Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2011

M4A34

Geometric Mechanics II

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This paper is also taken for the relevant examination for the Associateship.

M4A34

Geometric Mechanics II

Date:		Time:	

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. **Special Euclidean group,** $SE(3)$

Problem statement:

(a) List the subgroups of $SE(3)$.

(b) Write a 4×4 matrix representation of $SE(3)$.

(c) Give the definition of a semidirect-product Lie group $G = H \circledS N$.

(d) Show that $SE(3)$ is a semidirect-product Lie group.

(e) Compute the tangent space at the identity of $T_I SE(3)$ in its 4×4 matrix representation.

(f) Show that elements of $T_I SE(3)$ form a Lie algebra, $se(3)$.

(g) Compute the **adjoint actions**, $\mathrm{Ad} : SE(3) \times SE(3) \to SE(3)$, $\mathrm{Ad} : SE(3) \times se(3) \to se(3)$ and $\mathrm{ad} : se(3) \times se(3) \to se(3)$.

(h) Use the hat map to write the ad-action as vector multiplication.

2. **Monopole Kepler problem**

Consider the Kepler problem with a magnetic monopole, whose dynamical equation is,

$$\ddot{\mathbf{r}} + \frac{\lambda}{r^3}\mathbf{L} + \Big(\frac{\mu}{r^3} - \frac{\lambda^2}{r^4}\Big)\mathbf{r} = 0\,. \tag{1}$$

for real positive constants λ and μ. When $\lambda = 0$ this equation governs the Kepler problem for planetary motion.

(a) Does equation (1) conserve an energy? Prove it.

(b) Take vector cross products of equation (1) with $\mathbf{r}$ and $\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$ to find its two additional *conserved vectors*.

(c) Is angular momentum also conserved? Prove it.

(d) Are the orbits still conic sections? Prove it.

(e) Write the Lagrangian for which equation arises from Hamilton's principle. Hint: Assume there exists a generalised vector function $\mathbf{A}(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}^3$ whose curl satisfies $\mathrm{curl}\mathbf{A} = \mathbf{r}/r^3$.

(f) Write the Hamiltonian for equation (1). How does it relate to the energy?

(g) Write the Poisson brackets for equation (1) between position and velocity.

3. **Rigid body in a quadratic potential field**

The Lagrangian of an arbitrary rigid body rotating about a fixed point at the origin of spatial coordinates $x \in \mathbb{R}^n$ in a field with a quadratic potential

$$\phi(x) = \frac{1}{2}\mathrm{tr}\big(x^T \mathbb{S}_0 x\big)$$

is defined in the body coordinates by the difference between its kinetic and potential energies in the form,

$$l = \underbrace{\frac{1}{2}\mathrm{tr}(\Omega^T \mathbb{A}\Omega)}_{\text{Kinetic}} - \underbrace{\frac{1}{2}\mathrm{tr}(\mathbb{S}\mathbb{A})}_{\text{Potential}} . \tag{2}$$

Here $\Omega(t) = O^{-1}(t)\dot{O}(t) \in so(n)$, the $n \times n$ constant matrices $\mathbb{A}$ and $\mathbb{S}_0$ are symmetric, and $\mathbb{S}(t) = O^{-1}(t)\mathbb{S}_0 O(t)$.

Problem statement:

(a) Compute the reduced Euler-Lagrange equations for this Lagrangian by taking matrix variations in its Hamilton's principle $\delta S = 0$ with $S = \int l\, dt$.

(b) Following Manakov's idea discussed in class, combine these equations into a commutator of polynomials in a constant "spectral parameter", λ.

(c) Use Manakov's form of the equations as a Lax pair to determine the resulting set of constants of motion.

(d) Take the Legendre transformation and formulate these equations as a Lie-Poisson Hamiltonian system. Determine the Lie algebra involved.

(e) Determine the dimension of the generic solution of the reduced Euler-Lagrange equations for this Lagrangian? That is, what is the sum of the dimensions of $so(n)$ and the symmetric $n \times n$ matrices, minus the number of conservation laws?

4. **Momentum maps for cotangent lifts**

Review

The formula determining the momentum map for the cotangent-lifted action of a Lie group G on a smooth manifold Q may be expressed in terms of the pairings $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle : T^*Q \times TQ \to \mathbb{R}$ as

$$\langle\, J\,,\,\xi\,\rangle = \langle\langle\, p\,,\,\pounds_\xi q\,\rangle\rangle\,,$$

where $(q,p) \in T^*_qQ$ and $\pounds_\xi q$ is the infinitesimal generator of the action of the Lie algebra element ξ on the coordinate q.

Problem statement:

Define appropriate pairings and determine the momentum maps explicitly for the following four actions, *then compute their symmetric double-bracket canonical equations*.

[a] $\pounds_\xi q = \xi \times q$ for $\mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$

[b] $\pounds_\xi q = \mathrm{ad}_\xi q$ for ad-action $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$

[c] $UQU^\dagger$ for a unitary matrix $U \in U(n)$ satisfying $U^\dagger = U^{-1}$ acting on Hermitian $Q \in H(n)$ satisfying $Q = Q^\dagger$.