Mathematical Methods **Spring Term 2021**

Problem Sheet 8

1. i) Find the values of

$$a)\ \delta_{ij}\delta_{ij} \qquad b)\ \delta_{ij}\delta_{jk}\delta_{kl}\delta_{li} \qquad c)\ \epsilon_{ijk}\epsilon_{jki}.$$

   ii) Prove the identity $\epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$.

   iii) Show that the cross product of two axial vectors is axial.

2. i) Let A_{ij} be the entries of 3×3 matrix A. Show that

$$\det A = \epsilon_{ijk} A_{1i} A_{2j} A_{3j}.$$

   ii) Show that under proper rotations ϵ_{ijk} is a rank 3 tensor.

3. i) Prove the identity

$$\epsilon_{ijp}\epsilon_{klp} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}.$$

   ii) Use the result from part i) to prove the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

   iii) Use index notation to prove the vector calculus identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

4. In the lectures the rank 2 anti-symmetric tensor $F_{ij} = \partial_i A_j - \partial_j A_i$ was introduced (A_i is the vector potential). Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0.$$

   i) Write Maxwell's equations using F_{ij},, E_i, j_i, ρ and the operator ∂_i. In other words rewrite the equations using F_{ij} instead of $\mathbf{B}$.

   Hint: write $\nabla \cdot \mathbf{B} = 0$ as a tensor equation rather than a scalar equation!

   ii) The Poynting vector $\mathbf{S}$ and electromagnetic energy density are given by

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}, \qquad u = \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B}.$$

   Write S_i and u in terms of the vector E_i and the tensor F_{ij}. Is S_i polar or axial?

   iii) Show that

$$\frac{\partial u}{\partial t} + \partial_i S_i = -E_i j_i.$$

5. The Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i) Show that $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$.

ii) Under a rotation of angle θ about an axis with direction $\hat{\mathbf{n}}$

$$B_i'\sigma_i = e^{-\frac{1}{2}i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}\ B_i\sigma_i\ e^{+\frac{1}{2}i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$$

where B_i is the magnetic field. Show that for an infinitesimal rotation

$$B_i'\sigma_i = B_i\sigma_i - \frac{i\theta}{2}[\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}, B_i\sigma_i].$$

iii) Verify the infinitesimal rotation formula quoted in part ii)

Hint: an infinitesimal rotation matrix has the form $R_{ij} = \delta_{ij} - \theta\epsilon_{ijk}\, n_k$.