Answers to Problem Sheet 1

1. $f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$ with real and imaginary parts $u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$. $u_x = e^x \cos y$, $u_y = -e^x \sin y$, $v_x = e^x \sin y$, $v_y = e^x \cos y$. $u_x = v_y$ and $u_y = -v_x$.

2.
$$\sinh z = \frac{e^z - e^{-z}}{2} = \frac{e^{x+iy} - e^{-x-iy}}{2}$$
$$= \frac{(e^x \cos y - e^{-x} \cos y)}{2} + i\frac{(e^x \sin y + e^{-x} \sin y)}{2}$$
$$= \sinh x \cos y + i \cosh x \sin y.$$

   Since $\cosh x$ is non-zero for all real x the imaginary part of $\sinh z$ is zero only if $\sin y = 0$ so that y must be an integer multiple of π. $u(x, y = n\pi) = \sinh x\ (-1)^n$ which is zero only if $x = 0$. The zeros of $\sinh z$ are at the points $z = in\pi$ for integer n.

   The zeros of $\cosh z$ are at the points $z = \frac{1}{2}i\pi n$ for odd integer n.

3. To show that $f(z) = z\bar{z}$ is complex differentiable at $z = 0$ consider the quotient
$$\frac{f(0 + h) - f(0)}{h} = \frac{h\bar{h} - 0}{h} = \bar{h}.$$
   Since $\bar{h} \to 0$ as $h \to 0$ f is differentiable at $z = 0$ with $f'(0) = 0$. To show that f is not analytic at $z = 0$ it is sufficient to show that the Cauchy-Riemann equations fail at every point except $z = 0$ (so that f is not differentiable in any disc centred at $z = 0$). Here $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. $u_x = 2x$ and $u_y = 2y$ so the Cauchy-Riemann equations do not hold unless $x = 0$ and $y = 0$.

4. Let $u(x, y) = x^3 + 6x^2y - 3xy^2 - 2y^3$. Cauchy Riemann equations are
$$u_x = 3x^2 + 12xy - 3y^2 = v_y \quad \text{and } u_y = 6x^2 - 6xy - 6y^2 = -v_x.$$
   Integrating the first equation with respect to y
$$v(x, y) = 3x^2y + 6xy^2 - y^3 + C(x).$$
   Note that the 'constant of integration' may depend on x. Substituting this into the second Cauchy Riemann equation gives
$$6x^2 - 6xy - 6y^2 = -v_x = -(6xy + 6y^2 + C'(x))$$

which is consistent if $C'(x) = -6x^2$, so that $C(x) = -2x^3 + c$ where c is a constant. Hence

$$v(x, y) = 3x^2y + 6xy^2 - y^3 - 2x^3 + c.$$

One can also write

$$f(z) = (1 - 2i)z^3 + ic.$$

f is not unique (due to the arbitrary imaginary constant).

5.

$$v(x, y) = \frac{x}{x^2 + y^2}.$$

Cauchy-Riemann

$$u_x = v_y = -\frac{2xy}{(x^2 + y^2)^2}$$

Integrating with respect to x gives

$$u = \frac{y}{x^2 + y^2} + C(y).$$

Note that the 'constant of integration' may depend on y. Inserting this into the second Cauchy Riemann equation gives $C'(y) = 0$ so that

$$u(x, y) = \frac{y}{x^2 + y^2} + C.$$

The function can also be written in the form

$$f(z) = \frac{i}{z} + C,$$

where C is a real constant. f is not entire due to the (simple pole) singularity at the origin.

6. To prove that any analytic function f is harmonic start with the Cauchy-Riemann equations

$$u_x = v_y \quad (1) \qquad u_y = -v_x \quad (2).$$

Differentiating (1) with respect to x and (2) with respect to y gives

$$u_{xx} = v_{xy} \qquad u_{yy} = -v_{yx}.$$

Adding the two equations yields

$$u_{xx} + u_{yy} = 0.$$

Using $v_{xy} = v_{yx}$; mixed 2nd derivatives so not depend on order of partial differentiation.

To obtain

$$v_{xx} + v_{yy} = 0.$$

differentiate (1) with respect to y, differentiate (2) with respect to x and subtract