Mathematical Methods **Spring Term 2021**

Problem Sheet 2

1. Let $f(z) = \ln z$.

   i) Show that $f(z) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations.

   ii) Show that away from its branch cut

   $$\frac{df(z)}{dz} = \frac{1}{z}.$$

   Hint: the limit

   $$\lim_{w \to 0} \frac{\ln(1+w)}{w} = 1$$

   may be quoted without proof.

2. Use the Fundamental theorem of calculus to show that

   $$\oint_C \frac{dz}{z} = 2\pi i,$$

   where C encircles the origin once.

   Hint: Consider the integral over a non-closed curve C. Take two end points which are close together but separated by a branch cut of an anti-derivative for $f(z) = 1/z$.

3. i) Show that if C is a circular contour centred at w then

   $$\frac{1}{2\pi i} \oint_C \frac{\overline{f(z)}}{z-w} dz = \overline{f(w)}.$$

   Here f is analytic on and inside C.

   ii) Does the result from part i) hold for non-circular contours encircling w?

4. The Fundamental Theorem of Algebra states that a complex polynomial f has at least one root. Use Liouville's theorem to prove this.