Problem Sheet 5

1. i) Compute $(f \star g)(x)$ where $f(x) = \theta(x)e^{ax}$ and $g(x) = \theta(x)e^{bx}$ (a and b are constants).

   ii) Compute $(f \star g)(x)$ where $f(x) = 1/(x^2 + a^2)$ and $g(x) = 1/(x^2 + b^2)$ (a and b are non-zero constants).

   Hint: determine the Fourier transform of $f \star g$.

2. *Poisson Summation Formula*

   i) Use
   $$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} = \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m),$$
   to derive Poisson's summation formula
   $$\sum_{n=-\infty}^{\infty} \hat{f}(n) = \sum_{m=-\infty}^{\infty} f(2\pi m).$$

   ii) Use the Poisson summation formula to compute
   $$\sum_{p=-\infty}^{\infty} \frac{1}{a^2 + p^2} \qquad (a \ \text{ constant}).$$

3. i) Compute the Fourier transform of $\delta'(x - a)$ (a constant).

   ii) Express x^2 as a Fourier integral.

   iii) Write $\sin^2 x$ as a Fourier integral.

4. Obtain, in the form of Fourier integrals, particular solutions to the ODEs
   $$i) \ \ \ddot{x}(t) + 4x(t) = \frac{\sin t}{t} \qquad ii) \ \ \ddot{x}(t) + 2\dot{x}(t) + x(t) = \delta(t).$$
   For part ii) use contour integration to evaluate $x(t)$ explicitly.

5. i) Consider Laplace's equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

in the strip $-\infty < x < \infty$, $0 < y < 1$.

Show that

$$\phi(x, y) = \int_{-\infty}^{\infty} \hat{f}(k) \, \frac{e^{ikx} \sinh ky}{\sinh k} \, dk,$$

is harmonic in the strip and satisfies the boundary conditions

$$\phi(x, y = 0) = 0, \quad \phi(x, y = 1) = f(x).$$

Here $f(x)$ is a function of x and $\hat{f}(k)$ is its Fourier transform.

Solve Laplaces's equation in the strip subject to the boundary conditions

$$\phi(x, 0) = 0, \quad \phi(x, 1) = e^{-\frac{1}{2}x^2}.$$

ii) Obtain a solution to Laplace's equation in the half plane $-\infty < x < \infty$, $y \geq 0$ with the properties

$$\phi(x, y = 0) = e^{-|x|}, \quad \phi(x, y) \to 0 \text{ as } y \to \infty.$$

6. The two dimensional wave equation is

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0,$$

Writing $\phi(x, t)$ as a Fourier integral

$$\int_{-\infty}^{\infty} A(k, t) e^{ikx} dk,$$

where $A(k, t)$ is the Fourier transform of $\phi(x, t)$ with respect to x only (t is not Fourier-transformed) find the general form of $A(k, t)$. Use the result to show that the general form of $\phi(x, t)$ is

$$\phi(x, t) = f(x - ct) + g(x + ct)$$

where f and g are arbitrary functions.

Fourier Transform Conventions

$$\text{Fourier transform} \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \, e^{-ikx} \, dx$$

$$\text{Fourier integral} \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(k) \, e^{ikx} \, dk.$$