BIFURCATION THEORY

1 Preliminaries. Systems on a plane

We deal with autonomous systems of differential equations

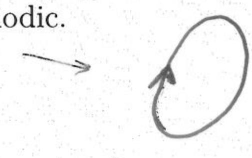
$$\frac{d}{dt}\underline{x} = \underline{f}(\underline{x}) \qquad (*)$$

where $\underline{x} = (x_1, \dots, x_n)$ is a vector in R^n and $\underline{f} = (f_1, \dots, f_n)$ is a sufficiently smooth vector-function $R^n \to R^n$. Since the right-hand side of (*) does not depend on time, it makes sense to consider *phase curves* (the term *orbits* is also used) of (*): if $\underline{x} = \underline{\phi}(t)$ is a solution of (*), then the curve run by the point $\underline{\phi}(t)$ in R^n as t changes is called a phase curve (the whole of R^n is called a phase space).

Exercise. Show that if $\underline{x} = \underline{\phi}(t)$ is a solution of (*), then $\underline{x} = \underline{\phi}(t+c)$ is also a solution (and corresponds to the same phase curve), given any constant c. Show that for every $\underline{x}_0 \in R^n$ there exists a unique phase curve that passes through it. Note that this curve depends continuously on $\underline{x}_0$: if $\underline{x}_t$ is the point where the phase curve starting at $\underline{x}_0$ arrives at the time moment t, then $\underline{x}_t$ depends on the initial condition continuously, in fact smoothly. Note also that we have a smooth dependence on parameters as well, in case f smoothly depends on some parameters.

There are 3 types of phase curves:
1. Equilibrium states: the points in R^n where $\underline{f}$ vanishes. Each such point $\underline{x}^*$ is a whole phase curve, as it corresponds to the constant solution $\underline{x}(t) = \underline{x}^*$, $t \in (-\infty, +\infty)$, of (*) (this is a solution since $\underline{f}(\underline{x}^*) = 0$).

2. Closed curves (periodic orbits): show that the phase curve is closed if and only if the corresponding solution of (*) is periodic.



3. All the others (one may show that each of them is a homeomorphic image of a straight line).

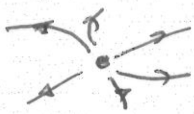
Equilibrium states are relatively easy to find and to study. Positions of the equilibria are found by solving the equation $\underline{f}(\underline{x}^*) = 0$. Once some equilibrium $\underline{x}^*$ is known, one can shift the coordinate origin to $\underline{x}^*$, i.e. make the following change of coordinates: $\underline{y} = \underline{x} - \underline{x}^*$. The system will take the form

$$\frac{d}{dt}\underline{y} = \underline{f}(\underline{x}^* + \underline{y}) = \underline{f}(\underline{x}^*) + \frac{\partial f}{\partial x}(\underline{x}^*)\underline{y} + o(\|\underline{y}\|) = A\underline{y} + o(\|\underline{y}\|).$$

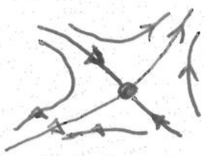
Here A is some constant matrix (it is the matrix of derivatives of $\underline{f}$ at $\underline{x}^*$). As we see, the behaviour in a small neighbourhood of the equilibrium (that is at small $\underline{y}$) is, to the leading order, given by the linear term $A\underline{y}$. It is known that the behaviour near an equilibrium state resembles indeed the behaviour of the linear system $\frac{d}{dt}\underline{y} = A\underline{y}$ in the case where the equilibrium is *hyperbolic* - this term means that the matrix A has no eigenvalues on the imaginary axis. For example, when $\underline{y} \in R^2$, there are 3 types of hyperbolic equilibria:

1. stable - both the eigenvalues $\lambda_{1,2}$ of A have negative real parts; all the orbits from a small neighbourhood of the stable equilibrium tend to it as $t \to +\infty$

2. unstable - both $\lambda_{1,2}$ have positive real parts; all the orbits from a small neighbourhood tend to the unstable equilibrium as $t \to -\infty$ and leave the neighbourhood as time grows

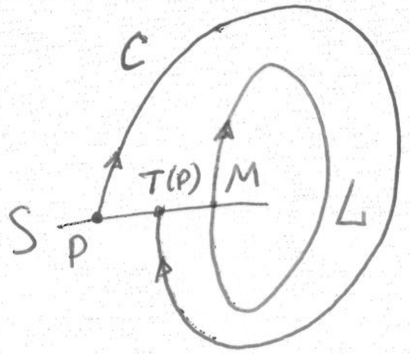


3. saddle - here $\lambda_1 < 0$ and $\lambda_2 > 0$; exactly 2 orbits tend to the saddle (from two opposite directions) as $t \to +\infty$, and 2 orbits tend to the saddle as $t \to -\infty$, all the other orbits leave a small neighbourhood of the saddle both as time grows and as time tends to $-\infty$. The two orbits that tend to the saddle as $t \to +\infty$ are called *stable separatrices*, the two orbits that tend to the saddle as $t \to -\infty$ are called *unstable separatrices*



Exercise. Show that for systems in R^2, the equilibrium state is a saddle if and only if $detA < 0$; it is stable if and only if $detA > 0$, $trA < 0$ and unstable if and only if $detA > 0$, $trA > 0$.

Analysis of the behaviour near closed phase curves can be done by the study of *the Poincare map*. If $(\phi_1(t), \ldots, \phi_n(t))$ is a τ-periodic solution of (*), then the curve $L : \underline{x} = \underline{\phi}(t), t \in [0, \tau]$, is closed. Let $M \in L$ be the point $\underline{x} = \underline{\phi}(0)$. We will move the coordinate origin to M and will rotate the coordinate axes in such a way that the x_n-axis will become tangent to L at M. In other words, we will assume that $M = (0, \ldots, 0)$ and $\underline{f}(M) = (0, \ldots, 0, 1)$. Let S be the cross-section to L defined as $x_n = 0$. The Poincare map $T : S \to S$ is defined as follows: given a sufficiently close to M point $P \in S$, issue a phase curve C through P, then the next point of intersection of C with S is the image of P by the map T. Since the iterations of the Poincare map are consecutive points of the intersection of the phase curve with S, the behaviour of phase curves near the periodic orbit L is completely determined by the behaviour of the iterations of the Poincare map T.



Denote as $\underline{\psi}(t, \underline{x}_0)$ the solution that starts at $t = 0$ at the point $\underline{x}_0 = (x_{10}, \ldots, x_{n-1,0}, 0)$ on the cross-section S. Let $\theta(\underline{x}_0)$ be the time of the first intersection of the phase curve $\underline{x} = \underline{\psi}(t, \underline{x}_0)$ with S. It is found from the condition

$$\psi_n(\theta, \underline{x}_0) = 0.$$

At $\underline{x}_0 = 0$ (which is the point of intersection of L with S) this equation has solution $\theta = \tau$ (the period of L). Since $\frac{d}{d\theta}\psi_n(\tau, 0) = f_n(0) = 1 \neq 0$, the Implicit Function Theorem guarantees us that $\theta(\underline{x}_0)$ is a well-defined and smooth function of $\underline{x}_0$ for all small $\underline{x}_0$. Thus we can right the Poincare map as

$$T : \underline{x} \mapsto \underline{\psi}(\theta(\underline{x}), \underline{x}).$$

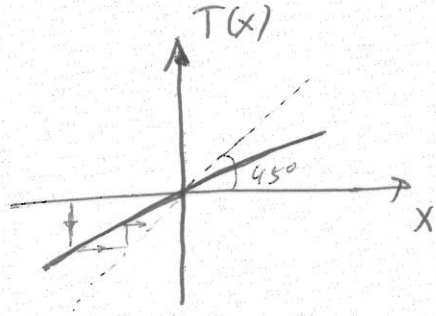
As we see, the Poincare map is smooth and has a fixed point at zero.

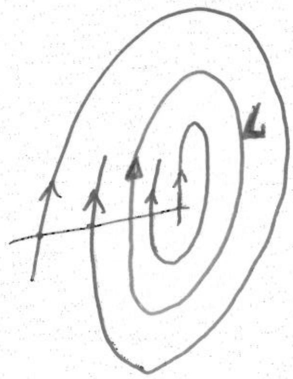
For a system on a plane (i.e. $\underline{x} = (x_1, x_2)$) the cross-section S is a segment of a straight-line, so the Poincare map is one-dimensional:

$$T : x_1 \mapsto \mu x_1 + o(x_1).$$

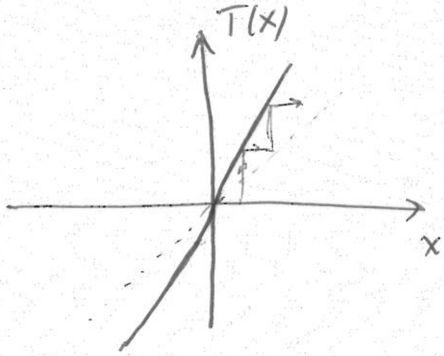
We explicitly take here into account that $T(0) = 0$. The derivative $T'(0)$ (which we denote as μ) is called *the multiplier* of L. It is clear that

1. if $\mu < 1$, then $|T(x_1)| < |x_1|$ at small $|x_1|$, which implies that the iterations of T starting close to zero converge to zero, i.e. phase curves from a small neighbourhood of L tend to L as $t \to +\infty$ - such periodic orbitis are *stable*;





2. if $\mu > 1$, then $|T(x_1)| > |x_1|$ at small $|x_1|$, which implies that the iterations of T starting close to zero diverge from zero, i.e. phase curves from a small neighbourhood of L leave the neighbourhood as time grows (and they tend to L as $t \to -\infty$) - such periodic orbitis are *unstable*.



For a periodic orbit $\{x = x_L(t), y = y_L(t)\}$ of a two-dimensional system

$$\frac{d}{dt}x = f(x,y), \qquad \frac{d}{dt}y = g(x,y), \qquad (**)$$

the following useful formula holds true:

$$\mu = \exp(\int_0^\tau \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)_{x=x_L(t),\, y=y_L(t)} dt).$$

Proof. The Poincare map is given by the formula

$$x \mapsto T(x) = \psi_1(\theta(x), x, 0),$$

where $\theta(x)$ is the time of return to the cross-section $S : \{y = 0\}$. Recall that we assume that the periodic orbit intersects S at $(x,y) = 0$ and that we assume $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 1$ at this point, i.e. $f(0,0) = 0$, $g(0,0) = 1$. As before, $\{\psi_1(t, x_0, y_0), \psi_2(t, x_0, y_0)\}$ denotes the solution with the initial condition (x_0, y_0). Thus, $\frac{\partial \psi_1}{\partial t} = f(\psi_1, \psi_2)$, $\frac{\partial \psi_2}{\partial t} = g(\psi_1, \psi_2)$. In particular, $\frac{\partial \psi_1}{\partial t}(0,0,0) = f(0,0) = 0$, so we find that

$$\mu = T'(0) = \frac{\partial \psi_1}{\partial t}(\theta(0), 0, 0)\theta'(0) + \frac{\partial \psi_1}{\partial x_0}(\theta(0), 0, 0) = \frac{\partial \psi_1}{\partial x_0}(\tau, 0, 0).$$

We will show below that $\frac{\partial \psi_1}{\partial y_0}(t,0,0) = f(x_L(t), y_L(t))$ and $\frac{\partial \psi_2}{\partial y_0}(t,0,0) = g(x_L(t), y_L(t))$, hence $\frac{\partial \psi_1}{\partial y_0}(\tau,0,0) = 0$ and $\frac{\partial \psi_2}{\partial y_0}(\tau,0,0) = 1$, so

$$\mu = det\Psi(\tau),$$

where we denote as $\Psi(t)$ the matrix $\begin{pmatrix} \frac{\partial \psi_1}{\partial x_0} & \frac{\partial \psi_1}{\partial y_0} \\ \frac{\partial \psi_2}{\partial x_0} & \frac{\partial \psi_2}{\partial y_0} \end{pmatrix}_{(x_0,\, y_0)=0}$. This matrix satisfies the differential equation

$$\frac{d}{dt}\Psi = B(t)\Psi$$

where $B(t) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x,y)=\psi(t,0,0)=(x_L(t),\, y_L(t))}$. The matrix differential equation can be written as a pair of equations for the columns:

$$\frac{d}{dt}\begin{pmatrix} \frac{\partial\psi_1}{\partial x_0} \\ \frac{\partial\psi_2}{\partial x_0} \end{pmatrix} = B(t)\begin{pmatrix} \frac{\partial\psi_1}{\partial x_0} \\ \frac{\partial\psi_2}{\partial x_0} \end{pmatrix}, \qquad \frac{d}{dt}\begin{pmatrix} \frac{\partial\psi_1}{\partial y_0} \\ \frac{\partial\psi_2}{\partial y_0} \end{pmatrix} = B(t)\begin{pmatrix} \frac{\partial\psi_1}{\partial y_0} \\ \frac{\partial\psi_2}{\partial y_0} \end{pmatrix}$$

(one gets these equations by diffrentiating both sides of (**) with respect to x_0 or y_0). It is easy to check that

$$\frac{d}{dt} det\Psi = (trB) \cdot \Psi = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\Psi.$$

As $\Psi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by definition, it follows that

$$\mu = det\Psi(\tau) = \exp(\int_0^\tau \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)_{x=x_L(t),\, y=y_L(t)} dt).$$

Thus, it remains to check that $\frac{\partial\psi_1}{\partial y_0}(t,0,0) = f(x_L(t), y_L(t))$ and $\frac{\partial\psi_2}{\partial y_0}(t,0,0) = g(x_L(t), y_L(t))$. In order to do it, we just note that it is easy to check that the vector $\underline{v} = \begin{pmatrix} f(x_L(t), y_L(t)) \\ g(x_L(t), y_L(t)) \end{pmatrix}$ satisfies the same differential equation $\frac{d}{dt}\underline{v} = B(t)\underline{v}$ as the vector $\begin{pmatrix} \frac{\partial\psi_1}{\partial y_0} \\ \frac{\partial\psi_2}{\partial y_0} \end{pmatrix}_{(x_0,y_0)=0}$, with the same initial conditions: $\underline{v}_{\,t=0} = \begin{pmatrix} f(0,0) \\ g(0,0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_0}{\partial y_0} \\ \frac{\partial y_0}{\partial y_0} \end{pmatrix} = \begin{pmatrix} \frac{\partial\psi_1}{\partial y_0} \\ \frac{\partial\psi_2}{\partial y_0} \end{pmatrix}_{(x_0,y_0)=0,t=0}$. Therefore these two vectors are indeed equal for all t due to uniqueness of the solution of the Cauchy problem. □

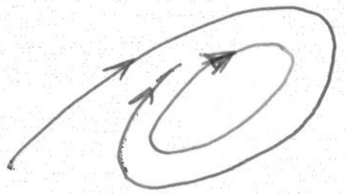
Note that this formula implies $\mu > 0$.

The basis of the theory of systems of differential equations on a plane is given by classical

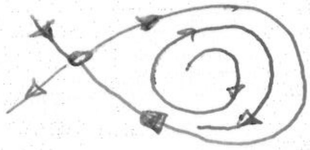
Poincare-Bendixson Theorem Every phase curve of system (**), which remains bounded as $t \to +\infty$, tends to either

1. an equilibrium state, or

2. a periodic orbit, or



3. a set which consists of equilibrium states and some orbits connecting them.



Note that the same statement is true as $t \to -\infty$: by changing the direction of time t to $-t$, we transform (**) to $\frac{d}{dt}x = -f(x,y), \quad \frac{d}{dt}y = -g(x,y)$ which is also an autonomous system of differential equations on a plane, hence it also satisfies the theorem.

Exercise. Prove the theorem. Start with reformulating it in an equivalent way as follows: if a bounded orbit $C : (x_C(t), y_C(t))$ is not periodic and has a point M, which is not an equilibrium, as a limit point (that means there exists a sequence of time values $t_n \to +\infty$ such that $M_n := (x_C(t_n), y_C(t_n)) \to M$), then no orbit can have any point of the curve C as its limit point as $t \to +\infty$. Now, in order to prove the latter statement, issue a curve S through M transverse to the vector $(f,g)_M$ (since M is not an equilibrium, $(f,g)_M \neq 0$). This curve is a cross-section, i.e. all phase curves (in particular, the curve C) that pass near M intersect S transversely. As C has M as a limit point, we may find a pair of points, $M^* = (x_C(t^*), y_C(t^*)) \in C \cap S$ and $M^* = (x_C(t^{**}), y_C(t^{**})) \in C \cap S$, $t^{**} > t^*$, close to M. The curve γ, which is the union of the piece of C between M^* and M^{**} and the segment of S between M^{**} and M^* is closed, so it divides the plane into 2 regions, which we denote as U_+ and U_-; by construction, the part C_+ of C that corresponds

to $t > t^{**}$ lies in U_+, while the part of C that correscponds to $t < t^*$ lies in U_-. Note that every orbit that starts on the boundary γ of U_+ goes inside U_+, which implies that no orbit from inside of U_+ can leave U_+ as time grows. This finishes the proof: if the statement is not true and there is a point $P = (x_C(t_0), y_C(t_0)) \in C$ such that P is a limit point of some orbit $Q : (x_Q(t), y_Q(t))$, i.e. $(x_Q(t_k), y_Q(t_k)) \to P$ for some sequence of time values $t_k \to +\infty$, then for some τ large enough we will have
a) $(x_Q(t_k - \tau), y_Q(t_k - \tau)) \to P_- = (x_C(t_0 - \tau), y_C(t_0 - \tau)) \in C_-$ and
b) $(x_Q(t_k + \tau), y_Q(t_k + \tau)) \to P_+ = (x_C(t_0 + \tau), y_C(t_0 + \tau)) \in C_+$.
However, as $C_+ \subset U_+$, it follows that $P_+ \in U_+$, hence b) implies that the point $(x_Q(t_{k_0} + \tau), y_Q(t_{k_0} + \tau))$ lies in U_+ for some sufficiently large k_0, and therefore every point $(x_Q(t), y_Q(t))$ lies in U_+ at $t \geq t_{k_0} + \tau$, which contradicts to a) that says that there exist arbitrarily large values of t for which $(x_Q(t), y_Q(t))$ lies close to the point P_- that lies in U_-, i.e. at a finite distance from U_+.
Further reading: Guckenheimer and Holmes, Ch.1