha)$ and use part 2(iv) to prove the Jacobi identity.

4. From the pull-back formula in part (e) of exercise (2.3), prove that

$$\begin{aligned}\mathcal{L}_X(\mathcal{L}_Y\alpha):=\frac{d}{dt}\Big|_{t=0}\big(\phi_t^*(\mathcal{L}_Y\alpha)\big)&=\frac{d}{dt}\Big|_{t=0}\big(\phi_t^*Y\,\lrcorner\,d\alpha+Y\,\lrcorner\,d(\phi_t^*\alpha)+d(\phi_t^*Y\,\lrcorner\,\alpha+Y\,\lrcorner\,\phi_t^*\alpha)\big)\\&=(\mathcal{L}_XY)\,\lrcorner\,d\alpha+Y\,\lrcorner\,d(\mathcal{L}_X\alpha)+d\big((\mathcal{L}_XY)\,\lrcorner\,\alpha+Y\,\lrcorner\,(\mathcal{L}_X\alpha)\big)\end{aligned}$$

You can use this identity to challenge your friends to derive calculus identities. Try evaluating with a scalar function, $\alpha=f$ and also a density in $\mathbb{R}^3$, $\alpha=\rho d^3x$. If you are brave, try letting α be a 1-form or a 2-form in $\mathbb{R}^3$, which will involve all four terms. Check your work with other some of the other students.