Ideal shallow water dynamics in a rotating frame

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Abstract

These lecture notes discuss various representations and approximations of ideal shallow water dynamics in a rotating frame. These rotating shallow water (RSW) equations possess a slow + fast decomposition in which they reduce approximately to quasigeostrophic (QG) motion (conservation of energy and potential vorticity) plus nearly decoupled equations for gravity waves in an asymptotic expansion in small Rossby number, $\epsilon \ll 1$. The solution properties of the RSW equations are discussed, and some alternative representations of the RSW equations which highlight the slow + fast interactions are given. At the conclusion, the RSW equations are re-derived as Euler–Poincaré equations from Hamilton's principle in the Eulerian fluid representation.

Contents

1 Rotating shallow water (RSW) equations

1.1 RSW motion equations

We consider dynamics of rotating shallow water (RSW) on a two dimensional domain with horizontal planar coordinates $\mathbf{x} = (x, y)$. This RSW motion is governed by the following nondimensional equations for horizontal fluid velocity vector $\mathbf{u} = (u, v)$ and the total depth η,

$$\epsilon \frac{D}{Dt}\mathbf{u} + f\hat{\boldsymbol{z}} \times \mathbf{u} + \boldsymbol{\nabla} h = 0\,, \qquad \frac{\partial \eta}{\partial t} + \boldsymbol{\nabla} \cdot (\eta \mathbf{u}) = 0\,, \tag{1.1}$$

with notation, cf. (??)

$$\frac{D}{Dt} := \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \right) \quad \text{and} \quad h := \left(\frac{\eta - b}{\epsilon \mathcal{F}} \right). \tag{1.2}$$

These equations include variable Coriolis parameter $f = f(\mathbf{x})$ and bottom topography $b = b(\mathbf{x})$. We will have more to say about the structure of these equations later, but for now we just think of them as nonlinear evolutionary PDEs in time and two-dimensional space, with homogeneous or periodic boundary conditions.

Dimensionless scale factors for RSW The dimensionless scale factors appearing in the RSW equations (1.1) and (1.2) are the Rossby number ϵ and the squared external Rossby ratio $\mathcal{F}$, given in terms of dimensional scales by

$$\epsilon = \frac{\mathcal{U}_0}{f_0 L} \ll 1 \quad \text{and} \quad \mathcal{F} = \frac{L^2}{L_R^2} = O(1) \quad \text{with} \quad L_R^2 = \frac{g b_0}{f_0^2}\,, \tag{1.3}$$

where L_R is the Rossby radius, which, as we will see in section 2.3, occurs at the peak of the dispersion curve for Rossby waves arising from perturbations of quasigeostrophic fluid at rest.

The dimensional scales $(b_0, L, \mathcal{U}_0, f_0, g)$ in RSW dynamics denote equilibrium fluid depth (b_0) horizontal length (L), horizontal fluid velocity $(\mathcal{U}_0)$, reference Coriolis parameter (f_0) and gravitational acceleration (g). Dimensionless quantities in equations (1.1) are unadorned and are related to their dimensional counterparts (primed), according to

$$\begin{aligned} &\mathbf{u}' = \mathcal{U}_0 \mathbf{u}, \quad \mathbf{x}' = L\mathbf{x}, \quad t' = \left(\frac{L}{\mathcal{U}_0} \right) t, \quad f' = f_0 f, \\ &b' = b_0 b, \quad \eta' = b_0 \eta, \quad \text{and} \quad \eta' - b' = b_0(\eta - b). \end{aligned} \tag{1.4}$$

Here, dimensional quantities are: $\mathbf{u}'$, the horizontal fluid velocity; η', the fluid depth; b', the equilibrium depth; and $\eta' - b'$, the free surface elevation.

For barotropic horizontal motions at length scales L in the ocean for which $\mathcal{F}$ is order $O(1)$ – as we shall assume – the Rossby number ϵ is typically quite small ($\epsilon \ll 1$) as indicated in equation (1.3) and, thus, the Rossby number is a natural parameter for making asymptotic expansions. For example, we shall assume $|\boldsymbol{\nabla} f| = O(\epsilon)$ and $|\boldsymbol{\nabla} b| = O(\epsilon)$, so we may write $f = 1 + \epsilon f_1(\mathbf{x})$ and $b = 1 + \epsilon b_1(\mathbf{x})$.

1.2 Geostrophic balance

At leading order in $\epsilon \ll 1$ the pressure gradient force in (1.1) may balance the Coriolis force, as

$$\hat{\boldsymbol{z}} \times \mathbf{u} = -\boldsymbol{\nabla} h\,. \tag{1.5}$$

Taking the cross product of the vertical unit vector $\hat{\boldsymbol{z}}$ with this ***geostrophic balance*** yields the ***geostrophic velocity***,

$$\mathbf{u}_G = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} h \,. \tag{1.6}$$

Thus, it makes sense to assume that the velocity has an ***ϵ-weighted Helmholtz decomposition***,

$$\mathbf{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi + \epsilon \boldsymbol{\nabla}\chi \quad \text{with} \quad \psi = h + O(\epsilon)\,. \tag{1.7}$$

In the oceans and atmosphere the geostrophic balance tends to be stable and small disturbances of it lead to waves, called Rossby waves. In seeking to establish the properties of these Rossby waves (such as their propagation velocity) we will analyse the linearised RSW equations. For this purpose, we first rewrite the nonlinear equations in their ***RSW curl form***

$$\partial_t(\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})) - \mathbf{u} \times \operatorname{curl}(\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})) + \boldsymbol{\nabla}\left(h + \frac{\epsilon}{2}|\mathbf{u}|^2\right) = 0\,, \tag{1.8}$$

where $\operatorname{curl}\mathbf{R}(\mathbf{x}) = f(\mathbf{x})\hat{\boldsymbol{z}} = (1+\epsilon f_1(\mathbf{x}))\hat{\boldsymbol{z}}$, so that $\mathbf{R}(\mathbf{x})$ is the vector potential for the divergence free rotation rate about the vertical direction.

1.3 Crucial vector calculus identities

To check that the RSW motion equations (1.1) and (1.8) are equivalent, one may use the fundamental vector identity of fluid dynamics,

$$(\operatorname{curl}\mathbf{a}) \times \mathbf{b} + \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + a_j \boldsymbol{\nabla} b^j\,. \tag{1.9}$$

Taking the curl of (1.8) yields the equation for vorticity dynamics,

$$\partial_t\varpi - \hat{\boldsymbol{z}} \cdot \operatorname{curl}\big(\mathbf{u} \times \varpi\hat{\boldsymbol{z}}\big) = 0\,, \quad \text{with} \quad \varpi := \epsilon\hat{\boldsymbol{z}}\cdot\operatorname{curl}\mathbf{u} + f(\mathbf{x}) \tag{1.10}$$

where we have defined the *ϵ-weighted total vorticity* as

$$\varpi := \epsilon\hat{\boldsymbol{z}}\cdot\operatorname{curl}\mathbf{u} + f(\mathbf{x}) = \epsilon\hat{\boldsymbol{z}}\cdot\operatorname{curl}(\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi + \epsilon\boldsymbol{\nabla}\chi) + f(\mathbf{x}) = \epsilon\Delta\psi + f(\mathbf{x}),$$

whereas the fluid vorticity is denoted as $\omega := \hat{\boldsymbol{z}}\cdot\operatorname{curl}\mathbf{u}$.

Expanding out the curl in equation (1.10) yields

$$\partial_t\varpi + \mathbf{u}\cdot\boldsymbol{\nabla}\varpi + \varpi\boldsymbol{\nabla}\cdot\mathbf{u} = 0\,. \tag{1.11}$$

by virtue of the vector identity,

$$\operatorname{curl}(\mathbf{a}\times\mathbf{b}) = -(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} - (\boldsymbol{\nabla}\cdot\mathbf{a})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + (\boldsymbol{\nabla}\cdot\mathbf{b})\mathbf{a}\,. \tag{1.12}$$

One may also write the RSW total vorticity equation (1.11) as a continuity equation,

$$\frac{\partial\varpi}{\partial t} + \boldsymbol{\nabla}\cdot(\varpi\mathbf{u}) = 0\,, \tag{1.13}$$

which implies conservation of integrated total vorticity $\int \varpi\, d^2x$, provided $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ on the boundary with unit normal vector $\hat{\mathbf{n}}$.

Order by order in an expansion in powers of $\epsilon \ll 1$, the total vorticity equation (1.11) and the ϵ-weighted Helmholtz decomposition (1.7) combine to yield,

$$\begin{aligned}
O(1):&\quad \boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot(\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi) = 0\,,\\
O(\epsilon):&\quad \partial_t\Delta\psi + (\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}(\Delta\psi + f_1(\mathbf{x})) + \Delta\chi = 0\,,\\
O(\epsilon^2):&\quad \boldsymbol{\nabla}\cdot\big((\Delta\psi + f_1(\mathbf{x}))\boldsymbol{\nabla}\chi\big) = 0\,.
\end{aligned}$$

Later, we will deal with the terms at each order, which will be important in deriving a sequence of approximations first at the level of the $O(\epsilon)$ quasigeostrophic approximation (QG) for Rossby waves and later at order $O(\epsilon^2)$ in discussing Poincaré gravity waves in two different decompositions of the fluid velocity.

1.4 Conservation laws for RSW

Exercise. Verify the following four properties of the RSW equations (1.1)

1. Energy conservation

$$E = \int \frac{\epsilon}{2}\eta|\mathbf{u}|^2 + \frac{(\eta-b)^2}{2\epsilon\mathcal{F}}\,d^{\,2}x \tag{1.14}$$

2. Kelvin circulation theorem

$$\frac{d}{dt}\oint_{c(u)}(\epsilon\mathbf{u}+\mathbf{R}(\mathbf{x}))\cdot d\mathbf{x} = 0\,,\quad \text{where}\quad \operatorname{curl}\mathbf{R}(\mathbf{x}) = f(\mathbf{x})\hat{\boldsymbol{z}}\,, \tag{1.15}$$

   and $c(u)$ is a closed planar loop moving with the fluid velocity $\mathbf{u}(\mathbf{x},t)$.

3. Conservation of potential vorticity (PV) on fluid parcels

$$\frac{Dq}{Dt} = \partial_t q + \mathbf{u}\cdot\nabla q = 0 \tag{1.16}$$

   where PV (q) is defined by

$$q := \frac{\varpi}{\eta}\,,\quad \text{and}\quad \varpi := \hat{\boldsymbol{z}}\cdot\operatorname{curl}(\epsilon\mathbf{u}+\mathbf{R}(\mathbf{x}))\,. \tag{1.17}$$

4. Infinite number of conserved integral quantities

$$\frac{d}{dt}\int \eta\,\Phi(q)\,d^{\,2}x = 0\,, \tag{1.18}$$

   for any differentiable function Φ.

Hints: The following alternative form of the RSW motion equation (1.1) may be helpful in verifying these four properties:

$$\partial_t(\epsilon\mathbf{u}+\mathbf{R}(\mathbf{x})) - \mathbf{u}\times\operatorname{curl}(\epsilon\mathbf{u}+\mathbf{R}(\mathbf{x})) + \boldsymbol{\nabla}\left(h+\frac{\epsilon}{2}|\mathbf{u}|^2\right) = 0\,, \tag{1.19}$$

where $\operatorname{curl}\mathbf{R}(\mathbf{x}) = f(\mathbf{x})\hat{\boldsymbol{z}}$ and $h := (\eta-b)/(\epsilon\mathcal{F})$.

You might also keep in mind the fundamental vector identity of fluid dynamics,

$$(\operatorname{curl}\mathbf{a})\times\mathbf{b} + \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + a_j\boldsymbol{\nabla}b^j\,. \tag{1.20}$$

★

Answer.

1.

$$
\begin{aligned}
\frac{dE}{dt} &= \int_{\mathcal{D}} \eta_t(\epsilon u^2/2 + h) + \eta \mathbf{u}\cdot \epsilon \mathbf{u}_t \, dx\,dy \\
&= -\int_{\mathcal{D}} \big(\operatorname{div}(\eta \mathbf{u})\big)(\epsilon u^2/2 + h) + \eta\mathbf{u}\cdot\nabla(\epsilon u^2/2 + h)\,dx\,dy \\
&= -\int_{\mathcal{D}} \operatorname{div}\big(\eta(\epsilon u^2/2 + h)\,\mathbf{u}\big)\;dx\,dy = -\oint_{\partial\mathcal{D}} \eta(\epsilon u^2/2 + h)\,\mathbf{u}\cdot\boldsymbol{\hat{n}}\,ds = 0\,,
\end{aligned}
$$

which vanishes for $\mathbf{u}$ tangent to the boundary $\partial\mathcal{D}$ of the domain of flow $\mathcal{D}$.

2.

$$
\begin{aligned}
&\frac{d}{dt}\oint_{c(u)} (\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x}))\cdot d\mathbf{x} \\
\text{By (1.9)}\quad &= \oint_{c(u)} \Big(\epsilon\partial_t\mathbf{u} + \epsilon\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + \epsilon u_j\boldsymbol{\nabla}u^j - \mathbf{u}\times\operatorname{curl}\mathbf{R} + \boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{R})\Big)\cdot d\mathbf{x} \\
\text{By (1.1)}\quad &= \oint_{c(u)} \boldsymbol{\nabla}\Big(\epsilon|\mathbf{u}|^2/2 + h + \mathbf{u}\cdot\mathbf{R}\Big)\cdot d\mathbf{x} = 0\,,
\end{aligned}
$$

which vanishes because the integral of a gradient over a closed loop is zero, by the fundamental theorem of calculus.

3. The curl of the alternative form of the RSW motion equation in (1.8) yields

$$
\begin{aligned}
0 &= \partial_t\varpi - \operatorname{curl}(\mathbf{u}\times\varpi) = \partial_t\varpi + \mathbf{u}\cdot\boldsymbol{\nabla}\varpi + \varpi\operatorname{div}\mathbf{u} \\
&= \frac{D\varpi}{Dt} - \varpi\eta^{-1}\frac{D\eta}{Dt} = \eta\frac{D}{Dt}\left(\frac{\varpi}{\eta}\right) = \eta\frac{Dq}{Dt}
\end{aligned}
$$

which verifies (1.16).

4. PV conservation on fluid parcels in (1.16) implies that the time derivative

$$
\begin{aligned}
\frac{d}{dt}\int \eta\,\Phi(q)\,dx\,dy &= \int \frac{\partial\eta}{\partial t}\,\Phi(q) + \eta\Phi'(q)\frac{\partial q}{\partial t}\,dx\,dy \\
&= -\int \operatorname{div}(\eta\mathbf{u})\,\Phi(q) + \eta\mathbf{u}\cdot\boldsymbol{\nabla}\Phi(q)\,dx\,dy \\
&= -\int \operatorname{div}\big(\eta\Phi(q)\mathbf{u}\big)\,dx\,dy \\
&= -\oint \eta\Phi(q)\,\boldsymbol{\hat{n}}\cdot\mathbf{u}\,ds \\
&= 0
\end{aligned}
$$

vanishes, because $\hat{\mathbf{n}}\cdot\mathbf{u}$ vanishes on the boundary.

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2 The Quasigeostrophic (QG) approximation

2.1 Derivation of QG

In this section, we derive the well known quasigeostrophic (QG) approximation [14] of the equations for RSW motion in a rotating frame into a form. Consistent with the QG approximation, we assume $f(\mathbf{x}) = 1 + \epsilon f_1(\mathbf{x})$ and $b(\mathbf{x}) = 1 + \epsilon b_1(\mathbf{x})$, with $\mathbf{x} = (x, y)$. We return to the RSW motion equation in (1.1), rewritten as

$$\epsilon \frac{D\mathbf{u}}{Dt} = -f\hat{\boldsymbol{z}} \times \mathbf{u} - \boldsymbol{\nabla} h \,, \tag{2.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}, \quad h = \frac{\eta - b}{\epsilon\mathcal{F}}. \tag{2.2}$$

Operating with $\hat{\mathbf{z}}\times$ on equation (2.1) and expanding in powers of ϵ yields

$$\begin{aligned}\mathbf{u} &= \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} h - \epsilon\, f_1\, \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} h - \epsilon \left(\frac{\partial}{\partial t} + \mathbf{u}_G\cdot\boldsymbol{\nabla}\right)\boldsymbol{\nabla} h + O(\epsilon^2) \\ &= \mathbf{u}_G + \epsilon\mathbf{u}_A + O(\epsilon^2),\end{aligned} \tag{2.3}$$

where the geostrophic and ageostrophic components of the velocity are defined, respectively, by

$$\mathbf{u}_G = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} h \quad \text{and} \quad \mathbf{u}_A = \left(\frac{\partial}{\partial t} + \mathbf{u}_G\cdot\boldsymbol{\nabla}\right)\hat{\boldsymbol{z}} \times \mathbf{u}_G - f_1\, \mathbf{u}_G \,. \tag{2.4}$$

The remainder of this section is devoted to studying the class of RSW flows that satisfy condition (2.3). In equation (2.4), $\mathbf{u}_G$ is divergenceless and $\mathbf{u}_A$ has divergence given by

$$\boldsymbol{\nabla}\cdot\mathbf{u}_A = -\left(\frac{\partial}{\partial t} + \mathbf{u}_G\cdot\boldsymbol{\nabla}\right)\Delta h - \mathbf{u}_G\cdot\boldsymbol{\nabla} f_1 \,, \tag{2.5}$$

in which Δh is the horizontal Laplacian of h.

Exercise. Verify equation (2.5) for $\boldsymbol{\nabla}\cdot\mathbf{u}_A$. ★

Substituting expression (2.5) for $\boldsymbol{\nabla}\cdot\mathbf{u}_A$ into the continuity equation

$$\frac{\partial\eta}{\partial t} + \boldsymbol{\nabla}\cdot(\eta\mathbf{u}) = 0 \,, \quad \text{rewritten as} \quad \epsilon\mathcal{F}h_{,t} = -\boldsymbol{\nabla}\cdot(\eta\mathbf{u}) \,, \tag{2.6}$$

and using the relations

$$\eta = b + \epsilon\mathcal{F}h \,, \quad \mathbf{u} = \mathbf{u}_G + \epsilon\mathbf{u}_A \quad \text{and} \quad b(\mathbf{x}) = 1 + \epsilon b_1(\mathbf{x}) \,, \tag{2.7}$$

yields at order $O(\epsilon)$ the QG equation for the dimensionless free surface height [14],

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_G\cdot\boldsymbol{\nabla}\right)(\mathcal{F}h - \Delta h + b_1 - f_1) = 0 \,. \tag{2.8}$$

Thus, in the QG approximation, the potential vorticity, defined by

$$q = \mathcal{F}h - \Delta h + b_1 - f_1 \,, \tag{2.9}$$

is advected by the divergenceless geostrophic velocity $\mathbf{u}_G = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} h$. That is,

$$\partial_t q + \mathbf{u}_G \cdot \boldsymbol{\nabla} q = 0\,. \tag{2.10}$$

The positive-definite symmetric operator $\mathcal{F} - \Delta$ is nondegenerate, so its operator inverse $1/(\mathcal{F} - \Delta)$ exists and is well defined on Fourier transformable functions, say. Therefore, the surface height h and its derivatives are determined uniquely from the potential vorticity q in QG theory.

Equations (2.8) and (2.9) combine into

$$\frac{\partial q}{\partial t} = -\hat{\boldsymbol{z}} \times \boldsymbol{\nabla} h \cdot \boldsymbol{\nabla} q = J(q,h)\,, \tag{2.11}$$

where

$$J(q,h) := \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} q \times \boldsymbol{\nabla} h \tag{2.12}$$

is the Jacobian of the transformation $(x,y) \to (q,h)$

$$dq \wedge dh = J(q,h)\, dx \wedge dy = (q_{,x}h_{,y} - h_{,x}q_{,y})\, dx \wedge dy =: \{q,h\}_{can}\, dx \wedge dy\,, \tag{2.13}$$

where

$$\{q,h\}_{can} := q_{,x}h_{,y} - h_{,x}q_{,y} \tag{2.14}$$

denotes the canonical Poisson bracket.

Thus, the PV equation for QG takes the canonical Hamiltonian form, $\partial_t q = \{q,h\}_{can}$.

2.2 The ageostrophic velocity in the QG approximation

Exercise. Show that the QG motion equation (2.8) implies

$$\frac{\partial}{\partial t}\left(\frac{\mathcal{F}h^2}{2} + \frac{|\boldsymbol{\nabla} h|^2}{2}\right) = \boldsymbol{\nabla} \cdot \Big(h\boldsymbol{\nabla} h_{,t} - h\mathbf{u}_G(\mathcal{F}h - \Delta h + b_1 - f_1)\Big). \tag{2.15}$$

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As a consequence of (2.15), *QG motion conserves the positive-definite energy,*

$$\begin{aligned} E_{QG} &= \int \left(\frac{\mathcal{F}h^2}{2} + \frac{1}{2}|\mathbf{u}_G|^2\right) d^2x \\ &= \frac{1}{2}\int (\mathcal{F}h^2 + |\nabla h|^2) d^2x \\ &= \frac{1}{2}\int \mu(\mathcal{F} - \Delta)^{-1}\mu\, d^2x =: H(\mu)\,, \end{aligned} \tag{2.16}$$

with $\mu := q - (b_1 - f_1) = (\mathcal{F} - \Delta)h$, provided the vector $\boldsymbol{\nabla} h$ in (2.15) is normal to the domain boundary (so $\mathbf{u}_G$ is tangential there) and also provided the boundary integral of the normal derivative of $\partial_t h$ vanishes [14].

Exercise. Show that the QG motion equation (2.8) yields the formal expression,

$$\frac{\partial h}{\partial t} = -(\mathcal{F}-\Delta)^{-1}(\mathbf{u}_G\cdot\boldsymbol{\nabla} q) = -(\mathcal{F}-\Delta)^{-1}J(h,q)\,, \tag{2.17}$$

where $(\mathcal{F}-\Delta)^{-1}$ denotes integration against the Green's function kernel $K(x,y)$ of the Helmholtz operator $(\mathcal{F}-\Delta)$. That is,

$$(\mathcal{F}-\Delta)^{-1}J = \int K(x,y)J(y)\,dy \quad\text{and}\quad (\mathcal{F}-\Delta)K(x,y) = \delta(x-y)\,,$$

where $\delta(x-y)$ is the Dirac delta distribution, defined by $\int\delta(x-y)f(y)dy = f(x)$. ★

The gradient of equation (2.17) provides an estimate for the quantity

$$\partial_t\Big(\hat{\boldsymbol{z}}\times\mathbf{u}_G\Big) = -\boldsymbol{\nabla} h_{,t}\,,$$

appearing in expression (2.4) for $\mathbf{u}_A$,

$$\mathbf{u}_A = (\mathbf{u}_G\cdot\boldsymbol{\nabla})\hat{\boldsymbol{z}}\times\mathbf{u}_G - \boldsymbol{\nabla} h_{,t} - f_1\mathbf{u}_G\,, \tag{2.18}$$

which may therefore be written as,

$$\mathbf{u}_A = (\mathbf{u}_G\cdot\boldsymbol{\nabla})\hat{\boldsymbol{z}}\times\mathbf{u}_G + \boldsymbol{\nabla}(\mathcal{F}-\Delta)^{-1}J(h,q) - f_1\mathbf{u}_G, \tag{2.19}$$

where $q = \mathcal{F}h - \Delta h + b_1 - f_1$, according to (2.9). Thus, in the QG approximation, the ageostrophic velocity $\mathbf{u}_A$ may be expressed via (2.17) entirely in terms of the geostrophic velocity $\mathbf{u}_G$ and other spatial derivatives of surface height elevation, h.

2.3 Rossby waves for QG and their dispersion relation

Exercise. Show that steady solutions (q_e, h_e) of (2.8) satisfy $J(q_e, h_e) = 0$, so that potential vorticity q_e and elevation h_e are functionally related.

Linearise the QG potential vorticity equation (2.8) around a steady solution h_e with $\hat{\boldsymbol{z}}\times\boldsymbol{\nabla} h_e = \mathbf{U} = U\hat{\mathbf{e}}_x = const$ and find the dispersion relation for the resulting wave equation. ★

Answer.

1. For an equilibrium solution q_e satisfying $\partial q_e/\partial t = 0$, equation (2.8) implies $J(q_e, h_e) = 0$, which means that $q_e(x,y) = \mathcal{F}h_e - \Delta h_e + b_1 - f_1$ and $h_e(x,y)$ are functionally related, so their gradients are collinear.
   We shall assume that
   $$\boldsymbol{\nabla} q_e = -U\hat{\mathbf{y}} \quad\text{and}\quad \boldsymbol{\nabla} h_e = -\beta\hat{\mathbf{y}}$$
   where U and β are positive constants.

2. Linearize equation (2.8) using $J(q,h) = \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla} q\times\boldsymbol{\nabla} h$ as
$$\frac{\partial q'}{\partial t} = -\hat{\boldsymbol{z}}\times\boldsymbol{\nabla} h_e\cdot\boldsymbol{\nabla} q' + \hat{\boldsymbol{z}}\times\boldsymbol{\nabla} q_e\cdot\boldsymbol{\nabla} h'. \tag{2.20}$$
Then insert $\hat{\boldsymbol{z}}\times\boldsymbol{\nabla} h_e = U\hat{\mathbf{e}}_x$ and $\hat{\boldsymbol{z}}\times\boldsymbol{\nabla} q_e = \beta\hat{\mathbf{e}}_x$, and select a solution proportional to $\exp(i(\mathbf{k}\cdot\mathbf{x}-\nu t))$ with $\mathbf{k}=(k,l)$, to find, upon using $q' = (\mathcal{F}-\Delta)h'$, that
$$\nu = Uk - \frac{\beta k}{k^2+l^2+\mathcal{F}} \tag{2.21}$$
This is the dispersion relation for the linearised QG equation.
3. The corresponding phase and group velocities are
$$c_p = \frac{\nu}{k} = U - \frac{\beta}{k^2+l^2+\mathcal{F}} \quad\text{and}\quad c_g = \frac{d\nu}{dk} = U + \frac{\beta(k^2-l^2)}{(k^2+l^2+\mathcal{F})^2}\,. \tag{2.22}$$

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Exercise.

1. Suppose we linearise equation (2.8) about a state of *rest*. How does the dispersion relation change?
2. Plot the dispersion relation, discuss its zonal phase and group speeds.

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Answer. A state of rest for equation (2.8) give $h_e = const$. Linearising then gives
$$\frac{\partial q'}{\partial t} + \hat{\boldsymbol{z}}\times\boldsymbol{\nabla} h'\cdot\boldsymbol{\nabla}\beta y = 0\,,\quad\text{or}\quad \frac{\partial q'}{\partial t} + \beta\partial_x h' = \partial_t(\mathcal{F}h' - \Delta h') + \beta\partial_x h' = 0\,. \tag{2.23}$$

1. For a solution proportional to $\exp(i(\mathbf{k}\cdot\mathbf{x}-\nu t))$ with $\mathbf{k}=(k,l)$ this yields
$$\nu = -\frac{\beta k}{k^2+l^2+\mathcal{F}} \tag{2.24}$$
Thus, the dispersion relation for the linearised QG equation about a state of rest amounts to setting $U=0$ in the dispersion relation for a state moving with constant velocity. This means that moving into a frame of motion with constant velocity produces a Doppler shift of the wave frequency, corresponding to the Galilean transformation of adding the moving frame velocity to the phase or group velocity.
2. The corresponding zonal ($l^2=0$) phase and group velocities are
$$c_p = \frac{\nu}{k} = -\frac{\beta}{k^2+\mathcal{F}} \quad\text{and}\quad c_g = \frac{d\nu}{dk} = \frac{\beta(k^2-\mathcal{F})}{(k^2+\mathcal{F})^2}\,. \tag{2.25}$$
Thus, the dispersion relation has a peak for $k^2=\mathcal{F}$, beyond which the slope changes sign, so the group velocity changes direction.

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The generation of Rossby waves is important in producing the deflections observed in the jet stream in the stratosphere, for example. See, e.g., http://en.wikipedia.org/wiki/File:Aerial_Superhighway.ogv for simulations of Rossby waves on the Jet Stream.

There are many good discussions of the meanings of the dispersion relation $\nu(\mathbf{k})$, phase velocity c_p, and group velocity c_g in the literature [19, 22].

3 Fundamental conservation laws of the QG equations

3.1 Energy and circulation

The conservation laws for energy and circulation are of great value in the analysis and understanding of their solution behaviour of the QG equations.

First, we consider the QG energy,

$$E_{QG} = \frac{1}{2}\int h\,(\mathcal{F}h - \Delta h)\,d^{\,2}x = \frac{1}{2}\int \mathbf{u}_G \cdot \big(\mathbf{u}_G - \mathcal{F}\Delta^{-1}\mathbf{u}_G\big)\,d^{\,2}x\,, \tag{3.1}$$

which is conserved, provided the vector $\boldsymbol{\nabla} h$ is normal to the domain boundary, so that the QG velocity $\mathbf{u}_G = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} h$ is tangent to the boundary. This may be seen by direct computation, as

$$\begin{aligned}
\frac{dE_{QG}}{dt} &= \int_S h\frac{\partial}{\partial t}\Big(\mathcal{F}h - \Delta h + b_1(\mathbf{x}) - f_1(\mathbf{x})\Big)\,d^2x\\
&= \int_S h\frac{\partial q}{\partial t}\,d^2x = -\int_S h(-\mathbf{u}_G\cdot\boldsymbol{\nabla} q)\,d^2x = -\int_S h\boldsymbol{\nabla}\cdot(\mathbf{u}_G q)\,d^2x\\
&= -\int_S \boldsymbol{\nabla}\cdot(h\mathbf{u}_G q)\,d^2x + \int_S \boldsymbol{\nabla} h\cdot(\hat{\boldsymbol{z}}\times\boldsymbol{\nabla} h)\,d^2x\\
\text{(By Gauss)}\quad &= -\oint_{\partial S} \hat{\boldsymbol{n}}\cdot(h\mathbf{u}_G q)\,ds = 0\,,\quad \text{provided}\quad \hat{\boldsymbol{n}}\cdot\mathbf{u}_G\big|_{\partial S} = 0\,.
\end{aligned}$$

Second, the Kelvin circulation integral for QG is defined by

$$\begin{aligned}
K &= \oint_{c(u_G)} (\epsilon\mathbf{u}_G - \epsilon\mathcal{F}\Delta^{-1}\mathbf{u}_G + \mathbf{R}(\mathbf{x}))\cdot d\mathbf{x}\\
&= \iint_{\partial S = c(u_G)} \Big(\Delta h - \mathcal{F}h - b_1 + f_1\Big)\,d^{\,2}x\\
&= \iint_{\partial S = c(u_G)} q\,d^{\,2}x\,,
\end{aligned} \tag{3.2}$$

where

$$\operatorname{curl}\mathbf{R}(\mathbf{x}) = \Big(1 + \epsilon\big(f_1(\mathbf{x}) - b_1(\mathbf{x})\big)\Big)\hat{\boldsymbol{z}}\,, \tag{3.3}$$

and $c(u_G)$ is a closed planar loop moving with the fluid velocity $\mathbf{u}_G(\mathbf{x}, t)$, which also coincides with the boundary ∂S of the surface integral. The Kelvin circulation theorem for QG may also be computed directly, as

$$\begin{aligned}
\frac{dK}{dt} &= \frac{d}{dt}\iint_{\partial S = c(u_G)} q\,d^{\,2}x\\
&= \iint_{\partial S = c(u_G)} (\partial_t q + \mathbf{u}_G\cdot\boldsymbol{\nabla} q + \boldsymbol{\nabla}\cdot\mathbf{u}_G)\,d^{\,2}x\\
&= 0\,,
\end{aligned} \tag{3.4}$$

since $\partial_t q + \mathbf{u}_G\cdot\boldsymbol{\nabla} q = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{u}_G = 0$.

Exercise. Verify equation (3.4) explicitly. ★

3.2 Potential vorticity (PV)

The conservation of potential vorticity (PV) on QG fluid parcels was proved as equation (2.10) in section 2.1. Namely,

$$\partial_t q + \mathbf{u}_G \cdot \nabla q = 0\,, \tag{3.5}$$

where PV is defined by using $\operatorname{curl}\mathbf{u}_G = \Delta h$ as

$$q := -\hat{\boldsymbol{z}} \cdot \operatorname{curl}(\epsilon \mathbf{u}_G - \epsilon \mathcal{F}\Delta^{-1}\mathbf{u}_G + \mathbf{R}(\mathbf{x})) = \mathcal{F}h - \Delta h + b_1 - f_1\,. \tag{3.6}$$

An alternative derivation of the QG motion equation (3.5) may be obtained by taking the curl of the following motion equation for the QG velocity $\mathbf{u}_G = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} h$,

$$\begin{aligned} &\partial_t(\epsilon \mathbf{u}_G - \epsilon \mathcal{F}\Delta^{-1}\mathbf{u}_G + \mathbf{R}(\mathbf{x})) \\ &- \mathbf{u}_G \times \operatorname{curl}(\epsilon \mathbf{u}_G - \epsilon \mathcal{F}\Delta^{-1}\mathbf{u}_G + \mathbf{R}(\mathbf{x})) + \boldsymbol{\nabla}\pi = 0\,, \end{aligned} \tag{3.7}$$

where $\operatorname{curl}\mathbf{R}(\mathbf{x})$ is given in (3.3) and $\boldsymbol{\nabla}\pi$ is a pressure force.

This calculation is facilitated by the fundamental vector identity of fluid dynamics,

$$(\operatorname{curl}\mathbf{a}) \times \mathbf{b} + \boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + a_j \boldsymbol{\nabla} b^j\,. \tag{3.8}$$

Remarkably, QG possesses an infinite number of conserved integral quantities (called enstrophies), in the form

$$\frac{dC_\Phi}{dt} = 0 \quad \text{for} \quad C_\Phi = \int \Phi(q)\, d^{\,2}x\,, \tag{3.9}$$

for any differentiable function Φ. To prove this statement, we compute directly,

$$\begin{aligned} \frac{d}{dt}\iint_S \Phi(q)\, d^{\,2}x &= \iint_S \partial_t \Phi(q)\, d^{\,2}x = -\iint_S \mathbf{u}_G \cdot \boldsymbol{\nabla}\Phi(q)\, d^{\,2}x \\ &= -\iint_S \boldsymbol{\nabla}\cdot(\mathbf{u}_G \Phi(q))\, d^{\,2}x \\ \text{(By Gauss)} \qquad &= -\oint_{\partial S} \hat{\boldsymbol{n}}\cdot(\mathbf{u}_G\Phi(q))\, ds = 0\,, \quad \text{provided} \quad \hat{\boldsymbol{n}}\cdot\mathbf{u}_G\big|_{\partial S} = 0\,. \end{aligned}$$

3.3 Casting QG into Hamiltonian form.

In this section we show that the QG equation (2.10) may be written in Hamiltonian form,

$$\partial_t q = \{q, H\}\,, \tag{3.10}$$

for the Hamiltonian $H(\mu) = E_{QG}$ in (2.16) and a Poisson bracket $\{F, H\}$ among functionals of $\mu := q + f_1 - b_1 = (\mathcal{F} - \Delta)h$, given by

$$\{F, H\} = \int q \left\{ \frac{\delta F}{\delta \mu}\,,\, \frac{\delta H}{\delta \mu} \right\}_{can} dxdy\,. \tag{3.11}$$

Here, the variational derivative $\frac{\delta H}{\delta \mu}$ of a functional of μ is defined by the limit

$$\begin{aligned} \lim_{\epsilon\to 0}\left(\epsilon^{-1}\Big(H(\mu + \epsilon\delta\mu) - H(\mu)\Big)\right) &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} H(\mu + \epsilon\delta\mu) \\ &=: \left\langle \frac{\delta H}{\delta \mu}\,,\, \delta\mu \right\rangle_{L^2}\,, \end{aligned} \tag{3.12}$$

where the angle brackets $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2}$ represent L^2 pairing of real functions.

To verify the Hamiltonian form of the QG equation in (3.10), one may begin by recalling the cyclic permutation formula

$$\int a\{b,c\}_{can}dxdy = \int b\{c,a\}_{can}dxdy\,,$$

for real functions a, b, c, on the (x, y) plane with homogeneous or periodic boundary conditions.

Then, one may verify that the variational derivative of the QG energy $H(\mu) = E_{QG}$ in (2.16) with respect to the quantity μ yields the surface height elevation. That is, verify the relation $\delta H/\delta\mu = h$ by using the definition of the the variational derivative (3.12) with $H(\mu)$ in (2.16).

Notice that the Poisson bracket in (3.11) is bilinear, skew-symmetric, and satisfies the Leibniz and Jacobi identities. In particular, it satisfies the Jacobi identity, because it is a linear functional of the canonical Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{can}$ in (2.13) which satisfies the Jacobi identity.

Using formula (3.11), one may compute the Poisson brackets $\{F, C_\Phi\}$ for the conserved quantities C_Φ in (3.9) with an arbitrary smooth functional F. In the next section, we will see that critical points of the conserved functional

$$H_\Phi(q) := E_{QG}(q) + C_\Phi(q)$$

are equilibrium solutions of the QG equations. Perturbations of these critical points will produce Rossby waves and Poincaré gravity waves whose stability naturally depends on the choice of the function Φ in the conserved functional $C_\Phi(q)$. Later, we will discuss transformations of variables and asymptotic expansions that will allow us to see how these waves in the RSW solutions couple to the circulations in the QG approximation.

4 Equilibrium solutions of QG

4.1 Critical point solutions of QG

As discussed earlier, two important properties of the QG equation (2.8) are conservation of energy E_{QG} in (3.1) and enstrophy $C_\Phi(q)$ in (1.18)

$$E_{QG}(h) = \frac{1}{2}\int h\,(\mathcal{F}h - \Delta h)\,d^{\,2}x \quad \text{and} \quad C_\Phi(q) = \int \Phi(q)\,d^{\,2}x = 0\,, \tag{4.1}$$

provided the vector $\boldsymbol{\nabla} h$ is normal to the domain boundary, so that the QG velocity $\mathbf{u}_G = \boldsymbol{\hat{z}}\times\boldsymbol{\nabla} h$ is tangent to the boundary. The potential vorticity q is related to the elevation h by

$$q = \mathcal{F}h - \Delta h + b_1 - f_1\,, \tag{4.2}$$

and it satisfies equation (3.5), recalled here as

$$\partial_t q + \mathbf{u}_G\cdot\nabla q = 0\,. \tag{4.3}$$

Therefore, the QG energy E_{QG} may be written in terms of the potential vorticity q as a quadratic form,

$$E_{QG}(q) = \frac{1}{2}\int (q - b_1 + f_1)(\mathcal{F}-\Delta)^{-1}(q - b_1 + f_1)\,d^{\,2}x\,, \tag{4.4}$$

where we have used $h = (\mathcal{F}-\Delta)^{-1}(q - b_1 + f_1)$. We define the conserved functional $H_\Phi(q) := E_{QG}(q) + C_\Phi(q)$ and find the condition for its first variation to vanish for a function q_e to be

$$\delta H_\Phi(q_e) = \int (h_e + \Phi'(q_e))\delta q\,d^{\,2}x = 0\,. \tag{4.5}$$

The critical point condition for $\delta H_\Phi(q_e) = 0$ is thus

$$h_e + \Phi'(q_e) = 0\,, \tag{4.6}$$

for a given choice of the function Φ.

Theorem

4.1. *Critical points of the conserved functional $H_\Phi(q) := E_{QG}(q) + C_\Phi(q)$ are equilibrium solutions of the QG equations.*

Proof. By the critical point condition (4.6), we have $\nabla h_e \times \nabla q_e = 0$ for $\Phi''(q_e)\neq 0$, because the functional relation between h_e and q_e implies that their gradients ∇h_e and ∇q_e are collinear. Thus, the PV evolution equation (4.3) implies that q_e is an equilibrium solution. Namely,

$$\partial_t q_e = -\boldsymbol{\hat{z}}\times\nabla h_e\cdot\nabla q_e = -\boldsymbol{\hat{z}}\cdot\nabla h_e\times\nabla q_e = 0\,. \tag{4.7}$$

□

Exercise. Explain how the stability of a QG equilibrium solution depends on the choice of the function Φ in the conserved functional $C_\Phi(q)$.

Hint: the second variation of the conserved functional $H_\Phi(q)$ is the Hamiltonian for the linearised evolution of perturbations in the neighbourhood of an equilibrium solution q_e. ★

5 Alternative representations of RSW: Part I

5.1 Vorticity, divergence and depth representation

We transform variables in equations (1.1) from fluid velocity and depth $(\mathbf{u}, \eta)$, to vorticity, divergence and depth, $(\omega = \hat{\boldsymbol{z}} \cdot \operatorname{curl} \mathbf{u}, \mathcal{D} = \operatorname{div} \mathbf{u}, \eta)$. After introducing the operator

$$\left(\frac{\partial}{\partial t} + \partial_j u^j\right) = \left(\frac{D}{Dt} + \mathcal{D}\right), \quad \text{so that} \quad \frac{\partial \mathcal{D}}{\partial t} + \partial_j(u^j \mathcal{D}) = \left(\frac{D}{Dt} + \mathcal{D}\right) D \,, \tag{5.1}$$

the RSW equations (1.1) take the following forms in the variables $\omega = \hat{\boldsymbol{z}} \cdot \operatorname{curl} \mathbf{u}$, $\mathcal{D} = \operatorname{div} \mathbf{u}$ and η,

$$\begin{aligned} \left(\frac{D}{Dt} + \mathcal{D}\right)(\epsilon\omega + f) &= 0, \\ \left(\frac{D}{Dt} + \mathcal{D}\right)\eta &= 0, \\ \left(\frac{D}{Dt} + \mathcal{D}\right)\epsilon\mathcal{D} &= -\operatorname{div}\left[f\hat{\boldsymbol{z}} \times \mathbf{u} + \boldsymbol{\nabla} h\right] + 2\epsilon J(u, v), \\ &=: \Omega + 2\epsilon J(u, v), \end{aligned} \tag{5.2}$$

where

$$\Omega := -\operatorname{div}\left(f\hat{\boldsymbol{z}} \times \mathbf{u} + \boldsymbol{\nabla} h\right), \quad \text{and} \quad J(u, v) := \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} u \times \boldsymbol{\nabla} v \,. \tag{5.3}$$

Here, $J(u, v)$ is the Jacobian of the velocity components $u(x, y)$ and $v(x, y)$.

1. We shall assume that the velocity has a weighted Helmholtz decomposition

$$\mathbf{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi + \epsilon\boldsymbol{\nabla}\chi \,. \tag{5.4}$$

   Inserting (5.4) into (5.2) shows that with this assumption the quantity Ω may be expressed as

$$\Omega = -\operatorname{div}\left[f\hat{\boldsymbol{z}} \times \mathbf{u} + \boldsymbol{\nabla} h\right] = \operatorname{div}(f\boldsymbol{\nabla}\psi) + \epsilon J(f, \chi) - \Delta h \,, \tag{5.5}$$

   which is called the ***imbalance*** [13, 20, 17, 18]. (Why is this a good name for Ω?)

2. In addition, since $\mathcal{D} = \epsilon\Delta\chi$ is order $O(\epsilon)$ once we assume (5.4), the quantity $\Omega + 2\epsilon J(u, v)$ in (5.2) must then be of order $O(\epsilon^2)$.

Exercise. Explicitly transform variables from (1.1)–(1.2) to (5.2)–(5.5).

Hint: For this calculation, you may want to recall that

$$\partial_j(u^j u^i_{,i}) - \partial_i(u^j u^i_{,j}) = 2J(u^1, u^2) \quad \text{for} \quad \mathbf{u} = (u^1, u^2) = (u, v) \,, \tag{5.6}$$

when you are taking the divergence of the motion equation (1.1) in the form,

$$\partial_t \mathbf{u} = -\,\mathbf{u} \cdot \nabla \mathbf{u} - (f\hat{\boldsymbol{z}} \times \mathbf{u} + \nabla h)) \,.$$

For more insight, see the standard literature [13, 20, 17, 18]. ★

Exercise. Prove equation (5.6) explicitly. ★

The operator $D/Dt+\mathcal{D}$ has an integrating factor $\exp\int \mathcal{D}dt$, where the integral is taken at constant Lagrangian fluid-parcel label $l^A(\mathbf{x},t)$, $A=1,2$, which satisfies,

$$\frac{Dl^A}{Dt} = \partial_t l^A + \mathbf{u}\cdot\boldsymbol{\nabla} l^A = 0\,, \tag{5.7}$$

since the fluid-parcel label is a Lagrangian tracer. Using this integrating factor in equations (5.2) gives

$$\begin{aligned}
e^{-\int \mathcal{D}dt}\frac{D}{Dt}\Big(e^{\int \mathcal{D}dt}(\epsilon\omega+f)\Big) &= 0,\\
e^{-\int \mathcal{D}dt}\frac{D}{Dt}\Big(e^{\int \mathcal{D}dt}\eta\Big) &= 0,\\
e^{-\int \mathcal{D}dt}\frac{D}{Dt}\Big(e^{\int \mathcal{D}dt}\epsilon\mathcal{D}\Big) &= \Omega+2\epsilon J(u,v).
\end{aligned} \tag{5.8}$$

Consequently, we find

$$\begin{aligned}
e^{\int \mathcal{D}dt}(\epsilon\omega+f) &= c_1(l^A),\\
e^{\int \mathcal{D}dt}\eta &= c_2(l^A),\\
\eta\frac{D}{Dt}\left(\frac{\epsilon\mathcal{D}}{\eta}\right) &= \Omega+2\epsilon J(u,v),
\end{aligned} \tag{5.9}$$

where c_1 and c_2 are functions of the Lagrangian labels l^A so they satisfy $Dc_1/Dt=0=Dc_2/Dt$. The ratio of the first pair of equations in (5.9) yields potential vorticity conservation,

$$q := \frac{\epsilon\omega+f}{\eta} = \frac{c_1(l^A)}{c_2(l^A)} \quad\Rightarrow\quad \frac{Dq}{Dt}=0\,. \tag{5.10}$$

We also find from (5.2) and (5.9) that

$$\frac{Dq}{Dt}=0,\quad \frac{D}{Dt}\left(\frac{1}{\eta}\right)=\frac{\mathcal{D}}{\eta},\quad \frac{D}{Dt}\left(\frac{\epsilon\mathcal{D}}{\eta}\right)=\Big(\Omega+2\epsilon J(u,v)\Big)\left(\frac{1}{\eta}\right). \tag{5.11}$$

Thus, the RSW equations transform without approximation (but at the cost of introducing higher spatial derivatives) to

$$\frac{Dq}{Dt}=0 \quad\text{and}\quad \epsilon\frac{D^2}{Dt^2}\left(\frac{1}{\eta}\right)=\Big(\Omega+2\epsilon J(u,v)\Big)\left(\frac{1}{\eta}\right). \tag{5.12}$$

According to these equations, the potential vorticity q is constant along a fluid parcel trajectory and the inverse depth $1/\eta$ either oscillates stably or evolves exponentially, depending on the sign of the quantity $\Omega+2\epsilon J(u,v)$. Of course, advection of q is well known. The oscillator equation for $1/\eta$ in (5.12) can be interpreted in a mixed Eulerian-Lagrangian fashion as saying that sufficient convergence of Eulerian force causes Lagrangian (advective) instability of the water depth.

Thus, the RSW equations may be separated into vortical motions and Rossby waves in q and Lagrangian oscillations in $1/\eta$, with no approximation. Recall that $\mathcal{D}$ is order $O(\epsilon)$ and the quantity $\Omega + 2\epsilon J(u, v)$ is of order $O(\epsilon^2)$. So the second equation in (5.12) is consistent with order $O(1)$ dynamics of η in (1.2) and order $O(\epsilon)$ variations in bottom topography

$$\eta = 1 + \epsilon(\mathcal{F}h + b_1)\,. \tag{5.13}$$

Unfortunately, there is no corresponding separation of time scales in equation (5.12).

Reduction of the q-equation in (5.12) to quasigeostrophy (QG). For $\epsilon \ll 1$ in the potential vorticity equation in (5.12), further assumptions can be imposed which decouple the slow vortical motion from the fast gravity waves. In this limit, one may write the Taylor expansion

$$q = \frac{1 + \epsilon\,(\omega + f_1)}{1 + \epsilon\,(\mathcal{F}h + b_1)} = 1 + \epsilon\,(\omega - \mathcal{F}h + f_1 - b_1) + O(\epsilon^2). \tag{5.14}$$

Thus, with $\omega = \Delta\psi$ and $\mathbf{u} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\psi + \epsilon\boldsymbol{\nabla}\chi$, the equation in (5.12) for the potential vorticity becomes

$$\frac{\partial}{\partial t}\Big(\Delta\psi - \mathcal{F}h\Big) + J\Big(\Delta\psi - \mathcal{F}h + f_1 - b_1, \psi\Big) = O(\epsilon). \tag{5.15}$$

The further assumption that

$$\mathbf{u} = \mathbf{u}_G + \epsilon\mathbf{u}_A + O(\epsilon^2)\,,$$

with $\mathbf{u}_G = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}h$, so that $\omega = \Delta\psi = \Delta h + O(\epsilon)$ reduces equation (5.15) to the quasigeostrophic (QG) motion equation [1, 14],

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\Delta h - \mathcal{F}h\Big) + J\Big(\Delta h - \mathcal{F}h + f_1 - b_1, h\Big) = O(\epsilon), \\ \text{or}\quad &\frac{\partial q}{\partial t} + \mathbf{u}_G \cdot \boldsymbol{\nabla} q = O(\epsilon) \end{aligned} \tag{5.16}$$

when terms of order $O(\epsilon)$ are neglected.

Exercise. Follow the reasoning above to derive equation (5.16) explicitly and find its conserved energy. ★

Estimating the imbalance $\Omega + 2\epsilon J(u, v) = O(\epsilon^2)$ using QG theory. As we have seen, QG theory [1, 14] sets

$$\mathbf{u} = \mathbf{u}_G + \epsilon\mathbf{u}_A + O(\epsilon^2)\,, \tag{5.17}$$

with geostrophic and ageostrophic velocities given respectively by

$$\begin{aligned} \mathbf{u}_G &= \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}h \quad \text{and} \\ \mathbf{u}_A &= -f_1\mathbf{u}_G + (\partial_t + \mathbf{u}_G \cdot \nabla)\,\hat{\mathbf{z}} \times \mathbf{u}_G. \end{aligned} \tag{5.18}$$

Consequently, one finds $\mathcal{D} = \epsilon\, \mathrm{div} \mathbf{u}_A$ and

$$
\begin{aligned}
\Omega + 2\epsilon J(u,v) &= \epsilon\, \mathrm{div}\Big[(\partial_t + \mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\Big] + 2\epsilon J(u,v) \\
\text{Using (5.17)} \quad &= \epsilon\, \mathrm{div}\Big[(\partial_t + \mathbf{u}_G\cdot\boldsymbol{\nabla})\mathbf{u}_G\Big] + \epsilon^2 \mathrm{div}\Big[(\partial_t + \mathbf{u}_G\cdot\boldsymbol{\nabla})\mathbf{u}_A + (\mathbf{u}_A\cdot\boldsymbol{\nabla})\mathbf{u}_G\Big] \\
&\quad + 2\epsilon J(u_G, v_G) + 2\epsilon^2\Big(J(u_A, v_G) + J(u_G, v_A)\Big) + O(\epsilon^3) \\
&= \epsilon^2\,(\partial_t + \mathbf{u}_G\cdot\boldsymbol{\nabla})\,\mathrm{div}\,\mathbf{u}_A + O(\epsilon^3)\,,
\end{aligned}
\tag{5.19}
$$

after cancellations at both orders $O(\epsilon)$ and $O(\epsilon^2)$.

The QG theory also gives

$$
\begin{aligned}
\frac{1}{\epsilon}\frac{D\eta}{Dt} &= (\partial_t + \mathbf{u}_G\cdot\boldsymbol{\nabla}\,)(\mathcal{F}h + b_1) + O(\epsilon) = -\,\mathrm{div}\,\mathbf{u}_A \\
&= (\partial_t + \mathbf{u}_G\cdot\boldsymbol{\nabla}\,)(\Delta h + f_1(x,y)\,),
\end{aligned}
\tag{5.20}
$$

which recovers the previous asymptotic equation (5.16) when terms of order $O(\epsilon)$ are neglected.

Exercise. Verify the computations in (5.19) and (5.20). ★

6 Alternative representations of RSW: Part II

6.1 Slow + Fast decomposition

Transformed variables for Rotating Shallow Water: Still with no approximation, we can transform the RSW equations into the following set of variables,

$$\begin{aligned}
&\omega := \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\mathbf{u}, \quad \mathcal{D} := \boldsymbol{\nabla}\cdot\mathbf{u},\\
&\Omega := -\boldsymbol{\nabla}\cdot[f\hat{\boldsymbol{z}}\times\mathbf{u}+\boldsymbol{\nabla}h] = \operatorname{div}(f\boldsymbol{\nabla}\psi)+\epsilon J(f,\chi)-\Delta h.
\end{aligned}\tag{6.1}$$

In these variables, with $q=(\epsilon\omega+f)/\eta=\varpi/\eta$, the RSW equations from (5.11) may be written, without approximation, as follows.

$$\begin{aligned}
\partial_t\eta &= -\nabla\cdot(\eta\mathbf{u})\,, \quad \text{for total depth} \quad \eta = 1+\epsilon b_1+\epsilon\mathcal{F}h\\
\partial_t q &= -\,\mathbf{u}\cdot\nabla q\,, \quad \text{for PV} \quad q=\frac{\epsilon\omega+f}{\eta}\\
\partial_t\mathcal{D}-\frac{1}{\epsilon}\Omega &= -\nabla\cdot(\mathcal{D}\mathbf{u})+2J(u,v)\\
\partial_t\Omega-\frac{1}{\epsilon\mathcal{F}}(\Delta-\mathcal{F})\mathcal{D} &= \Delta\nabla\cdot\big((h+\frac{b_1}{\mathcal{F}})\mathbf{u}\big)-\frac{1}{\epsilon}\nabla\cdot\big((f^2+\epsilon\omega f-1)\mathbf{u}\big)-J(\epsilon f_1,h+|\mathbf{u}|^2)
\end{aligned}\tag{6.2}$$

Exercise. Verify the calculation required to obtain equations (6.2) from equations (5.11).

Hint: It may be helpful to notice that for constant rotation and flat bottom topography one has $f_1=0$ and $b_1=0$. In this case, the imbalance Ω simplifies as $\Omega\to\Delta\psi-\Delta h$, and the last equation (6.2) simplifies to

$$\frac{\partial}{\partial t}\big(\Delta\psi-\Delta h\big)-\frac{1}{\epsilon\mathcal{F}}(\Delta-\mathcal{F})\mathcal{D} = \Delta\big(\operatorname{div}(h\mathbf{u})\big)-\operatorname{div}\big((\Delta\psi)\mathbf{u}\big)\,.\tag{6.3}$$

Two other convenient formulas which are useful to prove for this exercise are

$$\begin{aligned}
-\frac{\partial}{\partial t}\Delta h &= \frac{1}{\epsilon\mathcal{F}}\Delta\mathcal{D}+\frac{1}{\mathcal{F}}\Delta\Big(\operatorname{div}\big((b_1+\mathcal{F}h)\mathbf{u}\big)\Big)\,,\\
-\frac{\partial}{\partial t}\Delta\psi &= \operatorname{div}\big((\omega+f_1)\mathbf{u}\big)+\frac{\mathcal{D}}{\epsilon}\,.
\end{aligned}\tag{6.4}$$

★

Klein-Gordon equations: In terms of the fast time variable t/ϵ and up to order $O(\epsilon)$, the quantities $\mathcal{D}$ and Ω in equations (6.2) satisfy identical linear Klein-Gordon equations

$$\begin{aligned}
[\partial^2_{t/\epsilon}-(\mathcal{F}^{-1}\Delta-1)]\mathcal{D} &= O(\epsilon)+O(\epsilon^2),\\
[\partial^2_{t/\epsilon}-(\mathcal{F}^{-1}\Delta-1)]\Omega &= O(\epsilon)+O(\epsilon^2),
\end{aligned}\tag{6.5}$$

corresponding to rapidly fluctuating Poincaré-gravity waves of the RSW system (6.2), with fast frequency $\nu\approx\partial_{t/\epsilon}$ of order $O(1/\epsilon)$, driven by order $O(\epsilon)$ and $O(\epsilon^2)$ nonlinear slow + fast forcing terms on the right hand sides. Upon ignoring the nonlinear slow + fast forcing terms, the RSW linear Poincaré-gravity waves satisfy the dispersion relation,

$$\nu^2 = 1+\mathcal{F}^{-1}k^2\,, \quad \text{with} \quad c_p=\frac{\nu}{k}=\pm\sqrt{\mathcal{F}^{-1}+\frac{1}{k^2}}\,,\tag{6.6}$$

which admits both leftward and rightward travelling waves.

Exercise. The longer the wavelength, the greater the group velocity of shallow water waves. Compute the group velocity for the dispersion relation in (6.6). ★

The limiting linear PDE for RSW waves for $\epsilon \ll 1$ governing both $\mathcal{D}$ and Ω in (6.5)

$$\partial^2_{t/\epsilon}\phi - \mathcal{F}^{-1}\Delta\phi = \phi\,, \quad \text{for either} \quad \phi = \mathcal{D} \quad \text{or} \quad \phi = \Omega \tag{6.7}$$

is the celebrated ***Klein-Gordon*** (KG) equation. The KG equation is a relativistic version of the Schrödinger equation. Although KG was discovered a long time ago, it has received renewed interest in physics lately, because describes a spin-zero elementary particle, the famous Higgs boson, whose existence was verified at CERN in 2012. For a good account of its history and a few background references, see `http://en.wikipedia.org/wiki/Klein-Gordon_equation`.

Remark

6.1. *The leading order fast-time equations for the $\mathcal{D}$ and Ω system in (6.2) are expressed as by*

$$\frac{\partial}{\partial(t/\epsilon)}\begin{bmatrix}\mathcal{D}\\ \Omega\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}(-\mathcal{F}^{-1}\Delta + 1)\mathcal{D}\\ \Omega\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\delta\mathcal{H}/\delta\mathcal{D}\\ \delta\mathcal{H}/\delta\Omega\end{bmatrix}.$$

Thus, $\mathcal{D}$ and Ω appear as canonically conjugate variables in a Hamiltonian system for the fast-time dynamics in t/ϵ with Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H} = \int \Big(\frac{1}{2\mathcal{F}}|\boldsymbol{\nabla}\mathcal{D}|^2 + \frac{1}{2}\mathcal{D}^2 + \frac{1}{2}\Omega^2\Big)\,dx\,dy\,, \tag{6.8}$$

and canonical Poisson bracket given for functionals $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{D}$ and Ω by

$$\Big\{\mathcal{G}\,,\,\mathcal{H}\Big\} = \int \begin{bmatrix}\delta\mathcal{G}/\delta\mathcal{D}\\ \delta\mathcal{G}/\delta\Omega\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\delta\mathcal{H}/\delta\mathcal{D}\\ \delta\mathcal{H}/\delta\Omega\end{bmatrix}dxdy\,. \tag{6.9}$$

At orders $O(\epsilon)$ and $O(\epsilon^2)$, the right hand sides for $\epsilon \neq 0$ of equations (6.2) are coupled to the QG equation (5.16). As it turns out, the coupled system of equations (5.16) and (6.2) is also Hamiltonian [1]. This is not surprising, because the original RSW system (1.1) is Hamiltonian, as well.

Remark

6.2. *This observation suggests a two-timing approach $(t, t/\epsilon)$ in which, upon averaging over the fast time, $\mathcal{H}$ in (6.8) would emerge as an adiabatic invariant. For an extensive review of multiple time scale expansions from the current viewpoint, see, for example, Klein [12].*

Remark

6.3. *The slow + fast decomposition of the RSW solution into potential vorticity $q(t)$ governed by the QG equation (5.16) for the potential vorticity, interacting with wave variables $\mathcal{D}(t, t/\epsilon)$ and $\Omega(t, t/\epsilon)$ governed by coupled KG equations in (6.5), is the basis for many possible approximations in rotating shallow water dynamics. For example, the initialization in a "balanced"* *state to supress the time derivatives on the left sides of the last two equations in this set gives the "slow equations" due to Peter Lynch. See, for example, Lynch [13], as well as Warn et al. [20] and Browning and Kreiss [2, 3]. Many other useful approximate reduced equations are discussed in the references. For more information about such approximate "reduced" equations, read Chapters 4 and 5 of Vallis [19]. For more information about linear and nonlinear waves, see Whitham [22].*

7 Hamilton's principle for simple ideal fluids

7.1 Preparation for fluid dynamical variational principles

Definition

7.1. *The **variational derivative** $\frac{\delta F}{\delta \psi} \in V^*(M)$ of a real functional $F[\psi]$ of smooth functions $\psi \in \mathcal{F}(M)$ taking values in a vector space $V(M)$ over a manifold M is defined by*

$$\delta F[\psi] = \lim_{\varepsilon \to 0} \Big(F[\psi + \varepsilon \delta \psi] - F[\psi] \Big) = \left\langle \frac{\delta F}{\delta \psi} \,,\, \delta \psi \right\rangle \quad \textit{for} \quad \delta \psi \in \mathcal{F}(M)\,. \tag{7.1}$$

The angle brackets $\langle \,\cdot\,,\,\cdot\, \rangle$ here denote L^2 pairing, as in

$$\langle f \,,\, g \rangle = \int < f(x)\,,\, h(x) >_{V^* \times V} \, dx\,, \tag{7.2}$$

for integrable real functions $f \in V$ and $h \in V^*$, and pairing $< \,\cdot\,,\,\cdot\, >_{V^* \times V} \colon V^* \times V \to \mathbb{R}$, for a vector space V and its dual vector space V^*.

This definition of variational derivative applies for fluid dynamics, for example, with velocity $\mathbf{u}$ and depth (or density) $(\mathbf{u}, D) \in \mathfrak{X}(\mathbb{R}^2) \times Dens(\mathbb{R}^2)$, since both $\mathfrak{X}(\mathbb{R}^2)$ and $Dens(\mathbb{R}^2)$ are vector spaces.

Our strategy in applying Hamilton's principle $\delta S = 0$ with $S = \int l(\mathbf{u}, D)\, dt$ to derive ideal GFD approximations will be to perform variations at fixed $\mathbf{x}$ and t of the following action integral [7, 8, 6],

$$S = \int_0^T l(\mathbf{u}, D)\, dt\,, \tag{7.3}$$

whose Lagrangian $l(\mathbf{u}, D)$ depends on the horizontal fluid velocity $\mathbf{u}$ and the total depth D. We will first do the general case for an arbitrary choice of Lagrangian $l : \mathfrak{X}(\mathbb{R}^2) \times Dens(\mathbb{R}^2) \to \mathbb{R}$, for fluid velocity defined as a vector field over the plane $\mathbb{R}^2$, so that $\mathbf{u} \in \mathfrak{X}(\mathbb{R}^2)$, and depth defined as a density $D \in Dens(\mathbb{R}^2)$, so that variations in depth conserve the volume of water. That is, D satisfies the continuity equation,

$$\partial_t D + \mathrm{div}(D\mathbf{u}) = 0\,. \tag{7.4}$$

7.2 Explicitly varying the action integral

Hamilton's principle $\delta S = 0$ for the action S in (7.3) is derived by taking the variations,

$$0 = \delta S = \int_0^T \left(\left\langle \frac{\delta l}{\delta \mathbf{u}} \,,\, \delta \mathbf{u} \right\rangle + \left\langle \frac{\delta l}{\delta D} \,,\, \delta D \right\rangle \right) dt\,. \tag{7.5}$$

For fluid dynamics, the variations of the fluid velocity vector field $\mathbf{u}$ and the mass density D are given by

$$\delta \mathbf{u} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathbf{u} = \partial_t \boldsymbol{v} + \mathbf{u}\cdot\nabla\boldsymbol{v} - \boldsymbol{v}\cdot\nabla\mathbf{u} \quad \text{and} \quad \delta D = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} D = -\mathrm{div}(D\boldsymbol{v})\,, \tag{7.6}$$

where the variational vector field $\boldsymbol{v} \in \mathfrak{X}(\mathbb{R}^2)$ is assumed to vanish at the endpoints in time $[0,T]$.

7.3 Deriving the Euler–Poincaré motion equation

Substituting the expressions in (7.6) for the variations $\delta\mathbf{u}$ and δD into Hamilton's principle in (7.5) yields

$$\begin{aligned} 0 = \delta S &= \int_0^T \left\langle \frac{\delta l}{\delta u^i}\,,\, \partial_t v^i + u^j\partial_j v^i - v^j\partial_j u^i \right\rangle + \left\langle \frac{\delta l}{\delta D}\,,\, -\partial_i(Dv^i) \right\rangle dt \\ &= \int_0^T \left\langle -\partial_t \frac{\delta l}{\delta u^i} - \partial_j\left(\frac{\delta l}{\delta u^i}u^j\right) - \frac{\delta l}{\delta u^k}\partial_i u^k + D\partial_i\frac{\delta l}{\delta D}\,,\, v^i \right\rangle dt \\ &\quad + \left\langle \frac{\delta l}{\delta u^i}\,,\, v^i \right\rangle \Big|_0^T\,, \end{aligned} \tag{7.7}$$

where we have invoked natural boundary conditions ($\hat{\mathbf{n}}\cdot\mathbf{u} = 0$ on the boundary) when integrating by parts in space, so the corresponding boundary terms vanish. The last term vanishes, as well, because we have assumed that the variational vector field $\boldsymbol{v}$ vanishes at the endpoints in time. Consequently, for independent variational vector fields $\boldsymbol{v}$ we find from (7.7) that Hamilton's principle for fluids implies the following *Euler-Poincaré equation* for ideal fluid dynamics

$$\partial_t \frac{\delta l}{\delta u^i} + \partial_j\left(\frac{\delta l}{\delta u^i}u^j\right) + \frac{\delta l}{\delta u^k}\partial_i u^k = D\partial_i\frac{\delta l}{\delta D}\,. \tag{7.8}$$

To complete the dynamical system, we also have the auxiliary equation (7.4), rewritten in components now in three dimensions as

$$\partial_t D + \partial_j(Du^j) = 0\,. \tag{7.9}$$

Exercise. Verify equation (7.7) by substituting the variations (7.6) into equation (7.5) and integrating by parts. ★

Remark

7.2. *The class of equations derived in this section are called Euler-Poincaré equations. For an extensive discussion of Euler-Poincaré equations for fluid dynamics, see Holm, Marsden and Ratiu [9, 10].*

8 Hamilton's principle for RSW

8.1 Properties of the Euler–Poincaré motion equation

Hamilton's principle in (7.5) has produced in equation (7.8) an example of the ***Euler–Poincaré motion equation*** for fluids [9]. This equation may be expressed in three-dimensional vector form as

$$\frac{D}{Dt}\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} + \frac{1}{D}\frac{\delta l}{\delta u^j}\nabla u^j - \nabla\frac{\delta l}{\delta D} = 0, \tag{8.1}$$

upon using the continuity equation (7.4) for D to simplify the components form of the equation in (7.8).

One may also write equation (8.1) equivalently in three dimensional vector notation as,

$$\frac{\partial}{\partial t}\Big(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}\Big) - \mathbf{u}\times\operatorname{curl}\Big(\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}\Big) + \nabla\Big(\mathbf{u}\cdot\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}} - \frac{\delta l}{\delta D}\Big) = 0\,. \tag{8.2}$$

In writing the last equation, we have again used the fundamental vector identity of fluid dynamics, recalled from (1.9),

$$(\mathbf{b}\cdot\nabla)\mathbf{a} + a_j\nabla b^j = -\,\mathbf{b}\times(\nabla\times\mathbf{a}) + \nabla(\mathbf{b}\cdot\mathbf{a})\,, \tag{8.3}$$

for any three dimensional vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$ with, in this case, $\mathbf{a} = (\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}})$ and $\mathbf{b} = \mathbf{u}$.

Kelvin-Noether circulation theorem [9, 10]. Prove that equation (8.2) implies Kelvin's conservation law for circulation

$$\frac{d}{dt}\oint_{c(u)}\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}\cdot d\mathbf{x} = 0\,, \tag{8.4}$$

where $c(u)$ is a closed loop moving with the fluid velocity $\mathbf{u}(\mathbf{x},t)$ in three dimensions.

This form of the Kelvin circulation theorem generalises the result for RSW in (1.15) to the case of an arbitrary ideal compressible fluid moving in three dimensions.

★

Energy conservation. Verify that equations (8.1) or (8.2) with the continuity equation (7.4) conserve the energy

$$E(\mathbf{u},D) = \Big\langle \frac{\delta l}{\delta u^j}, u^j\Big\rangle - l(\mathbf{u},D)\,. \tag{8.5}$$

Explain why energy conservation is to be expected from Hamilton's principle $\delta S = 0$ for the Lagrangian $l(\mathbf{u},D)$. Hint: a symmetry of the Lagrangian is involved. ★

8.2 Specialising to the RSW equation

We now specialise the Lagrangian to the RSW case, by choosing, cf. (1.14),

$$S = \int_0^T l(\mathbf{u},D)\,dt = \int_0^T\!\!\int \frac{\epsilon}{2}D|\mathbf{u}|^2 + D\mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - \frac{(D-b(\mathbf{x}))^2}{2\epsilon\mathcal{F}}\,d^2x\,dt \tag{8.6}$$

where $\operatorname{curl}\mathbf{R}(\mathbf{x}) = f(\mathbf{x})\hat{\mathbf{z}}$. In this case, taking variations yields

$$0 = \delta S = \int_0^T\!\!\int D\big(\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})\big)\cdot\delta\mathbf{u} + \left(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - \frac{D - b(\mathbf{x})}{\epsilon\mathcal{F}}\right)\delta D\, d^2x\, dt\,, \tag{8.7}$$

where we denote $h := (D - b(\mathbf{x}))/\epsilon\mathcal{F}$. Substituting these variational derivatives into the Euler-Poincaré motion equation (8.2) yields

$$\partial_t(\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})) - \mathbf{u}\times\operatorname{curl}(\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})) + \boldsymbol{\nabla}\left(\frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{D - b(\mathbf{x})}{\epsilon\mathcal{F}}\right) = 0\,. \tag{8.8}$$

This recovers the RSW motion equation in its curl form (1.8). Thus, the RSW motion equation (1.8) is an Euler-Poincaré motion equation (8.2), which may be derived from Hamilton's principle $\delta S = 0$ with action integral in the form (7.3), $S = \int l(\mathbf{u}, D)\, dt$ with Lagrangian $l(\mathbf{u}, D)$ given in (8.6).

Exercise. Verify equation (8.8) by substituting the variational derivatives in (8.7) into the Euler-Poincaré motion equation (8.2). ★

8.3 Hamiltonian formulation of the Euler-Poincaré equations

The conserved energy for the EP equations in equation (8.5) may be rewritten in the following form

$$h(\mathbf{m}, D) = \big\langle m_j, u^j \big\rangle - l(\mathbf{u}, D) \quad \text{with components} \quad m_j := \frac{\delta l}{\delta u^j}\,, \tag{8.9}$$

in which the angle brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ still denote the L^2 pairing in (7.2). In this form, the conserved energy provides the Legendre transformation from the Lagrangian $l(\mathbf{u}, D)$ to the Hamiltonian $h(\mathbf{m}, D)$, in which fluid momentum density is defined by $\mathbf{m} := \delta l/\delta\mathbf{u}$. Taking variations of both sides of the Legendre transformation in equation (8.9) yields

$$\begin{aligned} \delta h(\mathbf{m}, D) &= \left\langle \frac{\delta h}{\delta m_j}, \delta m_j \right\rangle + \left\langle \frac{\delta h}{\delta D}, \delta D \right\rangle \\ &= \big\langle \delta m_j, u^j \big\rangle + \left\langle m_j - \frac{\delta l}{\delta u^j}, \delta u^j \right\rangle + \left\langle -\frac{\delta l}{\delta D}, \delta D \right\rangle\,. \end{aligned} \tag{8.10}$$

Identifying coefficients of the independent variations δm_j, δu^j and δD in (8.10) then yields the following variational relations,

$$\frac{\delta h}{\delta m_j} = u^j\,, \quad \frac{\delta h}{\delta D} = -\frac{\delta l}{\delta D} \quad \text{and} \quad m_j := \frac{\delta l}{\delta u^j}\,. \tag{8.11}$$

In terms of the variables m_i, $u^j = \delta h/\delta m_j$ and D, we may write the Euler-Poincaré equation (7.8) and its auxiliary continuity equation, respectively, as

$$\partial_t m_i + \partial_j\left(m_i\frac{\delta h}{\delta m_j}\right) + m_j\partial_i\frac{\delta h}{\delta m_j} = D\partial_i\frac{\delta l}{\delta D}\,, \tag{8.12}$$

and

$$\partial_t D + \partial_j\left(D\frac{\delta h}{\delta m_j}\right) = 0\,. \tag{8.13}$$

In turn, we may rewrite these component equations in skew-symmetric matrix operator form as

$$\partial_t \begin{bmatrix} m_i \\ D \end{bmatrix} = - \begin{bmatrix} (\partial_j m_i + m_j \partial_i) & D\partial_i \\ \partial_j D & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta m_j \\ \delta h/\delta D \end{bmatrix} . \tag{8.14}$$

This skew-symmetric matrix representation implies a Hamiltonian formulation in terms of the Lie–Poisson bracket

$$\partial_t g = \Big\{ g \,,\, h \Big\} = - \int \begin{bmatrix} \delta g/\delta m_i \\ \delta g/\delta D \end{bmatrix}^T \begin{bmatrix} (\partial_j m_i + m_j \partial_i) & D\partial_i \\ \partial_j D & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta m_j \\ \delta h/\delta D \end{bmatrix} d^n x \,, \tag{8.15}$$

for any differentiable functionals g and h. For the case of RSW with Lagrangian in (8.6), we have from the variational relations from equation (8.7) that

$$\begin{aligned} \mathbf{m} := \frac{\delta l}{\delta \mathbf{u}} &= D\big(\epsilon \mathbf{u} + \mathbf{R}(\mathbf{x})\big) \quad \text{and} \\ \frac{\delta h}{\delta D} = - \frac{\delta l}{\delta D} &= - \left(\frac{\epsilon}{2} |\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{R}(\mathbf{x}) - \frac{D - b(\mathbf{x})}{\epsilon \mathcal{F}} \right) . \end{aligned} \tag{8.16}$$

Exercise. Derive the Hamiltonian for RSW by computing its Legendre transformation from the Lagrangian in (8.6). ★

Exercise. Verify the results of the Euler-Poincaré theory introduced here, by substituting the variational derivatives (8.16) into Hamiltonian matrix form in equation (8.14) to recover the RSW equations in Hamiltonian form. ★

The Hamiltonian formulation of ideal fluid dynamics in terms of Lie–Poisson brackets has proven its utility many times in studying the qualitative and quantitative properties of GFD during the past 50 years. It is now part of the toolbox of every mathematician studying ideal fluid dynamics and it has accumulated a vast literature which is constantly being rediscovered.

The Euler–Poincaré approach discussed here for passing from Hamilton's variational principle to the Hamiltonian formulation of the fluid equations guarantees preservation of two key properties of the GFD balances which are responsible for large-scale ocean and atmosphere circulations. Namely:
(1) the Kelvin circulation theorem, leading to proper potential-vorticity (PV) dynamics; and
(2) the law of energy conservation.
These two important properties are preserved in the EP approach for Hamilton's-principle asymptotics at *every level of approximation*. However, they have often been *lost* when using the standard asymptotic expansions of the fluid equations.

The models pictured in the figure below form the ***main sequence*** of GFD model equations and they follow from the Euler–Poincaré form of Hamilton's principle for a fluid Lagrangian that depends parametrically on the advected quantities such as mass, salt and heat, all carried as material properties of the fluid's motion. Thus, the Euler–Poincaré theorem with advected quantities systematically selects and derives the useful GFD fluid models possessing the two main properties of energy balance and the circulation theorem. For fundamental references, see [9, 10].

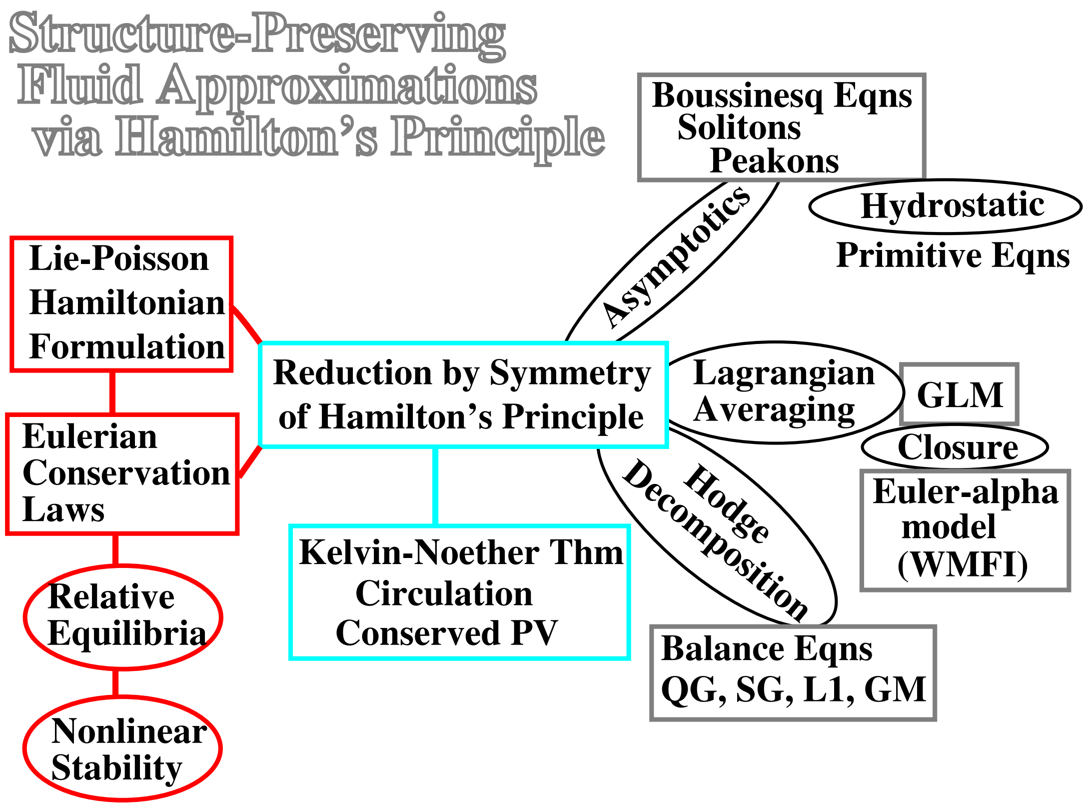

Figure 1: Asymptotics and averaging in Hamilton's principle in the Euler–Poincaré framework produces fluid approximations for GFD that preserve fundamental mathematical structures such as the Kelvin–Noether theorem for circulation leading to conservation of energy and potential vorticity (PV). Legendre transforming the resulting Euler–Poincaré Lagrangian yields the Lie–Poisson Hamiltonian formulation of geophysical fluid dynamics and its Eulerian conservation laws, which may be used to classify steady solutions as relative equilibria and determine sufficient conditions for their nonlinear stability [11].

8.4 Thermal Rotating Shallow Water (TRSW)

While the standard RSW model applies to a single layer of homogeneous fluid moving in vertical columns, the thermal rotating shallow water (TRSW) model describes an upper active layer of fluid motion with horizontally varying buoyancy and an inert lower layer. The TRSW model is an extension of the RSW model. This extension comprises an upper active layer of fluid motion with horizontally varying buoyancy and an inert lower layer. Since the lower layer is inert, the TRSW model is sometimes called a 1.5 layer model [21]. For a discussion of a fully multilayer model with nonhydrostatic pressure, see [4].

The TRSW equations are expressed in terms of the square root $\gamma = \sqrt{\theta}$ of the (nonnegative) buoyancy $\theta(\mathbf{x}, t) = (\bar\rho - \rho(\mathbf{x}, t))/\bar\rho$, where ρ is the (time and space dependent) mass density of the active upper layer, $\bar\rho$ is the uniform mass density of the inert lower layer. To distinguish from the RSW notation, we let $D = D(\mathbf{x}, t)$ be the thickness of the active layer, where $\mathbf{x} = (x, y)$ is the horizontal vector position, and t is time. The nondimensional TRSW equations are

$$\epsilon \frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} + \gamma \nabla (D\gamma) = 0\,, \qquad \frac{\partial D}{\partial t} + \nabla \cdot (D\mathbf{u}) = 0\,, \qquad \frac{D\gamma}{Dt} = 0\,. \tag{8.17}$$

with notation ϵ for Rossby number as in RSW and advective time derivative $\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla$, in equation (1.2). The boundary conditions are

$$\hat{\mathbf{n}} \cdot \mathbf{u} = 0 \quad \text{and} \quad \hat{\mathbf{n}} \times \nabla \theta = 0\,, \tag{8.18}$$

meaning that fluid velocity $\mathbf{u}$ is tangential and buoyancy θ is constant on the boundary of the domain of flow.

8.5 Conservation laws for TRSW

Exercise. Verify the following four properties of the TRSW equations (8.17)

1. Energy conservation
$$E(\mathbf{u}, D, \gamma) = \frac{1}{2}\int \epsilon D|\mathbf{u}|^2 + \gamma^2 D^2 \, d^2x \,. \tag{8.19}$$

2. Kelvin circulation theorem
$$\frac{d}{dt}\oint_{c(u)} (\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})) \cdot d\mathbf{x} = \oint_{c(u)} \frac{1}{2} D \boldsymbol{\nabla}\gamma^2 \cdot d\mathbf{x} \,, \tag{8.20}$$
where $\operatorname{curl}\mathbf{R}(\mathbf{x}) = f(\mathbf{x})\hat{\boldsymbol{z}}$ and $c(u)$ is a closed planar loop moving with the fluid velocity $\mathbf{u}(\mathbf{x}, t)$.

3. Evolution of potential vorticity (PV) on fluid parcels
$$\partial_t q + \mathbf{u} \cdot \boldsymbol{\nabla} q = \frac{1}{2D} J(D\,,\,\theta)\,, \tag{8.21}$$
where PV (q) is defined by
$$q := \frac{\varpi}{\eta}\,, \quad \text{and} \quad \varpi := \hat{\boldsymbol{z}} \cdot \operatorname{curl}(\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x}))\,, \tag{8.22}$$
and
$$J(D, \theta) = \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} D \times \boldsymbol{\nabla}\theta = -\operatorname{div}(D\hat{\boldsymbol{z}} \times \boldsymbol{\nabla}\theta)$$
is the Jacobian of the depth $D(\mathbf{x})$ and the buoyancy is $\theta(\mathbf{x}) = \gamma^2(\mathbf{x})$.

4. Infinite number of conserved integral quantities
$$C = \int D f(\gamma) + \varpi g(\gamma)\, d^2x\,, \tag{8.23}$$
for boundary conditions (8.18) and any differentiable functions f and g.

Hints: The following alternative form of the TRSW motion equation (8.17) may be helpful in verifying these four properties:
$$\partial_t \boldsymbol{v} - \mathbf{u} \times \operatorname{curl}\boldsymbol{v} + \boldsymbol{\nabla}\left(\frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{1}{2}D\gamma^2\right) - \frac{1}{2}D\boldsymbol{\nabla}\gamma^2 = 0\,, \tag{8.24}$$
where $\boldsymbol{v} := \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$ and $\operatorname{curl}\mathbf{R}(\mathbf{x}) = f(\mathbf{x})\hat{\boldsymbol{z}}$.

You might also keep in mind the fundamental vector identity of fluid dynamics,
$$(\operatorname{curl}\mathbf{a}) \times \mathbf{b} + \boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a} + a_j \boldsymbol{\nabla} b^j\,. \tag{8.25}$$

★

Answer.

1. The time derivative of the proposed TRSW energy is given by

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\frac{d}{dt}\int_{\mathcal{D}} \epsilon D|\mathbf{u}|^2 + \gamma^2 D^2\,dx\,dy \\
&= \int_{\mathcal{D}} D\mathbf{u}\cdot\partial_t\epsilon\mathbf{u} + \Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \gamma^2 D\Big)\partial_t D + \big(\gamma D^2\big)\partial_t\gamma\,dx\,dy \\
&= -\int_{\mathcal{D}} D\mathbf{u}\cdot\boldsymbol{\nabla}\left(\frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{1}{2}D\gamma^2\right) \\
&\qquad + \Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + D\gamma^2\Big)\mathrm{div}(D\mathbf{u}) + D\mathbf{u}\cdot\boldsymbol{\nabla}\left(D\frac{\gamma^2}{2}\right)\,dx\,dy \\
&= -\int_{\mathcal{D}} \mathrm{div}\left(D\mathbf{u}\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + D\gamma^2\Big)\right)\,dx\,dy\,,
\end{aligned}
$$

   which vanishes for $\mathbf{u}$ tangent to the boundary $\partial\mathcal{D}$ of the domain of flow $\mathcal{D}$.

2. The alternative form of the TRSW motion equation in (8.24) yields

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(u)}\boldsymbol{v}\cdot d\mathbf{x} &= \oint_{c(u)}\Big(\partial_t\boldsymbol{v} - \mathbf{u}\times\mathrm{curl}\boldsymbol{v} + \boldsymbol{\nabla}\big(\mathbf{u}\cdot\boldsymbol{v}\big)\Big)\cdot d\mathbf{x} \\
\text{By (8.24)}\quad &= -\oint_{c(u)}\left(\boldsymbol{\nabla}\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \frac{1}{2}D\gamma^2 - \mathbf{u}\cdot\boldsymbol{v}\Big) - \frac{1}{2}D\boldsymbol{\nabla}\gamma^2\right)\cdot d\mathbf{x} \\
&= \oint_{c(u)}\frac{1}{2}D\boldsymbol{\nabla}\gamma^2\cdot d\mathbf{x}\,,
\end{aligned}
$$

   where we have used the fundamental theorem of calculus in setting integrals of gradients over the closed loop to zero.

3. The curl of the alternative form of the TRSW motion equation in (8.24) yields

$$
\partial_t\varpi - \mathrm{curl}(\mathbf{u}\times\varpi) = \partial_t\varpi + \mathbf{u}\cdot\boldsymbol{\nabla}\varpi + \varpi\mathrm{div}\mathbf{u} = \frac{1}{2}\hat{\boldsymbol{z}}\cdot\mathrm{curl}(D\boldsymbol{\nabla}\theta)\,.
$$

   Using the continuity equation to write $\mathrm{div}\mathbf{u}$ in terms of D in this equation verifies (8.21) as

$$
D(\partial_t + \mathbf{u}\cdot\boldsymbol{\nabla})\frac{\varpi}{D} = \frac{1}{2}\hat{\boldsymbol{z}}\cdot\mathrm{curl}(D\boldsymbol{\nabla}\theta) = \frac{1}{2}J(D,\theta)\,.
$$

4. The expression for the Jacobian given in the remark immediately after equation (8.22) suggests rewriting (8.21) equivalently by using the continuity equation for D in (8.17) as

$$
\partial_t\varpi + \mathrm{div}\Big(\varpi\mathbf{u} + \frac{D}{2}\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\theta\Big) = 0\,. \tag{8.26}
$$

   The infinite family of constants of motion C in (8.23) may then be verified by taking the time derivative of C and using the boundary conditions for $\mathbf{u}$ and $\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\theta$ in (8.18) and the defining relation, $\theta := \gamma^2$. Thus,

$$
\begin{aligned}
\frac{d}{dt}\int Df(\gamma)\,d^2x &= -\int \mathrm{div}\Big(f(\gamma)D\mathbf{u}\Big)\,d^2x \\
\frac{d}{dt}\int \varpi g(\gamma)\,d^2x &= -\int \mathrm{div}\left(g(\gamma)\Big(\varpi\mathbf{u} + \frac{D}{2}\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\theta\Big)\right)\,d^2x \\
&\qquad + \int\frac{D}{2}\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}g(\gamma)\,d^2x\,.
\end{aligned}
$$

▲

Euler-Poincaré equation.

(a) Show that the *Euler-Poincaré equation*

$$\partial_t \frac{\delta l}{\delta u^i} + \partial_j \left(\frac{\delta l}{\delta u^i} u^j \right) + \frac{\delta l}{\delta u^k} \partial_i u^k = D \partial_i \frac{\delta l}{\delta D} - \gamma_{,i} \frac{\delta l}{\delta \gamma} \,, \tag{8.27}$$

arises from Hamilton's principle $\delta S = 0$ with action integral $S = \int l(\mathbf{u}, D, \gamma)\, dt$ given by

$$S = \int_0^T l(\mathbf{u}, D, \gamma)\, dt \,, \tag{8.28}$$

for variations of the action integral which depend on the horizontal fluid velocity $\mathbf{u}$, the depth of the active layer D and sqrt-buoyancy γ, as follows.

(i) In the variational formula,

$$0 = \delta S = \int_0^T \left(\left\langle \frac{\delta l}{\delta \mathbf{u}} \,, \delta \mathbf{u} \right\rangle + \left\langle \frac{\delta l}{\delta D} \,, \delta D \right\rangle + \left\langle \frac{\delta l}{\delta \gamma} \,, \delta \gamma \right\rangle \right) dt \,, \tag{8.29}$$

substitute variations given geometrically by the infinitesimal transformations,

$$\delta \mathbf{u} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathbf{u} = \partial_t \boldsymbol{v} + \mathbf{u} \cdot \nabla \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \mathbf{u} \,, \tag{8.30}$$

$$\delta D = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} D = -\mathrm{div}(D \boldsymbol{v}) \,, \tag{8.31}$$

$$\delta \gamma = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \gamma = -\boldsymbol{v} \cdot \nabla \gamma \,, \tag{8.32}$$

where the variational vector field $\boldsymbol{v} \in \mathfrak{X}(\mathbb{R}^2)$ which generates the flow parameterised by ϵ is assumed to vanish at the endpoints in time $[0, T]$ and the angle brackets $\langle \cdot \,, \cdot \rangle$ in (8.29) denote L^2 pairing, as in equation (7.2).

(ii) Integrate by parts in (8.29) using these variations to obtain the Euler-Poincaré equation (8.27).

★

Euler-Poincaré equation for TRSW system (8.17).

(a) Verify that one may also write the Euler-Poincaré equation (8.27) equivalently in three dimensional vector notation as,

$$\frac{\partial}{\partial t}\Big(\frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} \Big) - \mathbf{u} \times \mathrm{curl}\Big(\frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} \Big) + \nabla \Big(\mathbf{u} \cdot \frac{1}{D} \frac{\delta l}{\delta \mathbf{u}} - \frac{\delta l}{\delta D} \Big) + \frac{1}{D} \frac{\delta l}{\delta \gamma} \nabla \gamma = 0 \,. \tag{8.33}$$

In verifying the last equation, it may be helpful to use the fundamental vector identity of fluid dynamics, recalled from (1.9),

$$(\mathbf{b}\cdot\nabla)\mathbf{a} + a_j\nabla b^j = -\,\mathbf{b}\times(\nabla\times\mathbf{a}) + \nabla(\mathbf{b}\cdot\mathbf{a})\,, \tag{8.34}$$

for any three dimensional vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ with, in this case, $\mathbf{a} = (\frac{1}{D}\frac{\delta l}{\delta \mathbf{u}})$ and $\mathbf{b} = \mathbf{u}$.

(b) Evaluate the variational derivatives for the Lagrangian in the following action integral

$$S = \int_0^T l(\mathbf{u}, D, \gamma)\,dt = \int_0^T\!\!\int \frac{\epsilon}{2}D|\mathbf{u}|^2 + D\mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - \frac{1}{2}\gamma^2 D^2\,d^2x\,dt \tag{8.35}$$

and use formula (8.33) with curl $\mathbf{R}(\mathbf{x}) = f(\mathbf{x})\hat{\mathbf{z}}$ to obtain the motion equation for the TRSW system in (8.17).

★

Kelvin-Noether circulation theorem [9, 10].

(a) Prove that the Euler-Poincaré equation (8.33) implies the following Kelvin circulation law

$$\frac{d}{dt}\oint_{c(u)} \frac{1}{D}\frac{\delta l}{\delta \mathbf{u}}\cdot d\mathbf{x} = -\oint_{c(u)} \frac{1}{D}\frac{\delta l}{\delta \gamma}\nabla\gamma\cdot d\mathbf{x}\,, \tag{8.36}$$

where $c(u)$ is a closed loop moving with horizontal fluid velocity $\mathbf{u}(\mathbf{x}, t)$ in two dimensions.

(b) Evaluate this circulation law for the TRSW system (8.17).

★

Energy conservation.

Verify that equations (8.27) or (8.33) with the auxiliary equations for D and γ in the TRSW system (8.17) conserve the energy

$$E(\mathbf{u}, D, \gamma) = \left\langle \frac{\delta l}{\delta u^j}, u^j \right\rangle - l(\mathbf{u}, D, \gamma) = \frac{1}{2}\int \epsilon D|\mathbf{u}|^2 + \gamma^2 D^2\,d^2x\,. \tag{8.37}$$

Explain why energy conservation is to be expected from Hamilton's principle $\delta S = 0$ for the Lagrangian $l(\mathbf{u}, D, \gamma)$. Hint: a symmetry of the Lagrangian is involved. ★

Exercise. Derive the Hamiltonian for 1.5 layer TRSW by computing its Legendre transformation from the Lagrangian in (8.35) . ★

Hamiltonian formulation of TRSW.

Following section 8.3 in light of equation (8.27), we extend equations (8.12), (8.13) and (8.14), as follows. In terms of the variables m_i, $u^j = \delta h/\delta m_j$, D and γ, we may write the Euler-Poincaré equation (8.33) and its auxiliary equations, as

$$\partial_t m_i + \partial_j \left(m_i \frac{\delta h}{\delta m_j} \right) + m_j \partial_i \frac{\delta h}{\delta m_j} = -\, D \partial_i \frac{\delta h}{\delta D} + \frac{\delta h}{\delta \gamma} \partial_i \gamma \,, \tag{8.38}$$

$$\partial_t D + \partial_j \left(D \frac{\delta h}{\delta m_j} \right) = 0 \,, \tag{8.39}$$

and

$$\partial_t \gamma + \frac{\delta h}{\delta m_j} \partial_j \gamma = 0 \,. \tag{8.40}$$

In turn, we may rewrite these component equations in skew-symmetric matrix operator form as

$$\partial_t \begin{bmatrix} m_i \\ D \\ \gamma \end{bmatrix} = - \begin{bmatrix} (\partial_j m_i + m_j \partial_i) & D\partial_i & -\gamma_{,i} \\ \partial_j D & 0 & 0 \\ \gamma_{,j} & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta m_j \\ \delta h/\delta D \\ \delta h/\delta \gamma \end{bmatrix} . \tag{8.41}$$

This skew-symmetric matrix representation implies a Hamiltonian formulation in terms of the Lie–Poisson bracket

$$\partial_t g = \Big\{ g \,,\, h \Big\} = - \int \begin{bmatrix} \delta g/\delta m_j \\ \delta g/\delta D \\ \delta g/\delta \gamma \end{bmatrix}^T \begin{bmatrix} (\partial_j m_i + m_j \partial_i) & D\partial_i & -\gamma_{,i} \\ \partial_j D & 0 & 0 \\ \gamma_{,j} & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta h/\delta m_j \\ \delta h/\delta D \\ \delta h/\delta \gamma \end{bmatrix} d^2x \,, \tag{8.42}$$

for any differentiable functionals g and h. For the case of TRSW with Lagrangian in (8.35), we have via the variational relations from the Legendre transformation of the TRSW Lagrangian that

$$\begin{aligned} \mathbf{m} &:= \frac{\delta l}{\delta \mathbf{u}} = D\big(\epsilon \mathbf{u} + \mathbf{R}(\mathbf{x})\big) \quad \text{and} \quad \frac{\delta h}{\delta \mathbf{m}} = \mathbf{u} \,, \\ \frac{\delta h}{\delta D} &= - \frac{\delta l}{\delta D} = - \Big(\frac{\epsilon}{2} |\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{R}(\mathbf{x}) - \gamma^2 D \Big) \quad \text{and} \\ \frac{\delta h}{\delta \gamma} &= - \frac{\delta l}{\delta \gamma} = \gamma D^2 \,. \end{aligned} \tag{8.43}$$

★

Exercise. Verify the results of the Euler-Poincaré theory introduced here for TRSW, by substituting the variational derivatives (8.43) into the Hamiltonian matrix form in equation (8.41) to recover the TRSW equations in Hamiltonian form. ★

Hamilton's principle with auxiliary Clebsch constraints.
Show that the *Euler-Poincaré equation* in (8.27) arises from Hamilton's principle $\delta S = 0$ with constrained action integral given by

$$S = \int_0^T l(\mathbf{u}, D, \gamma) + \Big\langle \alpha\,,\, \partial_t D + \boldsymbol{\nabla}\cdot(D\mathbf{u})\Big\rangle + \Big\langle \beta\,,\, \partial_t\gamma + \mathbf{u}\cdot\boldsymbol{\nabla}\gamma\Big\rangle\, dt\,. \tag{8.44}$$

for free variations of the functions $\mathbf{u}$, D, γ, α and β, with L^2 pairing indicated by the angle brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ as in equation (7.2).

Hint: To prove this statement try taking the advective time derivative of the result for the $\delta\mathbf{u}$ variation,

$$\frac{1}{D}\frac{\delta l}{\delta\mathbf{u}}\cdot d\mathbf{x} = d\alpha - \frac{\beta}{D}d\gamma\,,$$

★

Answer. Taking variations wrt $\mathbf{u}$, D and γ yields

$$\begin{aligned}
\delta\mathbf{u}:&\quad \frac{\delta l}{\delta\mathbf{u}} - D\boldsymbol{\nabla}\alpha + \beta\boldsymbol{\nabla}\gamma = 0\,,\\
\delta D:&\quad \frac{\delta l}{\delta D} - \frac{D\alpha}{Dt} = 0\,,\\
\delta\gamma:&\quad \frac{\delta l}{\delta\gamma} - \partial_t\beta - \boldsymbol{\nabla}\cdot(\beta\mathbf{u}) = 0\,,
\end{aligned}$$

while variations wrt α and β yield the auxiliary equations for D and γ, respectively. Taking the advective time derivative $\frac{D}{Dt}$ along $\frac{D\mathbf{x}}{Dt} = \mathbf{u}$ of the result for the variation wrt the fluid velocity yields

$$\begin{aligned}
\frac{D}{Dt}\left(\frac{1}{D}\frac{\delta l}{\delta\mathbf{u}}\cdot d\mathbf{x}\right) &= \frac{D}{Dt}\left(d\alpha - \frac{\beta}{D}d\gamma\right)\\
\frac{D}{Dt}\left(\frac{1}{D}\frac{\delta l}{\delta\mathbf{u}}\right)\cdot d\mathbf{x} + \left(\frac{1}{D}\frac{\delta l}{\delta\mathbf{u}}\right)\cdot d\frac{D\mathbf{x}}{Dt} &= d\frac{D\alpha}{Dt} - \frac{D}{Dt}\left(\frac{\beta}{D}\right)d\gamma - \frac{\beta}{D}d\left(\frac{D\gamma}{Dt}\right)\\
\left(\frac{D}{Dt}\left(\frac{1}{D}\frac{\delta l}{\delta\mathbf{u}}\right) + \left(\frac{1}{D}\frac{\delta l}{\delta u^j}\right)\boldsymbol{\nabla}u^j\right)\cdot d\mathbf{x} &= \left(\boldsymbol{\nabla}\frac{\delta l}{\delta D} - \frac{1}{D}\frac{\delta l}{\delta\gamma}\boldsymbol{\nabla}\gamma\right)\cdot d\mathbf{x}
\end{aligned}$$

upon inserting the auxiliary equation for γ and the two other variational equations above. Hence, in addition to the auxiliary equations for buoyancy γ^2 and depth D, one obtains the motion equation

$$\frac{D}{Dt}\left(\frac{1}{D}\frac{\delta l}{\delta\mathbf{u}}\right) + \left(\frac{1}{D}\frac{\delta l}{\delta u^j}\right)\boldsymbol{\nabla}u^j = \boldsymbol{\nabla}\frac{\delta l}{\delta D} - \frac{1}{D}\frac{\delta l}{\delta\gamma}\boldsymbol{\nabla}\gamma\,,$$

which recovers equation (8.27) upon using the continuity equation for depth, D. and also recovers equation (8.33) upon using the fundamental vector identity of fluid dynamics (1.9)

▲

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