1. (i) Formulate necessary conditions of structural stability for systems of autonomous differential equations on a plane.

(ii) Show that the system $\begin{cases} \dfrac{dx}{dt} = x + y - x^2 + y^2 \\ \dfrac{dy}{dt} = -2x - y + xy \end{cases}$ is structurally unstable, and that the system $\begin{cases} \dfrac{dx}{dt} = 2x + y - x(x^2 + y^2) \\ \dfrac{dy}{dt} = -x + 2y - y(x^2 + y^2) \end{cases}$ is structurally stable.

(iii) Is the system $\begin{cases} \dfrac{dx}{dt} = xy \\ \dfrac{dy}{dt} = 1 - y^2 - x^2 \end{cases}$ structurally stable or not? Explain why.

2. (i) Draw the bifurcation diagram for the Andronov-Hopf bifurcation with a zero first Lyapunov value and a negative second Lyapunov value. Describe bifurcation curves and indicate which types of different dynamical behaviour correspond to the bifurcation curves and the regions between them.

(ii) Determine how many periodic orbits can be born from the zero equilibrium as ε becomes non-zero: $\begin{cases} \dfrac{dx}{dt} = \varepsilon x + y \\ \dfrac{dy}{dt} = -x + y^2 \end{cases}$

3. How many equilibria and periodic orbits can be born from zero equilibrium at bifurcations of the following systems:

(i) $$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -x + yz, \\ \dfrac{dz}{dt} = -2z + y^2, \end{cases}$$

(ii) $$\begin{cases} \dfrac{dx}{dt} = x - z + y^2, \\ \dfrac{dy}{dt} = x - 2y + z + y^2 + 2x^2 z, \\ \dfrac{dz}{dt} = -2x + 2y + z^2 - y^2 \end{cases}$$

4. The curve $\{y = \pm x\sqrt{1-x} \ \ (x \in (0,1]), \ \ z = 0\}$ is a homoclinic loop of the system

$$\begin{cases} \dfrac{dx}{dt} = -x + 2y + x^2, \\ \dfrac{dy}{dt} = 2x - y - 3x^2 + \dfrac{3}{2}xy, \\ \dfrac{dz}{dt} = -3z. \end{cases}$$

(i) What is the maximal possible number of periodic orbits that can be born from this loop at the bifurcations of this system?

(ii) What is the maximal possible number of periodic orbits that can be born from the homoclinic loop at the bifurcations of the following system:

$$\begin{cases} \dfrac{dx}{dt} = x - 2y - x^2, \\ \dfrac{dy}{dt} = -2x + y + 3x^2 - \dfrac{3}{2}xy, \\ \dfrac{dz}{dt} = -z \end{cases}$$

Answers to Bifurcation Theory exam (M3A24/M4A24), 2009

1. (i) Formulate necessary conditions of structural stability for systems of autonomous differential equations on a plane. (5pts)

1) All equilibria must be hyperbolic
2) For every periodic orbit the multiplier must be different from 1
3) There must be no orbits which connect saddles

(ii) (10pts= 4pts for the first system and 6pts for the second system) Show that the system $\begin{cases} \dfrac{dx}{dt} = x + y - x^2 + y^2 \\ \dfrac{dy}{dt} = -2x - y + xy \end{cases}$ is structurally unstable, and that the system $\begin{cases} \dfrac{dx}{dt} = 2x + y - x(x^2 + y^2) \\ \dfrac{dy}{dt} = -x + 2y - y(x^2 + y^2) \end{cases}$ is structurally stable.

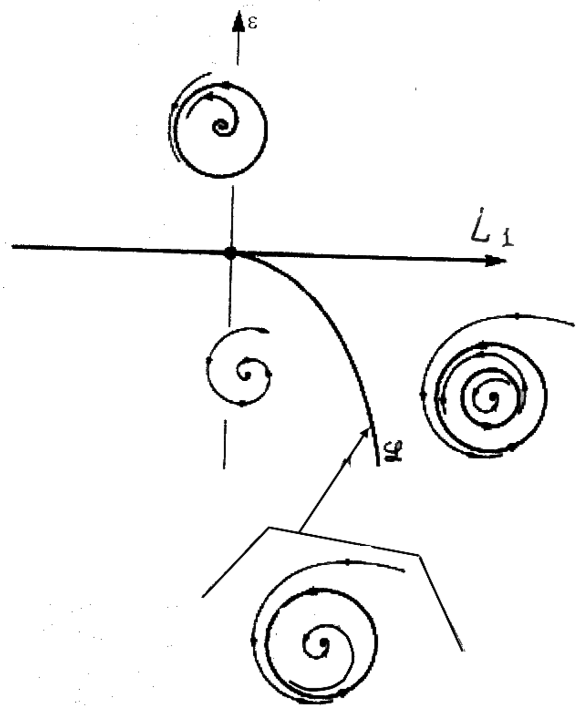
In the first system the linearisation matrix $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ at zero equilibrium has a pair of pure imaginary eigenvalues ($tr = 0$, $det > 0$) $\Longrightarrow$ no structural stability.
The second system in the polar coordinates $x = r\cos\phi, y = r\sin\phi$ acquires the form $\begin{cases} \dfrac{dr}{dt} = 2r - r^3 \\ \dfrac{d\phi}{dt} = -1 \end{cases}$ Obviously, every orbit with $r \neq 0$ tends to $r = \sqrt{2}$. Therefore, the original system has only one equilibrium - at zero, and one periodic orbit - at $x^2 + y^2 = 2$. The linearisation matrix at the equilibrium is $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. As $detA = 5 \neq 0$ and $trA = 4 \neq 0$, there are no eigenvalues on the imaginary axis, so the equilibrium is hyperbolic. Moreover, $detA > 0$, hence the equilibrium is not a saddle, hence there are no saddle connections. Thus, to prove the structural stability it remains to show that the multiplier μ of the periodic orbit is not 1. By plugging $x^2 + y^2 = 2$ in the system we find that the periodic orbit satisfies $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -x$, i.e. $x = \sin t, y = \cos t$ (so the period $T = 2\pi$). By formula $\ln\mu = \int_0^T (\partial f/\partial x + \partial g/\partial y)dt$, we find $\ln\mu = -2\int_0^{2\pi}(x^2 + y^2)_{x=\cos t, y=\sin t}dt = -4\pi \neq 0$. Thus, $\mu \neq 1$, which proves the structural stability.

(iii) (5pts) Is the system $\begin{cases} \dfrac{dx}{dt} = xy \\ \dfrac{dy}{dt} = 1 - y^2 - x^2 \end{cases}$ structurally stable or not? Explain why.

The system is structurally unstable because it has two saddle equilibria $(x = 0, y = -1)$ and $(x = 0, y = 1)$ and the phase curve $\{x = 0, y \in (-1, 1)\}$ connects them.

2. (i) (10 pts) Draw the bifurcation diagram for the Andronov-Hopf bifurcation with a zero first Lyapunov value and a negative second Lyapunov value.

Here ε is the real part of the eigenvalue that crosses the imaginary axis at the bifurcation moment, and L_1 is the first Lyapunov value.



(ii) (10pts) Determine how many periodic orbits can be born from the zero equilibrium as ε becomes non-zero: $\begin{cases} \dfrac{dx}{dt} = \varepsilon x + y \\ \dfrac{dy}{dt} = -x + y^2 \end{cases}$?

At $\varepsilon = 0$ the system has a center at zero: the eigenvalues of the linearisation matrix at zero are pure imaginary and the system is reversible $(t \to -t, y \to -y)$. As ε changes, the real part of the eigenvalues changes with non-zero velocity, hence no periodic orbit exist in a small neighbourhood of zero at small ε (by Hopf theorem).

3. How many equilibria and periodic orbits can be born from zero equilibrium at the bifurcations of the following systems:

(i) (10pts) $\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -x + yz, \\ \dfrac{dz}{dt} = -2z + y^2, \end{cases}$

Let us kill all quadratic terms in the equation for z by means of the transformation $u = z + \alpha x^2 + \beta xy + \gamma y^2$. We have $\frac{du}{dt} + 2u = y^2 + 2\alpha xy + \beta y^2 - \beta x^2 - 2\gamma xy + 2\alpha x^2 + 2\beta xy + 2\gamma y^2 + \text{h.o.t.}$, so the quadratic terms will indeed be killed if $1 + \beta + 2\alpha = 0$, $2\alpha - 2\gamma + 2\beta = 0$, $-\beta + 2\alpha = 0$ which gives us $\alpha = -1/5$, $\beta = -2/5$, $\gamma = -3/5$. After the transformation the system will take the form $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + y(u + x^2/5 + 2xy/5 + 3y^2/5) \\ \dfrac{du}{dt} = -2u + \dots \end{cases}$ where the dots stand for cubic and higher order terms. The center manifold will then have a form $u = f(x, y)$ where f is of a third order at least. Therefore, the system on the center manifold will be $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + yx^2/5 + 2xy^2/5 + 3y^3/5 + \dots \end{cases}$ where the dots stand for the terms of at least 4-th order. Denote $w = (x + iy)/2$, so $x = w + w^*$, $y = i(w^* - w)$. The system will take the form $\frac{dw}{dt} = -iw + i(i(w^* - w)(w + w^*)^2 - 2(w + w^*)(w^* - w)^2 - 3i(w^* - w)^3)/10 + \dots$. The first Lyapunov value L_1 is the real part of the coefficient of the term $w^2 w^*$, i.e. $L_1 = 1$. As $L_1 \neq 0$, no more than 1 periodic orbit can be born at the bifurcation.

(ii) (10pts) $\begin{cases} \dfrac{dx}{dt} = x - z + y^2, \\ \dfrac{dy}{dt} = x - 2y + z + y^2 + 2x^2 z, \\ \dfrac{dz}{dt} = -2x + 2y + z^2 - y^2 \end{cases}$

The linearisation matrix at zero is $\begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ -2 & 2 & 0 \end{pmatrix}$. The matrix has eigenvalues 0, 2 and -3. Since there is only one zero eigenvalue, no periodic orbits can be born at bifurcations. Equilibria are found by equating the right-hand sides to zero. Since

$$\frac{\partial(x - z + y^2, x - 2y + z + y^2 + 2x^2 z)}{\partial(x, z)}|_{(x,y,z)=0} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is a non-degenerate matrix, one can find x and z as functions of y from the first two equations:

$$x = y + y^3 + O(y^4), \ z = y - y^2 - y^3 + O(y^4).$$

Plugging this into the third equation we get $-4y^3 + O(y^4) = 0$. At small perturbations of this equation up to 4 roots can appear, i.e. the maximal possible number of equilibrium states that can be born is 4.

4. The curve $\{y = \pm x\sqrt{1-x}\ (x \in (0,1]),\ z = 0\}$ is a homoclinic loop of the system

$$\begin{cases} \dfrac{dx}{dt} = -x + 2y + x^2, \\ \dfrac{dy}{dt} = 2x - y - 3x^2 + \dfrac{3}{2}xy, \\ \dfrac{dz}{dt} = -3z. \end{cases}$$

(i) (10pts) What is the maximal possible number of periodic orbits that can be born from this loop at the bifurcations of this system?

The linearisation matrix at the saddle at zero has the eigenvalues $(1, -3, -3)$. as the saddle value $\sigma = 1 - 3 = -2$ is negative, at most 1 periodic orbit can be born as the loop splits.

(ii) (10pts) What is the maximal possible number of periodic orbits that can be born from the homoclinic loop at the bifurcations of the following system:

$$\begin{cases} \dfrac{dx}{dt} = x - 2y - x^2, \\ \dfrac{dy}{dt} = -2x + y + 3x^2 - \dfrac{3}{2}xy, \\ \dfrac{dz}{dt} = -z \end{cases}$$

The first two equations are obtained from the first two equations of the first system by time-reversal. So, in the plane $z = 0$ there exists the same homoclinic loop as in the first system. The linearisation matrix at the saddle has now the eigenvalues $(3, -1, -1)$. Since we have a multiple eigenvalue, a small perturbation can create a pair of complex eigenvalues with the real part -1. The saddle value $\sigma = 3 - 1$ will be positive, hence infinitely many periodic orbits will appear near the loop.