Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2013

M4A34

Geometric Mechanics II

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2013

This paper is also taken for the relevant examination for the Associateship.

M4A34

Geometric Mechanics II

Date: Time:

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Quaternions

Problem statement:

(a) De Moivre's theorem for unimodular complex numbers is

$$(\cos\theta + i\sin\theta)^m = (\cos m\theta + i\sin m\theta)\,.$$

Derive the analog of this theorem for unit quaternions.

(b) If $\mathfrak{r} = [0, \mathbf{r}]$ is a pure quaternion and $\hat{\mathfrak{q}} = [q_0\,,\,\mathbf{q}]$ is a unit quaternion, prove that under quaternionic conjugation,

$$\begin{aligned} \mathfrak{r}\,' &= \hat{\mathfrak{q}}\,\mathfrak{r}\,\hat{\mathfrak{q}}^* = \big[0, \mathbf{r}\,'\big] \\ &= \big[0, \mathbf{r} + c_1 q_0(\mathbf{q}\times\mathbf{r}) + c_2\mathbf{q}\times(\mathbf{q}\times\mathbf{r})\big]\,. \end{aligned} \tag{1}$$

That is, determine the constants c_1 and c_2.

(c) Prove the Euler-Rodrigues formula for quaternions, from quaternionic conjugation with $\hat{\mathfrak{q}} := \pm[\cos\frac{\theta}{2}\,,\,\sin\frac{\theta}{2}\,\hat{\mathbf{n}}]$.

(d) Write the isomorphism between the quaternions and the Pauli matrices.

(e) Use this isomorphism to write out the quaternionic version of the Hopf fibration.

2. Adjoint and coadjoint actions of semidirect product $(S \circledS v)$ acting on $\mathbb{R}$

The action of the scaling and translation group $(S \circledS v)$ on $\mathbb{R}$ may be represented by multiplying an *extended* vector $(r, 1)^T$ with $r \in \mathbb{R}$ by a 2×2 matrix, as

$$\begin{bmatrix} S & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} S\,r + v \\ 1 \end{bmatrix}$$

for a scaling parameter $S \in \mathbb{R}$ and a translation $v \in \mathbb{R}$.

The group composition rule for $(S \circledS v)$ is

$$(\tilde{S}, \tilde{v})(S, v) = (\tilde{S}S, \tilde{S}v + \tilde{v})\,, \tag{2}$$

which can be represented by multiplication of 2×2 matrices, as

$$\begin{pmatrix} \tilde{S} & \tilde{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{S}S & \tilde{S}v + \tilde{v} \\ 0 & 1 \end{pmatrix}. \tag{3}$$

Problem statement

(a) Derive the AD, Ad and ad actions for $(S \circledS v)$. Use the notation $(S'(0), v'(0)) = (\sigma, \nu)$ for Lie algebra elements.

(b) Introduce a natural pairing in which to define the dual Lie algebra and derive its Ad^* and ad^* actions. Denote elements of the dual Lie algebra as (α, β).

(c) Compute its coadjoint motion equations as Euler-Poincaré equations.

(d) Take the Legendre transform and, hence, find the corresponding canonical Poisson brackets.

(e) Choose the Hamiltonian $H = \frac{1}{2}\alpha^2 + \frac{1}{2}(\log\beta)^2$ and solve its coadjoint motion equations.

3. **Momentum maps for cotangent lifts**

Recall that the formula determining the momentum map for the cotangent-lifted action of a Lie group G on a smooth manifold Q may be expressed in terms of the pairings $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle : T^*Q \times TQ \to \mathbb{R}$ as

$$\langle\, J(q,p)\,,\,\xi\,\rangle = \langle\langle\, p\,,\,\pounds_\xi q\,\rangle\rangle\,,$$

where $(q,p) \in T^*_qQ$ and $\pounds_\xi q$ is the infinitesimal generator of the action of the Lie algebra element ξ on the coordinate q.

Problem statement:

(a) Consider the infinitesimal transformation defined by the Lie derivative action of a Lie algebra $\mathfrak{g}$ on a manifold, Q; namely, as $\xi_Q(q) = \pounds_\xi q$, for $\xi \in \mathfrak{g}$ and $q \in Q$.
In terms of appropriate pairings $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle : T^*Q \times TQ \to \mathbb{R}$ and $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, determine the corresponding cotangent-lift momentum map $J(q,p)$, $J : T^*Q \to \mathfrak{g}^*$.

(b) Compute the infinitesimal canonical transformations of q and p generated by $J^\xi(p,q) = \Big\langle J(p,q),\,\xi \Big\rangle$ for a fixed $\xi \in \mathfrak{g}$

(c) Use the Clebsch formulation with constraint $\dot q - \pounds_\xi q = 0$ imposed on Lagrangian $\ell(\xi,q)$ by Lagrange multiplier p with a pairing $\langle\!\langle p, \dot q - \pounds_\xi q\rangle\!\rangle$ to compute the *canonical* Hamiltonian equations in (q,p) phase space.

(d) Use these canonical equations to derive the dynamical equation for the momentum map J in this case.

4. The EPDiff(H^2) equation is obtained from the Euler-Poincaré reduction theorem for a right-invariant Lagrangian, when one defines this Lagrangian to be half the H^2 norm on the real line of the vector field of velocity $u = \dot{g}g^{-1}$, namely,

$$l(u) = \frac{1}{2}\|u\|^2_{H^2} = \frac{1}{2}\int_{-\infty}^{\infty} (u - u_{xx})^2\, dx\,.$$

(Assume u, u_x and u_{xx} all vanish as $|x| \to \infty$.)

(a) Derive the EPDiff(H^2) equation on the real line in terms of its velocity $u(x,t)$ and its momentum density $m(x,t) = \delta l/\delta u = u - 2u_{xx} + u_{4x}$ in one spatial dimension.

(b) Use the Clebsch approach with N constraints $\dot{q}_a(t) = u(q_a(t),t)$, $a = 1,\ldots,N$, to derive the peakon singular solution $m(x,t)$ of EPDiff(H^2) as a cotangent-lift momentum map in the variables $q_a(t)$ and $p_a(t)$.

(c) Solve for the velocity for the peakon solution in terms of the Green's function of the operator $(1-\partial_x^2)^2$.

(d) Legendre transform the Lagrangian to determine the Hamiltonian and express it in terms of the variables $q_a(t)$ and $p_a(t)$.

(e) Explain why the variables $q_a(t)$ and $p_a(t)$ are canonically conjugate and derive Hamilton's equations for them.