3 M345 PA16 Assessed Coursework #3 Spring Term 2020

Exercise 3.1 (Palais's Theorem)

(a) Prove that

$$[X\,,\,Y] \lrcorner\, \alpha = \pounds_X(Y \lrcorner\, \alpha) - Y \lrcorner\, (\pounds_X \alpha)\,.$$

(b) For a top form α and divergence free vector fields X and Y, use Cartan's formula $\pounds_X \alpha = d(X \lrcorner\, \alpha) + X \lrcorner\, d\alpha$ to show that

$$[X\,,\,Y] \lrcorner\, \alpha = d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big)\,. \tag{1}$$

(c) Write the equivalent of equation (1) as a formula in vector calculus.

(d) Show, for vector fields $u, v \in \mathfrak{X}(\mathbb{R}^3)$ and a 1-form $\alpha \in \Lambda^1(\mathbb{R}^3)$ that Palais's theorem holds,

$$\pounds_u(v \lrcorner\, \alpha) - \pounds_v(u \lrcorner\, \alpha) = [u, v] \lrcorner\, \alpha + v \lrcorner\, (u \lrcorner\, d\alpha)\,.$$

Hint: the proof follows by simply evaluating left and right sides of the equality.

Exercise 3.2 (Pull-back identities) The family of Lagrangian fluid trajectories $x_t = \phi_t(X)$ with initial positions X in $\mathbb{R}^3$ extends to include their deformations as $x_{t,\varepsilon} = \phi_{t,\varepsilon}(X)$, where t is time and ε is the deformation parameter.

Define three time-dependent vector fields $u_t(x_t), w_t(x_t)$ and $\delta u_t(x_t)$ in terms of the following three different types of tangents of the perturbed trajectories at the identity, $\varepsilon = 0$,

$$\begin{aligned}
u_t(x_t) &=: \left[\frac{\partial x_{t,\varepsilon}}{\partial t}\right]_{\varepsilon=0} := \left[\frac{\partial}{\partial t}(\phi_{t,\varepsilon}X)\right]_{\varepsilon=0}\,,\\
w_t(x_t) &=: \left[\frac{\partial x_{t,\varepsilon}}{\partial \varepsilon}\right]_{\varepsilon=0} := \left[\frac{\partial}{\partial \varepsilon}(\phi_{t,\varepsilon}X)\right]_{\varepsilon=0}\,,\\
\delta u_t(x_t) &:= \left[\frac{\partial\, u_{t,\varepsilon}(x_{t,\varepsilon})}{\partial \varepsilon}\right]_{\varepsilon=0}\,.
\end{aligned}$$

Thus, for example, the Eulerian vector field $u_t(x_t) = [\phi^*_{t,\varepsilon}(u_t(X))]_{\varepsilon=0}$ generates the flow $\phi_{t,0}$.

(a) By taking the difference of equal cross derivatives

$$\left[\frac{\partial}{\partial \varepsilon}\frac{\partial x_{t,\varepsilon}}{\partial t}\right]_{\varepsilon=0} = \left[\frac{\partial}{\partial t}\frac{\partial x_{t,\varepsilon}}{\partial \varepsilon}\right]_{\varepsilon=0}$$

show that

$$\delta u_t(x_t) - \partial_t w_t(x_t) = -\frac{\partial u_t}{\partial x_t}\cdot w_t + \frac{\partial w_t}{\partial x_t}\cdot u_t =: -\,\mathrm{ad}_{u_t} w_t$$

(b) Under the extended flow trajectory $x_{t,\varepsilon} = \phi_{t,\varepsilon}(X)$, advected quantities $a_{t,\varepsilon}(x_{t,\varepsilon})$ satisfy the pull-back relation, $[\phi^*_{t,\varepsilon} a_{t,\varepsilon}]_{\varepsilon=0} = a_{0,0}$, so that $a_t = a_{t,0} = [(\phi^{-1}_{t,\varepsilon})^* a_{0,0})]_{\varepsilon=0}$.

Prove the following sequence of equalities,

$$\partial_t a_t = \partial_t \big[a_{t,\varepsilon}\big]_{\varepsilon=0} = \partial_t \Big[(\phi^{-1}_{t,\varepsilon})^* a_{0,0})\Big]_{\varepsilon=0} = -\,\mathcal{L}_{u_t} a_t$$

where the vector field u_t is the generator of $\phi_{t,\varepsilon}$ and $\mathcal{L}$ is the Lie derivative. (Note the sign.) Recall the dynamical definition of the Lie derivative, $\big[\partial_t(\phi^*_t a(X))\big]_{t=0} = \big[\phi^*_t \mathcal{L}_{u_t} a(X)\big]_{t=0} = \mathcal{L}_u a$.

(c) Likewise, show that

$$0 = \delta a_{0,0} = \Big[\partial_\varepsilon(\phi^*_{t,\varepsilon} a_{t,\varepsilon})\Big]_{\varepsilon=0} = \delta w_t + \mathcal{L}_{w_t} a_t$$

Exercise 3.3 (Diamond)

The operation $\diamond : V \times V^* \to \mathfrak{X}^*$ between tensor space elements $a \in V^*(M)$ and $b \in V(M)$ produces $b \diamond a \in \mathfrak{X}(M)^*$, a one-form density, given by

$$\Big\langle b \diamond a, u \Big\rangle_{\mathfrak{X}} = -\int_{\mathcal{D}} b \cdot \pounds_u\, a =: \Big\langle b\,,\, -\mathcal{L}_u\, a \Big\rangle_V \,.$$

Here, the bracket $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{X}}$ denotes

$$\Big\langle b \diamond a, u \Big\rangle_{\mathfrak{X}} = \int_{\mathcal{D}} u \,\lrcorner\, (b \diamond a)\,.$$

The bracket $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{X}}$ is the symmetric, non-degenerate L^2 pairing between vector fields and one-form densities, which are dual with respect to this pairing. Likewise, $\langle\,\cdot\,,\,\cdot\,\rangle_V$ represents the corresponding L^2 pairing between elements of V and V^*. Also, $\mathcal{L}_u a$ stands for the Lie derivative of an element $a \in V^*$ with respect to a vector field $u \in \mathfrak{X}(M)$, and $b \cdot \mathcal{L}_u\, a$ denotes the contraction between elements of V and elements of V^*. E.g., for differential forms, if $V^* = \Lambda^k(M)$, then $V = \Lambda^{n-k}(M)$ for $\dim M = n$ and we have

$$\Big\langle b \diamond a, u \Big\rangle_{\mathfrak{X}} = -\int_{\mathcal{D}} b \cdot \pounds_u\, a = -\int_{\mathcal{D}} b \wedge \pounds_u\, a\,.$$

(a) Compute explicit formulae for the advection equations $\partial_t a = -\mathcal{L}_u\, a$ in the cases that the set of tensor fields $a \in V^*$ consists of elements with the following coordinate functions in a Euclidean basis on $\mathbb{R}^3$,

$$a \in \{s,\; A := \mathbf{A}\cdot d\mathbf{x},\; B := \mathbf{B}\cdot d\mathbf{S},\; D := \rho\, d^3x\}\,.$$

These apply to the Euler–Poincaré equations for fluids, when the reduced Lagrangian depends on $l(\mathbf{v}, s, A, B, D)$, where $\mathbf{v}$ is the fluid velocity.

(b) Compute explicit formulae for the diamond products (momentum maps) in the two cases $\big\langle s \diamond D\,,\, u\big\rangle_{\mathfrak{X}} = \big\langle s\,,\, -\mathcal{L}_u\, D\big\rangle_V$ and $\big\langle A \diamond B\,,\, u\big\rangle_{\mathfrak{X}} = \big\langle A\,,\, -\mathcal{L}_u\, B\big\rangle_V$. Also calculate the results in the two exchanged cases, $D \diamond s$ and $B \diamond A$. What conclusion can one draw from these comparisons?

Exercise 3.4 (Euler–Poincaré equations for fluids)

(a) Compute the Euclidean components of $\frac{\delta l}{\delta a} \diamond a$

$$a \in \{s,\; A := \mathbf{A}\cdot d\mathbf{x},\; B := \mathbf{B}\cdot d\mathbf{S},\; D := \rho\, d^3x\}\,.$$

These apply to the Euler–Poincaré equations for fluids, when the reduced Lagrangian depends on $l(\mathbf{v}, s, A, B, D)$, where $\mathbf{v}$ is the fluid velocity.

(b) Write the Euler–Poincaré motion equation

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right)\frac{\delta l}{\delta \mathbf{v}} = \frac{\delta l}{\delta a} \diamond a$$

in vector form for advected quantities $a \in \{s,\; A := \mathbf{A}\cdot d\mathbf{x},\; B := \mathbf{B}\cdot d\mathbf{S},\; D := \rho\, d^3x\}$.

Exercise 3.5 (EP equations for MHD)

Write the Euler–Poincaré motion equation

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{v}}\right)\frac{\delta l}{\delta \mathbf{v}} = \frac{\delta l}{\delta a} \diamond a$$

in vector form for advected quantities $a \in \{s,\; A := \mathbf{A}\cdot d\mathbf{x},\; B := \mathbf{B}\cdot d\mathbf{S},\; D := \rho\, d^3x\}$, when the Lagrangian is given by

$$l(\mathbf{v}, s, D, B) = \int_{\mathcal{D}} \frac{D}{2}|\mathbf{v}|^2 - De(D,s) - \frac{1}{2}|\mathbf{B}|^2\, d^3x\,.$$

Here, $e(D, s)$ is the fluid's specific internal energy, whose dependence on the density D and specific entropy s is given as the "equation of state" and which for an isotropic medium satisfies the thermodynamic First Law in the form $de = -pd(1/D) + Tds$ with pressure $p(D, s)$ and temperature $T(D, s)$. Taking the variation in D will produce the quantity $h = e + p/D$ which is called the specific enthalpy. The First Law implies the convenient relation $dh = (1/D)dp + Tds$. The EP equations for the Lagrangian $l(\mathbf{v}, s, A, B, D)$ describe adiabatic compressible magnetohydrodynamics (MHD).

Exercise 3.6 (Gyrostat: A rigid body with flywheel attached)

Consider a rigid body with flywheel attached along the intermediate principle axis in the body Just as for the isolated rigid body, the energy in this problem is purely kinetic; so one may define the kinetic energy Lagrangian for this system $L : TSO(3)/SO(3) \times TS^1 \to \mathbb{R}^3$ as

$$L(\boldsymbol{\Omega},\, \dot\phi) = \frac{1}{2}\lambda_1\Omega_1^2 + \frac{1}{2}I_2\Omega_2^2 + \frac{1}{2}\lambda_3\Omega_3^2 + \frac{1}{2}J_2(\dot\phi + \Omega_2)^2\,,$$

where $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the angular velocity vector of the rigid body, $\dot\phi$ is the rotational frequency of the flywheel about the intermediate principal axis of the rigid body,, and λ_1, I_2, J_2, λ_3 are positive constants corresponding to the principal moments of inertia, including the presence of the flywheel.

(a) Perform a partial Legendre transform in the flywheel variables $(\phi, \dot\phi) \in TSO(2)$ only. What can you say about the flywheel degree of freedom and its coupling to the rest of the system? In particular, prove that the angular momentum N conjugate to the angle ϕ is a constant of motion.

(b) Perform the complete Legendre-transform of this Lagrangian as

$$\begin{aligned} H(\boldsymbol{\Pi}, N) &= \boldsymbol{\Pi}\cdot\boldsymbol{\Omega} + N\dot\phi - L(\boldsymbol{\Omega},\, \dot\phi) \\ &= \frac{\Pi_1^2}{2\lambda_1} + \frac{\Pi_3^2}{2\lambda_3} + \underbrace{\frac{1}{2I_2}\big(\Pi_2 - N\big)^2}_{\text{offset along } \Pi_2} + \frac{N^2}{2}\Big(\frac{1}{I_2} + \frac{1}{J_2}\Big)\,. \end{aligned}$$

to express its Hamiltonian in terms of the angular momenta $\boldsymbol{\Pi} = \partial L/\partial\boldsymbol{\Omega} \in \mathbb{R}^3$ and $N = \partial L/\partial\dot\phi \in \mathbb{R}^1$ of the rigid body and flywheel, respectively.

Show that the resulting Hamiltonian $H(\boldsymbol{\Pi}; N)$ is an ellipsoid in coordinates $\boldsymbol{\Pi} \in \mathbb{R}^3$, whose centre is *offset* in the Π_2-direction by a constant amount equal to the conserved angular momentum N of the flywheel.

(c) Show that the motion equation in $\boldsymbol{\Pi} = \partial L/\partial\boldsymbol{\Omega} \in \mathbb{R}^3$ can be written as

$$\frac{d\boldsymbol{\Pi}}{dt} = -\,\boldsymbol{\Pi}\times\nabla_{\boldsymbol{\Pi}}\,H(\boldsymbol{\Pi}; N) = -\,\nabla_{\boldsymbol{\Pi}}\,|\boldsymbol{\Pi}|^2\times\nabla_{\boldsymbol{\Pi}}\,H(\boldsymbol{\Pi}; N)$$

Since the angular momentum N of the flywheel is constant, the full problem is still completely integrable and can be understood as motion in $\mathbb{R}^3$ along intersections of the Hamiltonian $H(\boldsymbol{\Pi}; N)$ with the angular momentum sphere, $|\boldsymbol{\Pi}|^2 = const$,

(d) **Qualitative analysis.** The presence of the flywheel offsets the Hamiltonian ellipsoid relative to the angular momentum sphere. What is the effect of this offset on the solution of the original rigid body, which has $N = 0$ and thus no offset? In particular, what does the offset do to the heteroclinic orbits on the angular momentum sphere for the rigid body? [1]

(e) Show that the equations of motion for $(\boldsymbol{\Pi}, \phi, N)$ can be written in Poisson bracket form in the variables $\boldsymbol{\Pi}, N, \phi \in so(3)^* \times T^*S^1$ as a direct sum of the rigid-body bracket for $\boldsymbol{\Pi} \in so(3)^* \simeq \mathbb{R}^3$

[1] Draw some pictures of intersections of an offset ellipsoid and a sphere, thinking about the comparison when there is no offset, then take a look at Elipe, A., Arribas, M. and Riaguas, A. [1997] Complete analysis of bifurcations in the axial gyrostat problem. Journal of Physics A: Mathematical and General, 30(2), p.587.

and the canonical bracket for the flywheel phase-space coordinates $(N, \phi) \in T^*S^1$. For this purpose, begin by writing the Hamiltonian equations in Poisson matrix form,

$$\frac{d}{dt}\begin{bmatrix} \mathbf{\Pi} \\ \phi \\ N \end{bmatrix} = -\begin{bmatrix} \mathbf{\Pi}\times & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \partial H/\partial \mathbf{\Pi} \\ \partial H/\partial \phi \\ \partial H/\partial N \end{bmatrix}.$$