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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

M3A16/M4A16/M5A16

GEOMETRIC MECHANICS I

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3A16/M4A16/M5A16

GEOMETRIC MECHANICS I

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) For the following Lagrangian $L(\dot{\mathbf{q}}) : T\mathbb{R}^3 \to \mathbb{R}$

$$L(\dot{\mathbf{q}}) = -\Big(1 - \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}\Big)^{1/2}$$

express the velocity $\dot{\mathbf{q}} \in T\mathbb{R}^3$ in terms of the position $\mathbf{q} \in \mathbb{R}^3$ and the fibre derivative of the Lagrangian.

(b) Write the Euler-Lagrange equation for this Lagrangian.

(c) Find the constants of the motion for the Euler-Lagrange equation and give their physical interpretations.

(d) Legendre transform this Lagrangian to determine its corresponding Hamiltonian and canonical equations.

(e) Explain the physical meaning of this motion.
Hint: suppose the Lagrangian were written as

$$L(\dot{\mathbf{q}}) = -\, m_0\Big(1 - \dot{\mathbf{q}} \cdot \dot{\mathbf{q}}/c^2\Big)^{1/2}$$

for particle rest mass m_0 and speed of light c.

2. Consider the dynamical system in $(x_1, x_2, x_3) \in \mathbb{R}^3$ for a smooth function f on the real line,

$$\begin{aligned}\dot{x}_1 &= f'(x_2) - f'(x_3)\\ \dot{x}_2 &= f'(x_3) - f'(x_1)\\ \dot{x}_3 &= f'(x_1) - f'(x_2)\end{aligned}$$

where $f'(x) = df/dx$. The vector field $(\dot{x}_1, \dot{x}_2, \dot{x}_3) \in T\mathbb{R}^3$ has zero divergence and its flow preserves the sum $H_2 = x_1 + x_2 + x_3$.

(a) Find the steady solutions of the system and determine their stability.

(b) Write this system in Nambu form, $\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2$.

(c) Restrict the motion to a level set of H_2, eliminate x_3 and write the equations of motion for x_1 and x_2 on that level set.

(d) Write the explicit solution of the system when $f(x) = x^2/2$.

(e) Give the geometrical interpretation of the result in the previous part.

3. Consider the (1,2,3) cyclically symmetric dynamical system,

$$\frac{da_1^*}{dt} = a_2 a_3 \,, \quad \frac{da_2^*}{dt} = a_3 a_1 \,, \quad \frac{da_3^*}{dt} = a_1 a_2 \,, \tag{1}$$

where $a_1, a_2, a_3 \in \mathbb{C}^3$ and a_k^* denotes the complex conjugate of a_k.

(a) Show that this system is Hamiltonian with canonical Poisson bracket $\{a_j \,,\, a_k^*\} = -2i\delta_{jk}$.

(b) Find two other constants of motion I_1 and I_2 that generate S^1 symmetries of the Hamiltonian. Show that they Poisson commute, so that $\{I_1, I_2\} = 0$.

(c) Show that the following transformation of variables is canonical,

$$a_1 = z e^{-i(\phi_2+\phi_3)} \,, \qquad a_2 = |a_2| e^{i\phi_2} \,, \qquad a_3 = |a_3| e^{i\phi_3} \,, \quad \text{with} \quad z = |z| e^{i\zeta} \in \mathbb{C}.$$

(d) Write the Hamiltonian for the cyclically symmetric system (1) solely in terms of z, z^*, I_2, I_3.

4. The Hamiltonian $H : (a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2 \to \mathbb{R}$

$$H = \tfrac{1}{2}|a_1|^2 - |a_2|^2 + \tfrac{1}{2}\mathrm{Im}(a_1^{*2}a_2)\,,$$

is invariant under the $1{:}2$ resonance S^1 transformation,

$$a_1 \to e^{i\phi} \quad \text{and} \quad a_2 \to e^{2i\phi}.$$

The variables $(a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2$ are canonical, with Poisson bracket relation

$$\{a_j, a_k^*\} = -2i\delta_{jk}, \quad \text{for} \quad j,k = 1,2\,.$$

(a) Write the motion equations generated by the Hamiltonian H, in terms of the canonical variables $(a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2$.

(b) Show that the following transformation is canonical:

$$a_1 = |a_1|e^{i\phi}, \qquad a_2 = ze^{2i\phi}, \qquad z = |z|e^{i\zeta}\,.$$

(c) Write the transformed equations in the new canonical variables and solve for $Q = |z|^2$ implicitly up to a quadrature integral for an elliptic function.

(d) Consider the orbit map

$$\pi : \ (a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2 \to (X, Y, Z, R) \in \mathbb{R}^4$$

from $\mathbb{C}^2$ to new variables that are invariant under the $1{:}2$ resonance S^1 transformation

$$\begin{aligned} R &= \tfrac{1}{2}|a_1|^2 + |a_2|^2 \\ Z &= \tfrac{1}{2}|a_1|^2 - |a_2|^2 \\ X - iY &= 2a_1^{*2}a_2 \end{aligned}$$

Find an algebraic relation among the variables $X, Y, Z, R \in \mathbb{R}^4$ and characterise the corresponding set of surfaces in $X, Y, Z \in \mathbb{R}^3$ parameterised by the value of R.

(e) Transform the Hamiltonian $H : \mathbb{C}^2 \to \mathbb{R}$ to $H \circ \pi$ in the new variables $X, Y, Z, R \in \mathbb{R}^4$.