1. (i) Consider the system

$$\begin{cases} \dot{x} = x + y, \\ \dot{y} = x^3 - y^2. \end{cases}$$

Find all equilibrium states in this system and determine how many periodic orbits can be born at the bifurcations of these equilibria at small perturbations of the system.

(ii) Consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon y + x^3. \end{cases}$$

Show that the zero equilibrium is stable at $\varepsilon = 0$. How many periodic orbits exists in a small neighbourhood of this equilibrium as ε becomes positive?

(iii) Consider the system

$$\begin{cases} \dot{x} = 2y + u^2, \\ \dot{y} = 2x - 3x^2 - y(y^2 - x^2 + x^3), \\ \dot{z} = -z + 3xu, \\ \dot{u} = -u + z^2. \end{cases}$$

What is the maximal possible number of periodic orbits that can be born from the homoclinic loop $\{y = \pm x\sqrt{1-x}(x \in [0,1]), z = u = 0\}$ at small perturbations of the system?

2. (i) Write the complex normal form up to the order three for the restriction of the system

$$\begin{cases} \dot{x} = y + \varepsilon, \\ \dot{y} = -x + 2uy, \\ \dot{u} = -2u + 16\varepsilon y + 8y^2. \end{cases}$$

onto the center manifold near the zero equilibrium at $\varepsilon = 0$.

(ii) Compute the first Lyapunov coefficient. Let U be a sufficiently small neighbourhood of zero. How many stable and unstable periodic orbits exist in U at small $\varepsilon > 0$ and at small $\varepsilon < 0$?

3. Study bifurcations in the system

$$\begin{cases} \dot{x} = a - x - y^2, \\ \dot{y} = b - y - 2xy. \end{cases}$$

as parameters a and b vary. Namely, do the following:

(i) Show that the system cannot have periodic orbits.

(ii) Show that the only bifurcations of equilibria of this system correspond to a single zero eigenvalue of the linearisation matrix.

(iii) Draw the bifurcation curve on the (a, b) plane.

(iv) For each of the regions, into which this curve divides the (a, b) plane, determine the number of equilibria and their stability.

4. Consider the following one-dimensional map

$$\bar{x} = a - \sqrt[3]{x}.$$

(i) Determine the set of a values for which the map has a stable fixed point. Which bifurcation corresponds to the boundary of this set? Is the fixed point stable at the bifurcation moment?

(ii) Show that points of period 2 are always unstable in this map.

(iii) Find all values of a for which the map has exactly one orbit of period 2.

Solutions

1. (i) Consider the system

$$\begin{cases} \dot{x} = x + y, \\ \dot{y} = x^3 - y^2. \end{cases}$$

Find all equilibrium states in this system and determine how many periodic orbits can be born at the bifurcations of these equilibria at small perturbations of the system.

Solution (7 points, seen similar). The equilibria are given by $x + y = 0$, $x^3 = y^2$, which gives $O_1(0,0)$ and $O_2(1,-1)$. The linearisation matrix at O_1 is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. It has a single zero eigenvalue - no periodic orbits can be born from such equilibrium. The linearisation matrix at O_2 is $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$. Its trace and determinant are both different from zero, so it has no eigenvalues on the imaginary axis. Hence, this equilibrium is structurally stable, e.g. no periodic orbits can be born from such equilibrium.

(ii) Consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon y + x^3. \end{cases}$$

Show that the zero equilibrium is stable at $\varepsilon = 0$. How many periodic orbits exists in a small neighbourhood of this equilibrium as ε becomes positive?

Solution (6 points, seen similar). At $\varepsilon = 0$ the system is Hamiltonian, with the integral $H = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}$. This is a Lyapunov function with the minimum at zero, so zero is a stable equilibrium. It is surrounded by closed orbits (the level lines of H). As ε becomes non-zero, the real part of the eigenvalues of the linearisation matrix at the zero equilibrium leaves the imaginary axis, so by Hopf theorem no periodic orbits remains at small non-zero ε (or notice that $H(x(t), y(t))$ is strictly increasing function of t at $\varepsilon > 0$).

(iii) Consider the system

$$\begin{cases} \dot{x} = 2y + u^2, \\ \dot{y} = 2x - 3x^2 - y(y^2 - x^2 + x^3), \\ \dot{z} = -z + 3xu, \\ \dot{u} = -u + z^2. \end{cases}$$

What is the maximal possible number of periodic orbits that can be born from the homoclinic loop $\{y = \pm x\sqrt{1-x}(x \in [0,1]), z = u = 0\}$ at small perturbations of the system?

Solution (7 points, seen similar). The eigenvalues of the linearisation matrix at the zero equilibrium are $(2, -1, -1, -2)$. The saddle value is positive, and by a small perturbation one can make the nearest to the imaginary axis eigenvalue complex - then infinitely many periodic orbits will exist near the homoclinic loop by Shilnikov theorem.

2. (i) Write the complex normal form up to the order three for the restriction of the system

$$\begin{cases} \dot{x} = y + \varepsilon, \\ \dot{y} = -x + 2uy, \\ \dot{u} = -2u + 16\varepsilon y + 8y^2. \end{cases}$$

onto the center manifold near the zero equilibrium at $\varepsilon = 0$.

Solution (14 points, seen similar). At $\varepsilon = 0$ the system is

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + 2uy \\ \dot{u} = -2u + 8y^2 \end{cases}$$

The linearisation matrix at zero is $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. The eigenvalues are $(i, -i, -2)$. The (x, y) plane is the invariant subspace that corresponds to the eigenvalues $\pm i$. Therefore, the center manifold is given by $u = f(x, y)$. Let $z = x - iy$. the system takes the form

$$\begin{cases} \dot{z} = iz + u(z - z^*) \\ \dot{u} = -2u - 2(z - z^*)^2 \end{cases}$$

The normalising transformation

$$v = u + \frac{z^2}{1+i} - 2zz^* + \frac{(z^*)^2}{1-i}$$

kills all the quadratic terms in the last equation, so the center manifold W^c is $v = O(|z|^3)$. This gives us

$$W^c = \{u = -\frac{z^2}{1+i} + 2zz^* - \frac{(z^*)^2}{1-i} + O(|z|^3)\}.$$

Thus the system on W^c is

$$\dot{z} = iz - \left[\frac{z^2}{1+i} - 2zz^* + \frac{(z^*)^2}{1-i}\right](z - z^*) + O(|z|^4).$$

By dropping the resonant cubic terms and terms of the orders 4 and higher, we obtain the sought normal form:

$$\dot{z} = iz + (2 - \frac{1}{1+i})z^2 z^*.$$

(ii) Compute the first Lyapunov coefficient. Let U be a sufficiently small neighbourhood of zero. How many stable and unstable periodic orbits exist in U at small $\varepsilon > 0$ and at small $\varepsilon < 0$?

Solution (6 points, partly unseen). The first Lyapunov coefficient

$$L = \mathrm{Re}(2 - \frac{1}{1+i}) = \frac{3}{2}.$$

As $L > 0$, exactly one unstable periodic orbit is born for those values of ε, for which the equilibrium is exponentially stable. The equilibrium is given by $(x = 8\varepsilon^3, y = -\varepsilon, u = -4\varepsilon^2)$. The linearisation matrix at the equilibrium is $\begin{pmatrix} 0 & 1 & 0 \\ -1 & -8\varepsilon^2 & -2\varepsilon \\ 0 & 0 & -2 \end{pmatrix}$, the characteristic polynomial is $P(\lambda) = (\lambda + 2)(\lambda^2 + 8\varepsilon^2\lambda + 1)$. At $\varepsilon \neq 0$ all the eigenvalues lie to the left of the imaginary axis, i.e. the equilibrium is exponentially stable. Thus, one unstable periodic orbit exists in U for all small $\varepsilon \neq 0$.

3. Study bifurcations in the system

$$\begin{cases} \dot{x} = a - x - y^2, \\ \dot{y} = b - y - 2xy. \end{cases}$$

as parameters a and b vary. Namely, do the following:

(i) Show that the system cannot have periodic orbits.

Solution (4 points, seen similar). This is a gradient system with $V(x, y) = -ax - by + \frac{y^2}{2} + \frac{x^2}{2} + xy^2$ being the Lyapunov function.

(ii) Show that the only bifurcations of equilibria of this system correspond to a single zero eigenvalue of the linearisation matrix.

Solution (6 points, seen similar). If (x, y) is an equilibrium, then the linearisation matrix is $\begin{pmatrix} -1 & -2y \\ -2y & -1 - 2x \end{pmatrix}$. Andronov-Hopf bifurcation and double zero bifurcation would correspond to zero trace and non-negative determinant in this matrix. However, when the trace is zero here, the determinant equals to $-1 - 4y^2$, which is strictly negative. Hence, these bifurcations are impossible.

(iii) Draw the bifurcation curve on the (a, b) plane.

Solution (4 points, seen similar). The equilibria are given by $x = a - y^2$ where $b - (1+2a)y + 2y^3 = 0$. Bifurcations correspond to multiple roots of the last equation, i.e. to $1 + 2a = 6y^2$ and $b = 4y^3$. This gives us the bifurcation curve

$$C : \{a = 2(\frac{b}{4})^{2/3} - \frac{1}{2}\}.$$

(iv) For each of the regions, into which this curve divides the (a, b) plane, determine the number of equilibria and their stability.

Solution (6 points, seen similar). The curve C divides the plane into two regions. We denote the region which contains the point $M(a = -1, b = 0)$ as D_1 and its complement as D_2. At M the system has only one equilibrium $(x = -1, y = 0)$. The linearisation matrix at this equilibrium is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so the equilibrium is a saddle. As bifurcations happen only upon crossing the bifurcation curve C, the same is true for every $(a, b) \in D_1$: the system has only one saddle equilibrium. When crossing C, two more equilibria are born, so we have 3 equilibria in D_2. In order to determine their stability, note that on C the determinant $(1 + 2x) - 4y^2$ of the linearisation matrix is zero, which implies $1+2x > 0$, hence the trace $-1-(1+2x)$ of the linearisation matrix is negative. This means that the saddle-node equilibrium, which emerges when $(a, b) \in C$ is stable, hence it decomposes into a saddle and a stable point when crossing into D_2. Altogether we count 2 saddles and 1 stable equilibrium for all $(a, b) \in D_2$.

4. Consider the following one-dimensional map

$$\bar{x} = a - \sqrt[3]{x}.$$

(i) Determine the set of a values for which the map has a stable fixed point. Which bifurcation corresponds to the boundary of this set? Is the fixed point stable at the bifurcation moment?

Solution (8 points, unseen). The right-hand side is a decreasing function, so the map always has a unique fixed point x_0. This point is stable, when its multiplier is less than 1 in the absolute value, i.e. we have

$$a = x0 + \sqrt[3]{x_0}, \qquad -\frac{1}{3}x_0^{-2/3} < 1,$$

which gives

$$|a| > \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{27}} = \frac{4}{3\sqrt{3}}.$$

The boundary corresponds to a multiplier equal to -1, a period-doubling bifurcation. The fixed point is unstable at the bifurcation moment, since the Schwarz derivative is positive:

$$\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 = \frac{1}{x^2}\left[\frac{2}{3}\cdot\frac{5}{3} - \frac{3}{2}\left(\frac{2}{3}\right)^2\right] = \frac{2}{3x^2}.$$

(ii) Show that points of period 2 are always unstable in this map.

Solution (6 points, unseen). Let (x_1, x_2) be an orbit of period 2, $x_1 \neq x_2$.Then

$$\begin{cases} x_2 = a - \sqrt[3]{x_1}, \\ x_1 = a - \sqrt[3]{x_2}. \end{cases}$$

This gives us

$$x_2 - x_1 = \sqrt[3]{x_2} - \sqrt[3]{x_1} \Longrightarrow u_1^2 + u_1 u_2 + u_2^2 = 1 \Longrightarrow (u_1 - u_2)^2 + 3u_1 u_2 = 1,$$

where we denote $u_{1,2} = \sqrt[3]{x_{1,2}}$. This implies

$$\sqrt[3]{x_1 x_2} < \frac{1}{3}.$$

The multiplier of the period-2 orbit equals to

$$(-\frac{1}{3}x_1^{-2/3}) \cdot (-\frac{1}{3}x_2^{-2/3}) = \frac{1}{(3\sqrt[3]{x_1 x_2})^2} > 1,$$

so the orbit is unstable.

(iii) Find all values of a for which the map has exactly one orbit of period 2.

Solution (6 points, unseen). In the map with the decreasing right-hand side, orbits of period 2 are nested and must have alternating stability. As there are no stable orbit of period 2, there can be only one such orbit. Moreover, it may exist only when the fixed point is stable. Thus, by (i), the orbit of period 2 exists at

$$|a| > \frac{4}{3\sqrt{3}}$$

(the orbit is born from the fixed point at the period-doubling bifurcation at $a = \pm\frac{4}{3\sqrt{3}}$).