Mathematical Methods **Spring Term 2021**

Problem Sheet 3

1. i) What is the radius of convergence of the Taylor expansion for $f(z) = (1 - \cos z)/z^2$ about $z = 0$?

   ii) Obtain the Taylor expansion of $f(z) = \ln z$ about $e^{2\pi i/3}$ (take $\ln e^{2\pi i/3} = \frac{2}{3}\pi i$). What is the radius of convergence of this series?

2. Find the poles and associated residues of the meromorphic functions

$$i)\ \ f(z) = \frac{e^{iz}}{1+z^2}, \quad ii)\ \ f(z) = \frac{1}{(z+1)(z+2)(z+3)}$$

3. Let C be the unit circle with the orientation taken anti-clockwise. Evaluate the contour integrals

$$a)\ \oint_C \frac{e^z - 1}{z}\, dz \qquad b)\ \oint_C \frac{\cos 2z}{z^5}\, dz \qquad c)\ \oint_C z^2 e^{1/z}\, dz.$$

4. Consider the integral

$$\oint_C \frac{e^{iz}}{(z-i)^2} dz$$

   taken around a semi-circular path in the upper-half plane, with straight line section between $z = -R$ and $z = R$, $R > 1$. By letting $R \to \infty$ show that

$$\int_0^\infty \frac{2x \sin x - (x^2 - 1)\cos x}{(1+x^2)^2} dx = \frac{\pi}{e}.$$

5. Use residues to compute the integrals

$$i)\ \int_{-\infty}^{\infty} \frac{1}{1+x^6} dx \quad ii)\ \int_0^\infty \frac{1}{1+x^3} dx \quad iii)\ \int_0^\infty \frac{\ln x}{1+x^3} dx.$$

$$iv)\ \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} \quad (a > b > 0) \qquad v)\ \int_0^\infty \frac{\sin^2 x}{x^2} dx.$$

6. A particular solution of the ODE $\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = \delta(t)$ is (see also Problem Sheet 4 Q3 ii))

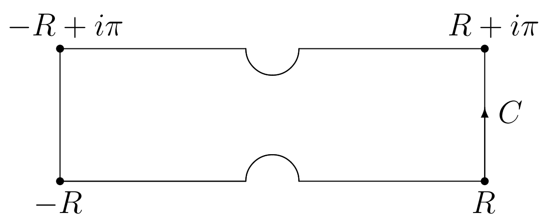
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{-\omega^2 + 3i\omega + 2} d\omega.$$

   Evaluate this using residues (hint: treat the cases $t < 0$ and $t > 0$ separately).

7. Locate the poles and determine the associated residues of the meromorphic function

$$f(z) = \frac{e^{iz}}{\sinh z}$$

Integrate f over the contour, C, depicted below



The semi-circles are centred at $z = 0$ and $z = i\pi$. Hence, or otherwise, compute the integral

$$\int_0^\infty \frac{\sin x}{\sinh x}\, dx.$$