Mathematical Methods **Spring Term 2021**

Problem Sheet 6

1. Show that if the integral

$$S = \int_a^b L(y(x), y'(x))dx$$

is stationary for fixed $y(a)$ and $y(b)$ then

$$H = y'\frac{\partial L}{\partial y'} - L,$$

is a constant.

Remark: the result is not true if L has explicit x-dependence, ie. if it is a function of y, y' *and* x.

2. *Double Bubble Problem*

The area of a surface of revolution obtained by rotating the curve $y = y(x)$ around the x-axis for $a \leq x \leq b$ is

$$A = 2\pi \int_a^b y(x)\sqrt{1 + (y'(x))^2}dx.$$

i) Take the end points to be $-L/2$ and $L/2$. A is to be minimised with $y(-L/2) = y(L/2) = R$ where R is a positive constant. Write down the Euler-Lagrange equation for $y(x)$. Show that $y(x)$ has the form $y(x) = p^{-1}\cosh(px)$ where p is constant. This is a catenary.

Hint: Do not solve the Euler-Lagrange equation directly - use the result of question 1.

ii) Show that if R/L is too small the solution from part i) cannot be matched to the boundary conditions (a graphical plot may help here).

3. A curve of fixed length has fixed endpoints on a line. Show that the area enclosed by the curve and the line is maximised if the curve is circular.

Hint: Take the line to be the x-axis and the endpoints to be (a, 0) and (b, 0).
Maximize the integral $\int_a^b y(x)\,dx$ with y(a) = y(b) = 0,

subject to the constraint that the length of the curve is fixed.