Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

M3/4A16

DYNAMICS II

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This paper is also taken for the relevant examination for the Associateship.

M3/4A16

DYNAMICS II

Date: Time:

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider the Poisson bracket defined for smooth real functions F, H of $\mathbf{x} \in \mathbb{R}^3$ by

$$\begin{aligned} \{F, H\}_{\mathsf{K}} &:= -\mathbf{x} \cdot \mathsf{K}\big(\nabla \mathsf{F} \times \nabla \mathsf{H}\big) \\ &=: -\big\langle \mathbf{x} \,,\, \big[\, \nabla F \,,\, \nabla H \,\big]_{\mathsf{K}} \big\rangle \,. \end{aligned}$$

Here, $\mathsf{K}^{\mathsf{T}} = \mathsf{K}$ is a symmetric 3×3 matrix, $\nabla = \partial/\partial\mathbf{x}$, the bracket $[\,\cdot\,,\,\cdot\,]_{\mathsf{K}}$ denotes

$$\big[\,\mathbf{y}, \mathbf{z}\,\big]_{\mathsf{K}} := K(\mathbf{y} \times \mathbf{z}) \quad \text{for all} \quad \mathbf{y},\, \mathbf{z} \in \mathbb{R}^3 \,,$$

and $\times$ is the vector cross product on $\mathbb{R}^3$.

(a) Determine the family of Casimirs for this Poisson bracket.

(b) Explain how to show that the Poisson bracket $\{F, H\}_{\mathsf{K}}$ satisfies the Jacobi identity. Hint: Do not verify it directly!

(c) Show that the Hamiltonian vector field $X_H = \{\,\cdot\,, H\}_{\mathsf{K}}$ for this Poisson bracket is a divergenceless vector field on $\mathbb{R}^3$.

(d) Recall that the **commutator** of the divergenceless vector fields X_F and X_H is written symbolically as

$$[X_F \,,\, X_H] = X_F X_H - X_H X_F = G(H) - H(G) \,.$$

Use this symbolic notation to verify that the commutator of the divergenceless vector fields satisfies the Jacobi identity.

(e) Put the Poisson bracket $\{F,\, H\}_{\mathsf{K}}$ into one-to-one correspondence with the commutator of the corresponding Hamiltonian vector fields X_F and X_H by proving the equality

$$X_{\{F\,,H\}} = -\,[X_F \,,\, X_H] \,.$$

Hint: You may wish to invoke the Jacobi identity.

(f) Use the Poisson bracket $\{F,\, H\}_{\mathsf{K}}$ to write the equation of motion for $\mathbf{x}$ when the Hamiltonian is $H = |\mathbf{x}|^2/2$.

(g) Describe the solutions geometrically in $\mathbb{R}^3$.

(h) Explain how these solutions are related to rigid-body motion.

2. A steady Euler fluid flow in a rotating frame satisfies

$$\pounds_u(\mathbf{v}\cdot d\mathbf{x}) = -\,d(p+\tfrac{1}{2}|\mathbf{u}|^2-\mathbf{u}\cdot\mathbf{v})\,,$$

where $\pounds_u$ is Lie derivative with respect to the divergenceless vector field $u=\mathbf{u}\cdot\nabla$, with $\nabla\cdot\mathbf{u}=0$, and $\mathbf{v}=\mathbf{u}+\mathbf{R}$, with Coriolis parameter curl$\mathbf{R}=2\boldsymbol{\Omega}$.

(a) Write out this Lie-derivative relation in Cartesian coordinates.

(b) By taking the exterior derivative, show that this relation implies that the exact two-form

$$\mathrm{curl}v \,\lrcorner\, d^{\,3}x = \mathrm{curl}\mathbf{v}\cdot\nabla \,\lrcorner\, d^{\,3}x = \mathrm{curl}\mathbf{v}\cdot d\mathbf{S} = d(\mathbf{v}\cdot d\mathbf{x}) =: d\Xi\wedge d\Pi$$

is invariant under the flow of the divergenceless vector field u.

(c) Show that Cartan's formula for the Lie derivative in the steady Euler flow condition implies that

$$u \,\lrcorner\, \Big(\mathrm{curl}v \,\lrcorner\, d^{\,3}x\Big) = dH(\Xi,\Pi)$$

and identify the function H.

(d) Use the result of (2c) to write $\pounds_u\Xi=\mathbf{u}\cdot\nabla\Xi$ and $\pounds_u\Pi=\mathbf{u}\cdot\nabla\Xi$ in terms of the partial derivatives of H.

(e) What do the results of (2d) mean geometrically? Hint: Is a symplectic form involved?

3. (a) Compute the Poisson bracket table among the axisymmetric optical variables

$$X_1 = |\mathbf{q}|^2 \geq 0\,, \quad X_2 = |\mathbf{p}|^2 \geq 0\,, \quad X_3 = \mathbf{p}\cdot\mathbf{q}\,,$$

with $(\mathbf{q},\mathbf{p}) \in T^*\mathbb{R}^2$.

(b) Derive the Poisson bracket for smooth functions on $\mathbb{R}^3$ by changing variables $(\mathbf{q},\mathbf{p}) \in T^*\mathbb{R}^2 \to (X_1, X_2, X_3) \in \mathbb{R}^3$ by using the chain rule. Show that the result may be expressed as

$$\frac{dF}{dt} = \{F, H\} \;=\; \nabla F \cdot \nabla S^2 \times \nabla H = \frac{\partial F}{\partial X_k}\,\epsilon_{klm}\,\frac{\partial S^2}{\partial X_l}\,\frac{\partial H}{\partial X_m}\,. \tag{1}$$

with

$$S^2 = X_1 X_2 - X_3^2 \geq 0\,, \tag{2}$$

Explain why $S^2 \geq 0$.

(c) Explain why the Poisson bracket (1) with definition (2) satisfies the Jacobi identity.

(d) Consider the Hamiltonian

$$H = aY_1 + bY_2 + cY_3 \tag{3}$$

with the linear combinations

$$Y_1 = \frac{1}{2}(X_1 + X_2)\,, \quad Y_2 = \frac{1}{2}(X_2 - X_1)\,, \quad Y_3 = X_3\,,$$

and constant values of (a, b, c). Compute the motion generated by Hamiltonian (3) by the Poisson bracket (1) on the space of variables $\mathbf{Y} \in \mathbb{R}^3$. In particular, what is the motion for $(a, b, c) = (1, 0, 0)$?

4. A three dimensional spatial rotation is described by multiplication of a spatial vector by a 3×3 special orthogonal matrix, denoted $O \in SO(3)$,

$$O^T \mathbb{I} O = \mathbb{I}\,, \quad \text{so that} \quad O^{-1} = O^T \quad \text{and} \quad \det O = 1\,,$$

where $\mathbb{I}$ is the 3×3 identity matrix. Geodesic motion on the space of rotations in three dimensions may be represented as a curve $O(t) \in SO(3)$ depending on time t. Its angular velocity is defined as the 3×3 matrix $\widehat{\Omega}$,

$$\widehat{\Omega}(t) = O^{-1}(t)\frac{dO(t)}{dt} =: O^{-1}(t)\dot{O}(t)\,.$$

(a) Show that $\widehat{\Omega}(t)$ is skew symmetric. How would this change if $\mathbb{I} = \mathbb{I}^T$ were only a constant symmetric 3×3 matrix?

(b) Show that the variational derivative $\delta\widehat{\Omega}$ of the angular velocity $\widehat{\Omega} = O^{-1}\dot{O}$ satisfies

$$\delta\widehat{\Omega} = \frac{d\widehat{\Xi}}{dt} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega}\,,$$

in which $\widehat{\Xi} = O^{-1}\delta O$.

(c) Compute the Euler-Lagrange equations for Hamilton's principle

$$\delta S = 0 \quad \text{with} \quad S = \int L(\widehat{\Omega})\,dt\,,$$

using the quadratic Lagrangian $L : TSO(3) \to \mathbb{R}$,

$$L(\widehat{\Omega}) = -\frac{1}{2}\mathrm{tr}(\widehat{\Omega}\mathbb{A}\widehat{\Omega})\,,$$

in which $\mathbb{A}$ is a symmetric, positive-definite 3×3 matrix.

5. The real-valued Maxwell-Bloch system on $\mathbb{R}^3$ is given by

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = x_1 x_3\,, \quad \dot{x}_3 = -x_1 x_2\,.$$

(a) Write this system in three-dimensional vector $\mathbb{R}^3$-bracket notation as

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where H_1 and H_2 are two conserved functions. Show that the level sets of one of these (let it be H_2) are **parabolic cylinders** oriented along the x_2-direction.

(b) Restrict the equations and their $\mathbb{R}^3$ Poisson bracket to a level set of H_2. Show that the Poisson bracket on the parabolic cylinder $H_2 = const$ is symplectic.

(c) Derive the equation of motion on a level set of H_2 and express them in the form of Newton's Law. Do they reduce to something familiar?

(d) Identify steady solutions and determine which are unstable (saddle points) and which are stable (centers).