Mathematical Methods **Spring Term 2021**

Problem Sheet 4

1. Express $e^{-a|x|}$ as a Fourier integral ($a > 0$).

2. The function f is defined by

$$f(x) = \begin{cases} \cos x, & -\frac{1}{2}\pi < x < \frac{1}{2}\pi \\ 0, & \text{otherwise} \end{cases}$$

   Compute $\hat{f}(k)$. Sketch the graph of $\hat{f}(k)$.

3. Prove the following properties of the Fourier transform

   i) The Fourier transform of an even function is even.

   ii) The Fourier transform of a real odd function is imaginary.

   iii) $\widehat{f'}(k) = ik\hat{f}(k)$.

   iv) Acting with the Fourier transform four times reproduces the original function apart from an overall constant.

4. Use the standard integral

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right), \qquad (a > 0, \quad b \in \mathbb{C})$$

   to compute the Fourier transform of $f(x) = e^{-ax^2}$ and $g(x) = xe^{-ax^2}$ (here a is a positive constant).

5. Use residues to compute the Fourier transform of

$$f(x) = \frac{1}{1+x^4}.$$

   Hint: compute the cases $k > 0$ and $k < 0$ separately.

6. Write

$$\frac{\sin x}{x}$$

   as a Fourier integral.

7. Compute

   i)
$$\int_{-\infty}^{\infty} x^2 \ \delta(x-3) \ dx$$

   ii)
$$\int_{-\infty}^{\infty} \delta(x^2+x) \ dx$$

   iii)
$$\int_{0}^{2} e^x \ \delta'(x-1) \ dx$$

   iv)
$$\int_{0}^{\infty} e^{-ax} \delta(\cos x) \ dx$$

   v)
$$\int_{0}^{\infty} \delta(e^{ax} \cos x) \ dx.$$

   In parts iv) and v) a is a constant.

8. Compute the derivatives

   i)
$$\frac{d}{dx} \left(\theta(x)\right)^3$$

   ii)
$$\frac{d}{dx} e^{a\theta(x)},$$

   where a is a constant.

9. i) Compute the Fourier transform of $\delta'(x-a)$ (a constant).

   ii) Express x^2 as a Fourier integral.

   iii) Write $\sin^2 x$ as a Fourier integral.

Fourier Transform Conventions

$$\text{Fourier transform} \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \ e^{-ikx} \ dx$$

$$\text{Fourier integral} \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(k) \ e^{ikx} \ dk.$$