1 Center manifold theorem. Hyperbolic equilibria. Saddle-node

Behaviour near equilibrium states of multidimensional systems of differential equations is analysed with the help of the so-called center manifold theorem. Essentially, the theorem says that the local behaviour (that is the behaviour in a small neighbourhood of an equilibrium) can be non-trivial only if some eigenvalues of the linearisation matrix lie on the imaginary axis. Moreover, in case such eigenvalues exist (they are called critical eigenvalues), one has a significant reduction in the dimension of the problem: local dynamics and bifurcations can be completely recovered from the analysis of some reduced system of differential equations whose dimension equals to the number of the critical eigenvalues. Namely, let the spectrum of the eigenvalues of the linearisation matrix of a certain system near an equilibrium be $\{\lambda_1, \ldots, \lambda_k, \gamma_1, \ldots, \gamma_m, \eta_1, \ldots, \eta_n\}$ where $Re\ \lambda_j < 0$, $Re\ \gamma_j > 0$ and $Re\ \eta_j = 0$ (i.e. we have n critical eigenvalues). One can bring the linearisation matrix to a diagonal form or to a Jordan form - in any case it will be block-diagonal, with one block having the eigenvalues λ, another block having the eigenvalues γ and the last block with eigenvalues η. Thus, we can write the system near the equilibrium in the following form:

$$\begin{cases} \dfrac{dx}{dt} = Ax + f(x, y, z) \\ \dfrac{dy}{dt} = By + g(x, y, z) \qquad (*) \\ \dfrac{dz}{dt} = Cz + h(x, y, z) \end{cases}$$

where the eigenvalues of the matrix A are $\{\lambda_1, \ldots, \lambda_k\}$, the eigenvalues of B are $\{\gamma_1, \ldots, \gamma_m\}$ and the eigenvalues of C are $\{\eta_1, \ldots, \eta_n\}$; the functions f, g, h are nonlinearities, i.e. they vanish at zero along with the first derivatives. Note that x here is a k-dimensional vector, y is an m-dimensional vector, z is n-dimensional.

Center manifold theorem *In a small neighbourhood U of the origin there exists a pair of smooth manifolds, the center-stable one W^{cs} : $y = \varphi(x, z)$ and the center-unstable W^{cu} : $x = \psi(y, z)$ where the functions φ and ψ vanish at zero along with the first derivatives, such that every orbit which stays in U for all positive times necessarily lies in W^{cs} and every orbit which stays in U for all negative times necessarily lies in W^{cu}. The manifolds W^{cs} and W^{cu} are locally invariant, i.e. for every point in any of these manifold the orbit of this point lies in the same manifold all the time it stays in U*(see a proof in Shilnikov,Shilnikov,Turaev,Chua, Ch.5).

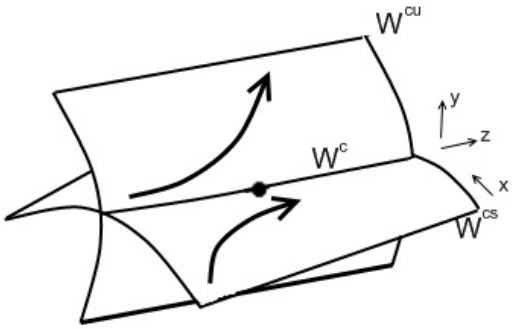
According to this theorem, every orbit that stays in a small neighbourhood of zero for all times, from minus infinity to plus infinity, must lie entirely in both W^{cu} and W^{cs}, i.e. in their intersection. Given z, the coordinates x and y of the corresponding intersection point of W^{cs} and W^{cu} are found from the system

$$y - \varphi(x, z) = 0, \quad x - \psi(y, z) = 0.$$

At $z = 0$ this system has a solution $(x, y) = 0$, and the derivative $\dfrac{\partial(y - \varphi(x, z), x - \psi(y, z))}{\partial(x, y)}$ at $(x, y, z) = 0$ is the identity matrix (since the derivatives of φ and ψ vanish at zero), i.e. its determinant is non-zero. Therefore, by the Implicit Function Theorem, for every small z the corresponding value (x, y) is determined uniquely. In other words, the intersection $W^c := W^{uc} \cap W^{sc}$ is a smooth manifold of the form

$$(x, y) = \xi(z),$$

where $\xi(0) = 0, \xi'(0) = 0$. The manifold W^c is called *center manifold*. As the intersection of locally invariant manifolds, it is also locally invariant.



Importantly, if the system depends on some parameters ε such that it has form (*) at $\varepsilon = \varepsilon_0$, the center-stable and center-unstable manifolds persist, so does their intersection. In order to see this, one adds formally an equation $d\varepsilon/dt = 0$ to system (*) (where the functions f, g, h may now depend on ε in a smooth way with the only requirement that f, g, h vanish at $\varepsilon = \varepsilon_0$ along with the first derivatives with respect to (x, y, z); the equation $d\varepsilon/dt = 0$ simply means that the values of the parameters ε do not change with time). Then $(x = 0, y = 0, z = 0, \varepsilon = \varepsilon_0)$ is an equilibrium of the augmented system, and the linearisation matrix acquires a zero block corresponding to the new artificial variables ε. Thus, one may consider the parameters ε as part of the z-variables - the Center Manifold Theorem than gives the center-stable, center-unstable and center manifolds in the form

$$y = \varphi(x, z, \varepsilon), \qquad x = \psi(y, z, \varepsilon), \qquad (x, y) = \xi(z, \varepsilon).$$

By the same theorem, for every ε close to ε_0, every orbit $(x(t), y(t), z(t))$ which stays in a small neighbourhood of $(x, y, z) = 0$ for all times must lie entirely in the center manifold, i.e. the following identity is satisfied for all t:

$$(x(t), y(t)) = \xi(z(t), \varepsilon).$$

It is said that the non-center variables (x and y) are *enslaved* by z: they rigidly follow the z-variables. Once we know the function ξ, we do not need to solve the full system of differential equations (*), we need to solve only the z part, restricted to the center manifold $(x, y) = \xi(z, \varepsilon)$:

$$\frac{dz}{dt} = Cz + h(\xi(z, \varepsilon), z, \varepsilon). \qquad (*\mathrm{C})$$

While solving or analysing system (*C) may still be not easy, we often have here a tremendous reduction in the dimension of the problem: from $(n + m + k)$-dimensional system (*) to an n-dimensional system (*C). The number n equals to the number of the eigenvalues on the imaginary axis. In the most typical case there are no such eigenvalues at all (an eigenvalue is, essentially, a point situated anywhere in the complex plane, and this is a probability zero event when such a point falls on a given line). If the system depends on a number of parameters, then the eigenvalues of the linearisation matrix at the equilibrium also depend on the parameters and for some parameter values some eigenvalues can be put on the imaginary axis. In general, to have p zero eigenvalues and q pairs of non-zero pure-imaginary

eigenvalues, one needs at least $(p+q)$-parameters (on the complex plane, each parameter governs a motion either of one real eigenvalue or of a pair of complex-conjugate ones). As we see, for a parameter-dependent family of systems of differential equations the complexity of local bifurcations is restricted foremost by the number of parameters, even if the number of variables is arbitrarily high. Caution: these arguments are valid in the general case, the system under consideration may, however, have additional symmetries or some other special structures (e.g. Hamiltonian one) - in that case the symmetry or the structure may force the eigenvalues to reside on the imaginary axis, and they may remain there even when parameters vary (provided the special structure remains intact).

In applications, in order to make a reduction to the center manifold, we need to know the function ξ: it enters the right-hand sides of (*C); the function h is also there - however h is a given function, i.e. we know it, while the only thing we know about the function ξ so far is that it exists. Recall, however, that by the Center Manifold Theorem, ξ vanishes at $(z=0, \varepsilon=\varepsilon_0)$ with the first derivatives, i.e. its Taylor expansion at this point starts with qudratic terms with respect to z and $(\varepsilon-\varepsilon_0)$. As, by definition, the function h also has no linear terms in the Taylor expansion, it follows that $h(\xi(z,\varepsilon),z,\varepsilon)$ coincides with $h(0,z,\varepsilon)$ up to cubic and higher order terms. Thus we may write system (*C) as

$$\frac{dz}{dt} = Cz + h(0,z,\varepsilon) + \text{ cubic and higher order terms.} \qquad (**C)$$

Thus, if we may perform the analysis of local dynamics and bifurcations without knowing the cubic terms, (**C) is a good approximation for the system on the center manifold.

Sometimes we need a better approximation. For example, when there is a pair of non-zero eigenvalues on the imaginary axis, system (*C) is 2-dimensional and the matrix C has these two pure-imaginary eigenvalues. Thus the system on the center manifold undergoes the Andronov-Hopf bifurcation - and we know that understanding of what is going on there requires knowing the terms of the Taylor expansion up to the 3d order at least. In order to obtain the necessary better approximations to the center manifold, one may choose the following way. First, bring system (*) to a normal form up to a certain order r; namely, we will kill all non-resonant terms of orders

less or equal to r in the functions f and g. When this is done, the function ξ (whose graph $(x, y) = \xi(z, \varepsilon)$ is the center manifold) will vanish at $z = 0$ along with all derivatives with respect to z up to the order r, i.e.

$$\xi = O(\|z\|^{r+1}).$$

To see this, we recall that a monomial $Kx_1^{r_1} \cdots x_k^{r_k} \cdot y_1^{r_{k+1}} \cdots y_m^{r_{k+m}} \cdot z_1^{r_{k+m+1}} \cdots z_n^{r_{k+m+n}}$ in the equation for $\frac{d}{dt}x_j$ is resonant if

$$\lambda_j = \lambda_1 r_1 + \ldots + \lambda_k r_k + \gamma_1 r_{k+1} + \ldots + \gamma_m r_{k+m} + \eta_1 r_{k+m+1} + \ldots + \eta_n r_{k+m+n}.$$

Since $Re\ \lambda < 0$ while $Re\ \gamma > 0$ and $Re\ \eta = 0$, it follows that at least one of the indices $r_1, \ldots, r_k$ is positive, i.e. those monomials in the function f which do not contain x-variables are non-resonant and, hence, can be killed by normalising transformations. Thus, when the equations for $\frac{d}{dt}x$ in (*) are brought to the normal form up to the order r, all monomials of order less or equal to r in the Taylor expansion of the function f will contain x-variables, i.e. we can write

$$f(x, y, z) = \tilde{f}(x, y, z)x + O(\|x, y, z\|^{r+1}),$$

where $\tilde{f}$ vanishes at zero. Imagine for a moment that we just drop the $O(\|x, y, z\|^{r+1})$-terms. Then the first line in (*) will recast as

$$\frac{d}{dt}x = (A + \tilde{f}(x, y, z))x. \qquad (*\mathrm{x})$$

In this case, $x = 0$ will be an invariant manifold (as $x = 0$ solves this equation). Moreover, one can show (we will make a similar computation below) that if $x(t)$ is non-zero, it will decay exponentially all the time $\tilde{f}$ remains small, i.e. all the time the orbit stays in a small neighbourhood of the equilibrium. Namely, for any orbit which remains in a small neighbourhood of the equilibrium for all negative times, we will have $\|x(0)\| \leq \|x(t)\|e^{-\alpha|t|}$, where t is any negative number and α is some positive coefficient. As $x(t)$ remains bounded, we may take a limit $t \to -\infty$ and get $x(0) = 0$, i.e. every orbit which remains in a small neighbourhood of the equilibrium for all negative times must lie in the invariant manifold $x = 0$. This means that if the x-equation in (*) has the form (*x), then the manifold W^{cu} will be just $x = 0$. Adding $O(\|x, y, z\|^{r+1})$-terms to the right-hand side of (*x) will only change the equation of W^{cu} to $x = O(\|y, z\|^{r+1})$ (see a proof in

Shilnikov,Shilnikov,Turaev,Chua, Ch.5). Analogously, when the y-equation in (*) is brought to the normal form up to the order r, the equation of W^{cs} is $y = O(\|x, z\|^{r+1})$. By taking an intersection of W^{cu} with W^{cs} we find that the equation for W^c will be $(x, y) = O(\|z\|^{r+1})$ indeed. The system on the center manifold takes then the form

$$\frac{dz}{dt} = Cz + h(0, z, \varepsilon) + O(\|z\|^{r+2})$$

(we get $(r+2)$ instead of $(r+1)$ as $h'(0) = 0$). As we see, the method of noemal forms gives us an algorithm of computing the system on the center manifold with arbitrary accuracy.

Example. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = -x + z^2, \\ \\ \dfrac{dz}{dt} = x^2. \end{cases}$$

The matrix of the linear part at the zero equilibrium is $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, so there is one eigenvalue $\lambda = -1$ to the left of the imaginary axis and one critical eigenvalue $\eta = 0$. The center manifold is, hence, one-dimensional. It is tangent to the z-axis. Suppose we need an approximation of order 6 to the system on the center manifold. In order to find it, we start making normalising transformations in the first equation. The z^2 term in this equation is non-resonant, so it can be killed by a coordinate transformation $x_{new} = x + az^2$ with a coefficient a which has to be found. We have $\frac{d}{dt}x_{new} = \frac{d}{dt}x + 2az\frac{d}{dt}z = -x + z^2 + 2azx^2 = -x_{new} + (a+1)z^2 + 2az(x_{new} - az^2)^2$. As we see, by choosing $a = -1$ we will kill the z^2 term indeed. The system will take the following form (we omit the index "new"):

$$\begin{cases} \dfrac{dx}{dt} = -x - 2z(x + z^2)^2 = -(1 + 2xz + 4z^3)x - 2z^5, \\ \\ \dfrac{dz}{dt} = (x + z^2)^2. \end{cases}$$

If we drop the term $(-2z^5)$ in the x-equation, the center manifold will be $x = 0$. Therefore, when we restore this term, we will get the center manifold

$x = O(z^5)$. Plugging this into the z-equation will give

$$\frac{dz}{dt} = z^4 + O(z^7)$$

which is the sought 6th order approximation to the system on the center manifold. If we need a better approximation, we may continue killing the x-independent terms in the x-equation. Thus by putting $x_{new} = x + bz^5$ we will get $\frac{d}{dt}x_{new} = \frac{d}{dt}x + 5bz^4\frac{d}{dt}z = -(1+2xz+4z^3)x - 2z^5 + 5bz^4(x+z^2)^2 = -(1+\ldots)x_{new} + (b-2)z^5 + O(z^8)$. As we see, by choosing $b = 2$ we will kill the $O(z^5)$-term and this will give us the center manifold equation $x_{new} = O(z^8)$. In the old variables we get $x = -2z^5 + O(z^8)$. Plugging this into the equation for $\frac{d}{dt}z$ we find

$$\frac{d}{dt}z = (z^2 - 2z^5 + O(z^8))^2 = z^4 - 4z^7 + O(z^{10}),$$

which is the 9th order approximation for the system on the center manifold.

Exercise Consider the system

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = z, \\ \dfrac{dz}{dt} = -x - y - z + x^2 + az^3. \end{cases}$$

Check that the zero equilibrium has a pair of pure imaginary eigenvalues. Make a linear coordinate transformation which makes the linear part of the system block-diagonal. Make normal form transformations which allow to find a 3d order approximation for the system on the center manifold. Compute the first Lyapunov value as a function of a. For which values of a a stable periodic orbit can be born at bifurcations? (Hint: to produce a stable periodic orbit at the Andronov-Hopf bifurcation the first Lyapunov value has to be non-positive.)

In the analysis of a finite time behaviour near an equilibrium a helpful method is the local *straightening of invariant manifolds*. Namely, if the equation of the center-stable manifold is $y = \varphi(x, z)$ and $x = \psi(y, z)$, then one can introduce new coordinates $x_{new} = x - \psi(y, z)$, $y_{new} = y - \varphi(x, z)$. In these coordinates the equations of W^{cs} and W^{cu} will be, respectively, $y_{new} = 0$ and $x_{new} = 0$. The invariance of the manifold $W^{cs} : y_{new} = 0$ means that if $y_{new} = 0$ at some t, it remains equal to zero for all t (for which the orbit stays in a small neighbourhood of the equilibrium). Therefore, $y_{new} = 0$ implies $\frac{d}{dt} y_{new} = 0$. Analogously, $x_{new} = 0$ implies $\frac{d}{dt} x_{new} = 0$ by virtue of the invariance of W^{cu}. In other words, in the new coordinates, the right-hand side of the x-equation of (*) vanish identically at $x = 0$ and the right-hand side of the y-equation of (*) vanish identically at $y = 0$. Thus, system (*) recasts as follows:

$$\begin{cases} \dfrac{dx}{dt} = (A + \tilde{f}(x, y, z))x \\ \\ \dfrac{dy}{dt} = (B + \tilde{g}(x, y, z))y \qquad (**) \\ \\ \dfrac{dz}{dt} = Cz + h(x, y, z) \end{cases}$$

where $\tilde{f}$ and $\tilde{g}$ vanish at zero, i.e. they are small in a small neighbourhood of zero. Recall the following fact:

Lemma Given a matrix A with the eigenvalues $\lambda_1, \ldots, \lambda_k$ and a number $\bar{\lambda}$ such that $\bar{\lambda} > \max\{Re\lambda_1, \ldots, Re\lambda_k\}$, one can choose coordinates $x = (x_1, \ldots, x_k)$ such that the inner product (x, Ax) satisfies

$$Re(x, Ax) \leq \bar{\lambda}(x, x).$$

Proof. In the standard Jordan base, for the jth component of the vector Ax, we have $(Ax)_j = \lambda_j x_j + \delta_j x_{j+1}$ where δ_j is either 0 or 1. Take a sufficiently small $\varepsilon > 0$ and introduce new coordinates x by the rule $x_{j,new} = x_j \varepsilon^{-j}$. In the new coordinates we get $(Ax)_j = \lambda_j x_j + \varepsilon \delta_j x_{j+1}$. Thus $Re(x, Ax) =$

$$= Re \sum_{j=1}^{k} x_j^* (Ax)_j = \sum_{j=1}^{k} (\lambda_j |x_j|^2 + \varepsilon \delta_j x_j^* x_{j+1}) \leq \sum_{j=1}^{k} (Re\lambda_j |x_j|^2 + \frac{\varepsilon}{2}(|x_j|^2 + |x_{j+1}|^2)) \leq$$

$$\leq (\max\{Re\lambda_1, \ldots, Re\lambda_k\} + \varepsilon) \sum_{j=1}^{k} |x_j|^2, \text{ which proves the lemma. } \square$$

It follows that in system (**), all the time the orbit stay in a small neighbourhood of zero, we have

$$\frac{d}{dt}x^2 = 2Re(x, \frac{dx}{dt}) = 2Re(x, (A + \tilde{f})x) \leq 2(\bar{\lambda} + \beta)x^2$$

where β is the maximum of $\|\tilde{f}\|$ over the small neighbourhood of zero, i.e. it can be taken arbitrarily small if the size of the neighbourhood if sufficiently small. Therefore, as the spectrum of the matrix A lies strictly to the left of the imaginary axis, the coefficient $2(\bar{\lambda} + \beta)$ is strictly negative. This means that x^2 decays exponentially all the time the orbit stays in a small neighbourhood of zero, i.e. $\|x(t)\| = \sqrt{x(t)^2}$ decays exponentially:

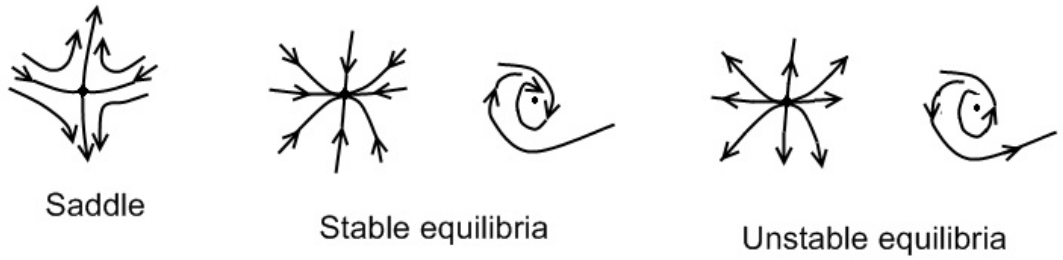
$$\|x(t)\| \leq \|x(0)\|e^{-\alpha t} \quad (t \geq 0)$$

where $\alpha > 0$. Analogously, $\|y(t)\|$ increases exponentially all the time the orbit stays in a small neighbourhood of zero:

$$\|y(t)\| \leq \|y(0)\|e^{-\alpha|t|} \quad (t \leq 0).$$

As we see, the orbits in $W^{cs} : \{y = 0\}$ exponentially tend to $W^c : \{x = 0, y = 0\}$ as time increases, provided $z(t)$ remains small. The orbits in $W^{cu} : \{x = 0\}$ exponentially tend to W^c with the decrease of time, all the time $z(t)$ is small. This allows us to recover completely the picture of behaviour near *hyperbolic equilibria*. These are the equilibria which do not have critical exponents. Therefore, the center manifold is zero-dimensional here, i.e. it is just the equilibrium itself. The center-stable and center-unstable manifolds are called resp. stable and unstable manifolds in this case, W^s and W^u. Every orbit in W^s exponentially tends to the equilibrium as $t \to +\infty$ while every orbit in W^u exponentially tends to the equilibrium as $t \to -\infty$. Orbits which do not lie in W^u or in W^s leave a small neighbourhood of the equilibrium both as $t \to +\infty$ and $t \to -\infty$. The dimension of the stable manifold equals to the number of eigenvalues to the left of the imaginary axis, the dimension of the unstable manifold equals to the number of eigenvalues to the right of the imaginary axis. When both these numbers are non-zero, i.e. when there are eigenvalues both to the left and to the right of the imaginary axis, the hyperbolic equilibrium is called *a saddle*. If all the eigenvalues lie to the left of the imaginary axis, then the dimension of the stable manifold equals to the dimension of the phase space, so the stable manifold is a whole

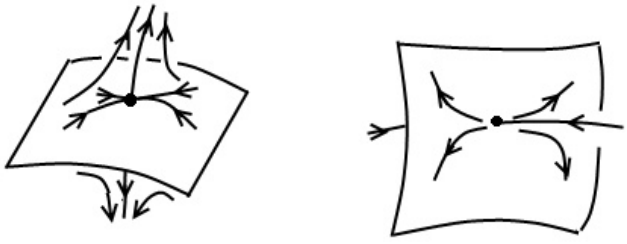
neighbourhood of the equilibrium state. This means that all the orbits from a small neighbourhood of such equilibrium exponentially tend to it as $t \to +\infty$. Such equilibrium is called *exponentially stable*. In the opposite case where all the eigenvalues have positive real parts the equilibrium is *exponentially unstable* - all the orbits from its small neighbourhood exponentially tend to it as $t \to -\infty$. Note that if the equilibrium is hyperbolic, it remains hyperbolic for every close system: as parameters of the system vary, the eigenvalues change continuously, i.e. they cannot just jump to the imaginary axis, or to another half of the complex plane - so the dimensions of the stable and unstable manifolds also stay constant. By this reason, hyperbolic equilibria (saddle, stable and unstable ones) are called also *structurally stable*.



Saddle

Stable equilibria

Unstable equilibria



Saddles in 3D

The simplest case of non-hyperbolic equilibria corresponds to the case where there is only one critical eigenvalue $\eta = 0$. In this case the center manifold is one-dimensional and matrix C in (**) with the zero eigenvalue is 1×1, i.e. it is just zero. The system on the center manifold is written as

$$\frac{dz}{dt} = \phi(z)$$

where $\phi(0) = \phi'(0) = 0$. Let us write the Taylor expansion of $\phi(z)$ at zero:

$$\phi(z) = \ell_2 z^2 + \ell_3 z^3 + \dots.$$

The numbers ℓ_k are called *Lyapunov values*. In the most typical case $\ell_2 \neq 0$, however more degenerate cases can also be considered. Let $\ell_q \neq 0$ be the first non-zero Lyapunov value. We have then $\phi^{(q)}(0) \neq 0$. When we consider bifurcations near the equilibrium, we imbed our system in a family of systems depending on some parameters ε. The system on the center manifold will be

$$\frac{d}{dt} z = \phi(z, \varepsilon).$$

Its dynamics is very simple for every given ε: zeros of $\phi(z)$ correspond to equilibria; on any interval between two consecutive equilibria $\phi(z)$ has a definite sign; if $\phi > 0$, then $z(t)$ is a monotonically increasing function of time, i.e. the orbit tends to the right end of the interval as $t \to +\infty$ and to the left end as $t \to -\infty$; if $\phi < 0$, then $z(t)$ decreases from the right end of the interval to the left end. Thus, the problem reduces to determining zeros of ϕ and their behaviour as the parameters change. Since $\frac{\partial^{q-1}}{\partial z^{q-1}}\phi(0, \varepsilon_0) = 0$ and $\frac{\partial}{\partial z}\frac{\partial^{q-1}}{\partial z^{q-1}}\phi(0, \varepsilon_0) \equiv \frac{\partial^q}{\partial z^q}\phi(0, \varepsilon_0) \neq 0$ by assumption, for every ε close to ε_0 there exists $z^*(\varepsilon)$ uniquely defined by the condition

$$\frac{\partial^{q-1}}{\partial z^{q-1}}\phi(z^*, \varepsilon) = 0.$$

Thus, if we move the origin into the point z^*, the coefficient of $(q-1)$th power of z in the Taylor expansion of the right-hand side will be zero for all ε (and the coefficient of the qth power will remain non-zero). So, the system on the center manifold for small ε is written as follows:

$$\frac{d}{dt} z = \mu_0 + \dots + \mu_{q-2} z^{q-2} + \ell_q z^q + o(z^q)$$

where μ_j are small (they vanish at $\varepsilon = \varepsilon_0$) and $\ell_q \neq 0$. As the qth derivative of the right-hand side does not vanish at small z (it is equal to $q!\ell_q + o(1)$), *the right-hand side cannot have more than q roots*,
and, in fact, q roots may indeed exist for appropriate (small) values of μ_j. Thus,
bifurcations of an equilibrium with one critical (zero) eigenvalue and $(q-2)$ zero first Lyapunov values: $\ell_2 = \ldots = \ell_{q-1} = 0$, $\ell_q \neq 0$, lead to a birth of up to q equilibrium states.

Example System

$$\begin{cases} \dfrac{dx}{dt} = \varepsilon_1 + x + y + 2x^2 - y^2, \\ \dfrac{dy}{dt} = \varepsilon_2 + x + y + y^2 \end{cases}$$

has at $\varepsilon_1 = \varepsilon_2 = 0$ an equilibrium at zero. The linearisation matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; its eigenvalues are 0 and 2, i.e. the above theory is applicable. In order to find equilibria we may try to find a reduction to the center manifold, or may search for the equilibria directly, by equating the right-hand sides to zero. Namely, the system for the equilibria is

$$0 = \varepsilon_1 + x + y + 2x^2 - y^2, \qquad 0 = \varepsilon_2 + x + y + y^2 + x^3.$$

The first equation, for all small ε_1 and x can be resolved with respect to y:

$$y = -\varepsilon_1 - x - 2x^2 + y^2 = -\varepsilon_1(1-\varepsilon_1) - x(1-2\varepsilon_1) - x^2(1-4(\varepsilon_1+x)) - 2(\varepsilon_1+x)^3 + \ldots$$

where the dots stand for fourth and higher order terms in x and ε_1. By plugging this expression for y into the second equation, we obtain the following equation for x:

$$\varepsilon_2 - \varepsilon_1(1 - 2\varepsilon_1 + 4\varepsilon_1^2) + 4\varepsilon_1(1 - 3\varepsilon_1)x - 4\varepsilon_1 x^2 + 5x^3 + \ldots = 0.$$

This equation cannot have more than three small roots (as the third derivative of the left-hand side is nonzero at small x and ε). At $\varepsilon_1 = \varepsilon_2 = 0$ the left-hand side is $5x^3 + O(x^4)$, this function changes sign when x changes sign. Hence, at sufficiently small ε the left-hand side will still change sign when

x varies from negative to positive values. Therefore, one root always exists. The boundary between the regions (in the $(\varepsilon_1, \varepsilon_2)$-plane) corresponding to 1 and 3 roots (i.e. to 1 and 3 equilibria of the system under consideration) is given by the curve which corresponds to a multiple root - at this root the derivative of the function vanishes along with the function itself. We obtain the following equations for the multiple root:

$$\varepsilon_2 - \varepsilon_1(1 - 2\varepsilon_1 + 4\varepsilon_1^2) + 4\varepsilon_1(1 - 3\varepsilon_1)x - 4\varepsilon_1 x^2 + 5x^3 + \ldots = 0,$$

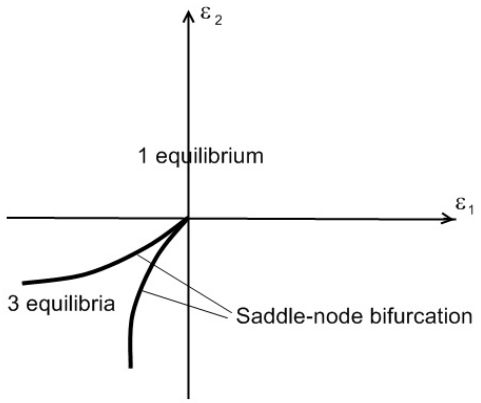
$$4\varepsilon_1(1 - 3\varepsilon_1) - 8\varepsilon_1 x + 15x^2 + \ldots = 0.$$

From the second equation, we find

$$x = \pm\sqrt{\frac{-4\varepsilon_1}{15}} + O(\varepsilon_1).$$

By plugging this into the first equation, we find the following formula for the bifurcation curve which separates the regions of 1 and 3 equilibria:

$$\varepsilon_2 = \varepsilon_1 \mp \frac{8}{3}\varepsilon_1\sqrt{\frac{-4\varepsilon_1}{15}} + O(\varepsilon_1^2), \quad \varepsilon_1 < 0.$$



Exercise Determine, how many equilibria can be born at the bifurcations of the zero equilibrium of the following system:

$$\frac{dx}{dt} = x + y + x^3, \quad \frac{dy}{dt} = x + z + z^4, \quad \frac{dz}{dt} = 2x + y + z - y^3.$$

Hint: write a system for determining equilibria, reduce it to one equation and find how many derivatives vanish at zero.

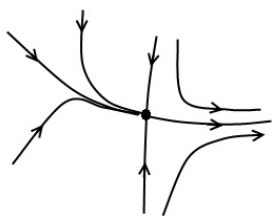
In the most typical case of the equilibrium state with one critical eigenvalue we have $\ell_2 \neq 0$. In this case such equilibrium is called a *saddle-node* (or, if there exist eigenvalues both with positive real parts and negative real parts, it is also called a *saddle-saddle*). When all non-critical eigenvalues have negative real part (so there are no y-variables in (**)), a neighbourhood of the saddle-node is divided by an invariant manifold tangent to the $\{z = 0\}$-space into two halves: one half is similar to a neighbourhood of a saddle, and the other half looks like a neighbourhood of a stable equilibrium (see in Shilnikov,Shilnikov,Turaev,Chua, Ch.9). This picture, essentially, follows from the exponential convergence of the x-variables to the center manifold $x = 0$ and from the fact that the motion on the center manifold is just a slow drift in one direction: the system on the center manifold is

$$\frac{dz}{dt} = \ell_2 z^2 + o(z^2),$$

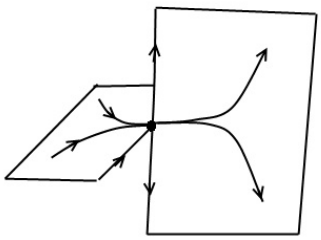
so if $\ell_2 > 0$, then $z(t)$ increases monotonically at $z \neq 0$, and if $\ell_2 < 0$, then $z(t)$ decreases monotonically at $z \neq 0$. Bifurcations of the saddle-node are described by the system

$$\frac{dz}{dt} = \mu + \ell_2 z^2 + o(z^2) :$$

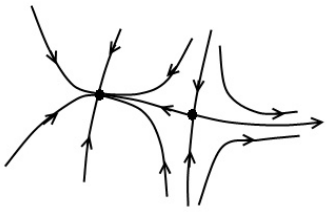
when $\mu\ell_2 < 0$ there are two equilibrium states, which collide at $\mu = 0$ and disappear at $\mu\ell_2 > 0$.



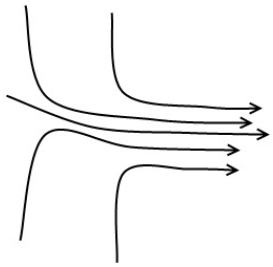
Saddle-node



Saddle-saddle



$\mu\ell_2<0$



$\mu\ell_2>0$