1 M3-4-5A16 Un-assessed homework to discuss Spring Term 2020

Exercise 1.1 Find any errors in computing the Euler-Lagrange equations from Hamilton's principle for the following simple mechanical systems: $L(q,\dot q) = T(\dot q) - V(q) = KE - PE$.

1. Planar isotropic oscillator, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$:
   $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k}{2}|\mathbf{x}|^2 \quad\Longrightarrow\quad \ddot{\mathbf{x}} = -\,\omega^2\mathbf{x} \quad\text{with}\quad \omega^2 = k/m$

2. Planar anisotropic oscillator, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$:
   $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k_1}{2}x_1^2 - \frac{k_2}{2}x_2^2 \quad\Longrightarrow\quad \ddot{x}_i = -\,\omega_i^2 x_i \quad\text{with}\quad \omega_i^2 = k_i/m \quad i=1,2$

3. Planar pendulum in polar coordinates, $(\theta,\dot\theta) \in TS^1$:
   $L = \frac{m}{2}R^2\dot\theta^2 - mgR(1-\cos\theta) \quad\Longrightarrow\quad \ddot\theta = -\omega^2\sin\theta \quad\text{with}\quad \omega^2 = g/R$

4. Planar pendulum, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$, constrained to $TS^1 = \{\mathbf{x},\dot{\mathbf{x}} \in T\mathbb{R}^2 \,|\, 1-|\mathbf{x}|^2 = 0 \;\&\; \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$:
   $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} - \frac{\mu}{2}(1-|\mathbf{x}|^2) \quad\Longrightarrow\quad m\ddot{\mathbf{x}} = -\,mg\hat{\mathbf{e}}_3(\mathrm{Id} - \mathbf{x}\otimes\mathbf{x}) - \,\mathrm{m}|\dot{\mathbf{x}}|^2\mathbf{x}$, (gravity & centripetal force). Hint: Find Lagrange multiplier μ by requiring that the constraint is preserved.

5. Charged particle in a magnetic field, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^2$:
   $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + \frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) \quad\Longrightarrow\quad \ddot{\mathbf{x}} = \frac{e}{mc}\dot{\mathbf{x}}\times\mathbf{B} \quad\text{with}\quad \mathbf{B} = \mathrm{curl}\,\mathbf{A}$

6. Kepler problem in Cartesian coordinates, $(\mathbf{r},\dot{\mathbf{r}}) \in T\mathbb{R}^3$:
   $L(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}|\dot{\mathbf{r}}|^2 - V(r)$ with $V(r) = -\,\mu/r$ and $r := |\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}}$. $\quad\Longrightarrow\quad \ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = 0.$

7. Kepler problem in polar coordinates, $(r,\dot r,\theta,\dot\theta) \in T\mathbb{R}_+ \times TS^1$: $|\dot{\mathbf{r}}|^2 = \dot r^2 + r^2\dot\theta^2$
   $L = \frac{1}{2}\Big(\dot r^2 + r^2\dot\theta^2\Big) + \frac{\mu}{r} \quad\Longrightarrow\quad \ddot r = -\frac{\mu}{r^2} + \frac{J^2}{r^3} \quad\text{with}\quad J = r^2\dot\theta = const$

8. Free motion on a sphere, $(\mathbf{x},\dot{\mathbf{x}}) \in T\mathbb{R}^3$, with $TS^2 = \{(\mathbf{x},\dot{\mathbf{x}}) \in \mathbb{R}^6 : |\mathbf{x}|^2 = 1 \text{ and } \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$:
   $L = \frac{1}{2}|\dot{\mathbf{x}}|^2 + \frac{\mu}{2}(1-|\mathbf{x}|^2) \quad\Longrightarrow\quad \ddot{\mathbf{x}} = -\,|\dot{\mathbf{x}}|^2\mathbf{x}$

9. Spherical pendulum (b), set $\mathbf{x}(t) = O(t)\mathbf{x}_0\,,\quad \dot{\mathbf{x}}(t) = \dot O(t)\mathbf{x}_0$ for $(O,\,\dot O) \in TSO(3)\,,$ where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial position of the particle and $O^T = O^{-1}$

   $$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} = \frac{m}{2}|\dot O(t)\mathbf{x}_0|^2 - mg\,O^T(t)\hat{\mathbf{e}}_3\cdot\mathbf{x}_0\,.$$

   Setting $\mathbf{x}(t) = O(t)\mathbf{x}_0$ avoids the need for the constraint $|\mathbf{x}|^2 = 1$, since rotations preserve length. $\Longrightarrow$

   $$\dot{\boldsymbol{\Pi}} + \boldsymbol{\Omega}\times\boldsymbol{\Pi} = -\,g\,\boldsymbol{\Gamma}\times\mathbf{x}_0 \quad\text{with}\quad \boldsymbol{\Pi} := \mathbf{x}_0\times(\boldsymbol{\Omega}\times\mathbf{x}_0) = \boldsymbol{\Omega}|\mathbf{x}_0|^2 - \mathbf{x}_0(\mathbf{x}_0\cdot\boldsymbol{\Omega})\,.$$

   Set $g = 0$ to get free motion on the sphere. Finally, from its definition, $\boldsymbol{\Gamma} := O^{-1}(t)\hat{\mathbf{e}}_3$ satisfies

   $$\dot{\boldsymbol{\Gamma}} := -\,\widehat{\Omega}\boldsymbol{\Gamma} = -\,\boldsymbol{\Omega}\times\boldsymbol{\Gamma}\,.$$

10. Rotating rigid body, $\widehat{\Omega} = O^{-1}\dot O \in T(SO(3) \simeq \mathfrak{so}(3)$:
    $\ell(\boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\Omega}\cdot I\boldsymbol{\Omega} \quad\text{with}\quad \boldsymbol{\Omega}\times = \widehat{\Omega}\,,\quad\text{that is,}\quad -\,\epsilon_{ijk}\Omega_k = \widehat{\Omega}_{ij}\,. \quad\Longrightarrow\quad I\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}\times I\boldsymbol{\Omega} = 0\,.$

Exercise 1.2

(a) Find any errors in computing the Legendre transforms

$$p := \frac{\partial L}{\partial \dot q}\,, \quad H(q,p) = \langle p,\, \dot q\rangle - L(q,\dot q) = T(p) + V(q) = KE + PE$$

for the simple mechanical systems in Exercise 1.1.

(b) Compute the canonical Hamiltonian equations for each system and show equivalence with their corresponding Euler-Lagrange equations.

1. Planar isotropic oscillator, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k}{2}|\mathbf{x}|^2$

2. Planar anisotropic oscillator, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k_1}{2}x_1^2 + \frac{k_2}{2}x_2^2$

3. Planar pendulum in polar coordinates, $(\theta, p_\theta) \in T^*S^1$: $\quad H = \frac{1}{2mR^2}p_\theta^2 + mgR(1-\cos\theta)$

4. Planar pendulum, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$, constrained to $S^1 = \{\mathbf{x}\in\mathbb{R}^2 : 1 - |\mathbf{x}|^2 = 0\}$:
$H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} + \frac{1}{2}\mu(1-|\mathbf{x}|^2)$.

5. Charged particle in a magnetic field, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})|^2 \quad \mathbf{p} := \partial L/\partial\dot{\mathbf{q}} = m\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}(\mathbf{x}) \in T^*M$

6. Kepler problem, $(r, p_r, \theta, p_\theta) \in T^*\mathbb{R}_+ \times T^*S^1$: $\quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$ with $\quad p_\theta = r^2\dot\theta = const$

7. Free motion on a sphere, $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x}\in\mathbb{R}^3 : 1-|\mathbf{x}|^2 = 0\}$:
$H = \frac{1}{2m}|\mathbf{p}|^2 - \mu(1-|\mathbf{x}|^2)$

8. Spherical pendulum (a), $(\mathbf{x},\mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x}\in\mathbb{R}^3 : 1-|\mathbf{x}|^2 = 0\}$:
$H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_3\cdot\mathbf{x} - \mu(1-|\mathbf{x}|^2)$

9. Spherical pendulum (b), $(O,\dot O) \in TSO(3)$, $\widehat{\xi} = O^{-1}\dot O \in T(SO(3) \simeq \mathfrak{so}(3)$, $\mathbf{\Pi} = \partial\ell/\partial\mathbf{\Omega} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3 \quad H = \frac{1}{2}\mathbf{\Pi}\cdot I^{-1}\mathbf{\Pi} + g\,\mathbf{\Gamma}\cdot\mathbf{x}_0$ with $\mathbf{\Pi} = \frac{\partial\ell}{\partial\mathbf{\Omega}} = I\mathbf{\Omega}$. Set $g=0$ to get freely rotating rigid body motion.

10. Rotating rigid body, $\mathbf{\Pi} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3 \quad H = \frac{1}{2}\mathbf{\Pi}\cdot I^{-1}\mathbf{\Pi}$ with $\mathbf{\Pi} = \frac{\partial\ell}{\partial\mathbf{\Omega}} = I\mathbf{\Omega}$.

Classical mechanics relations may be visualised as equivariant transformations.

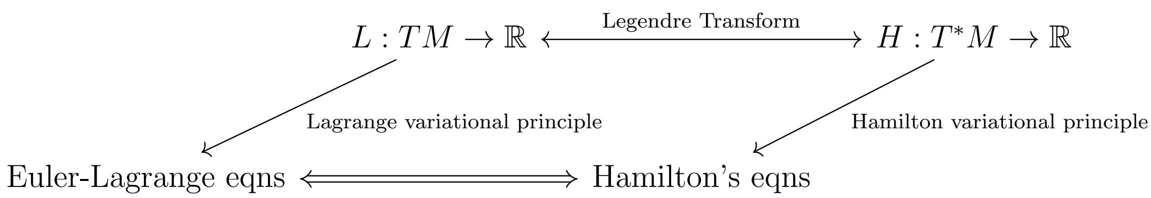

Figure 1: Framework for Classical Mechanics

Exercise 1.3 (Examples of Noether's theorem)

(a) What conservation law does Noether's theorem imply for symmetries of the action principle given by $\delta \mathsf{S} = 0$ with

$$\mathsf{S} = \int_a^b L(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)\, dt\,, \quad \text{for} \quad \mathbf{q} \in \mathbb{R}^3 \quad \text{and} \quad L : T\mathbb{R}^3 \to \mathbb{R}\,,$$

when the Lagrangian $L(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)$ is invariant under infinitesimal azimuthal rotations about $\hat{\mathbf{z}}$ given by

$$\mathbf{q}(t, \epsilon) = \mathbf{q}(t) + \epsilon\, \hat{\mathbf{z}} \times \mathbf{q}(t) + O(\epsilon^2) \quad \text{so that} \quad \delta \mathbf{q} = \frac{d\mathbf{q}}{d\epsilon}\bigg|_{\epsilon=0} = \hat{\mathbf{z}} \times \mathbf{q}(t)\,.$$

(b) What additional conservation law is implied by Noether's theorem when the Lagrangian in the form $L(\dot{\mathbf{q}}(t), \mathbf{q}(t))$ is translation invariant in time, $t \to t + \epsilon$; that is, $\partial_t L = 0$.
Hint: What does this question have to do with Exercise 1.2?

(c) What are the symmetries and corresponding Noether conservation laws of the Lagrangians for the simple mechanical systems in Exercise 1.1?