Question I. (Do not reprove the local existence and uniqueness theorem, you may use it)

(a) Prove that given any $(n \times n)$-matrix $A(t)$ and an n-vector $b(t)$ that depend continuously on t, every solution $x(t)$ of the equation

$$\frac{dx}{dt} = A(t)x + b(t), \qquad x \in R^n,$$

is defined for all $t \in (-\infty, +\infty)$.

(b) Prove that every solution of the equation

$$\frac{dx}{dt} = \sqrt{x^2+1} + t^2, \qquad x \in R^1,$$

is defined for all $t \in (-\infty, +\infty)$.

(c) Prove that every solution of the system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^7, \qquad (x, y) \in R^2,$$

is defined for all $t \in (-\infty, +\infty)$.

(d) Prove that no solution of the equation

$$\frac{dx}{dt} = x^2 + t^2, \qquad x \in R^1,$$

is defined for all $t \in R^1$.

Solutions (5 points each, all seen or seen similar). I(a): Define $u = x^2$, note that u is a nonnegative scalar. We have

$$\frac{du}{dt} = 2x \cdot \frac{dx}{dt} = 2x \cdot A(t)x + 2x \cdot b(t),$$

so

$$\frac{du}{dt} \leq 2\|A(t)\|\|x\|^2 + 2\|x\|\ \|b(t)\| = 2\|A(t)\|u + 2\|b(t)\|\sqrt{u} \leq (2\|A(t)\| + \|b(t)\| + 1)u.$$

By comparison principle, $u(t) \le v(t)$ at $t \ge 0$ where v solves

$$\frac{dv}{dt} = (2\|A(t)\| + \|b(t)\| + 1)v,$$

i.e.

$$x^2(t) = u(t) \le C \exp[\int_0^t (2\|A(s)\| + \|b(s)\| + 1)ds].$$

Thus, $x(t)$ cannot tend to infinity at a finite positive time. By the change $t \to -t$ we obtain an equation of the same form, so $x(t)$ cannot tend to infinity at any finite negative time too. Hence, $x(t)$ remains defined for all t.

I(b). The right-hand side grows not faster than linearly with x:

$$|\frac{dx}{dt}| \le 2|x| + t^2,$$

so, by comparison principle, the solution is bounded by a solution of a linear equation, which cannot tend to infinity at a finite t (see I(a)). Hence, the solution is globally defined.

I(c). The energy $H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^8}{8}$ is conserved:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} = (x^7 - x)y + y(x - x^7) = 0.$$

Therefore $x(t)$ and $y(t)$ remain bounded for all t (otherwise $H(x,y)$ would grow). Hence, $(x(t), y(t))$ is globally defined.

I(d). If a solution is defined for all t, it is defined for $t \ge 1$. In this interval we have

$$\frac{dx}{dt} \ge x^2 + 1,$$

hence $x(t) \ge v(t)$ where v is a solution of

$$\frac{dv}{dt} = v^2 + 1,$$

i.e. $x(t) \ge \tan(t + C)$ for some C, hence $x(t) \to +\infty$ at a finite moment of time, a contradiction.

Question II. Consider a system $\dfrac{dx}{dt} = f(x), \quad x \in R^n$. Let a bounded and connected region U be defined by condition $F(x) < 0$ where $F : R^n \to R^1$ is a smooth scalar function. The boundary ∂U of the region U is given by $F(x) = 0$. Assume that

$$F'(x) \cdot f(x) < 0$$

everywhere on ∂U.

(a) Prove that every orbit that starts in the closure of U belongs to U for all positive times.

(b) We define the maximal attractor in U as the set A of all points whose orbits stay in U for all $t \in (-\infty, +\infty)$. Prove that A is non-empty, closed, and connected.

(c) Prove that the ω-limit set of each point of the closure of U is a subset of A.

Solutions (a- 6 points, b,c - 7 points each, all seen or seen similar). II(a): For any initial condition x_0 on the boundary of U, we have $\dfrac{d}{dt}F(x(t)) = F'(x) \cdot f(x) < 0$, hence $F(x_t) < F(x_0) = 0$ for $t > 0$ small enough, and $F(x_t) > 0$ at < 0 small enough, i.e. the orbit of x_0 must enter U as t grows and get outside of U as t decreases. In particular, it also shows that once the phase point is inside U its forward orbit cannot leave U: to do this, it must hit the boundary, which would mean, as we just proved, that the orbit was outside of U before, a contradiction.

II(b). Denote X_t the time-t shift map by the flow of the system. If x_t is an orbit, then $x_0 = X_t(x_{-t})$. Thus, by our definition, $x_0 \in A$ if and only if $x_0 \in \bigcap\limits_t X_t(U)$. Since $X_t(U) \subset U$ for all $t > 0$ (by II(a)), it follows also that $U = X_0(U) \subset X_t(U)$ for all $t < 0$, so we may rewrite the definition of A as

$$A = \bigcap_{t>0} X_t(U).$$

Let us prove

$$A = \bigcap_{t>0} X_t(cl(U)).$$

As $U \subset cl(U)$, it follows that

$$A \subseteq \bigcap_{t>0} X_t(cl(U)).$$

On the other hand, given any $t_2 > t_1 \geq 0$, we have $X_{t_2-t_1}(cl(U)) \subset U$ (by II(a)), which implies $X_{t_2}(cl(U)) \subset X_{t_1}(U)$, hence

$$A \supseteq \bigcap_{t>0} X_t(cl(U)).$$

By these two inclusions we get the sought equality. As we have already proved,

$$X_{t_2}(cl(U)) \subset X_{t_1}(U) \subset X_{t_1}(cl(U))$$

for any $t_2 > t_1 > 0$, hence A is the intersection of an ordered family of nested closed, bounded, connected sets. Thus, A is non-empty, closed an connected.

II(c). By definition, if x_t is the orbit of x_0, then $y \in \Omega(x_0) \iff y \in \bigcap_{t>0} cl(\bigcup_{\tau>0} x_{t+\tau})$. As we have shown, $x_0 \in cl(U)$ implies that $x_\tau \in U$ for all $\tau > 0$, hence $\bigcup_{\tau \geq 0} x_{t+\tau} \subset X_t(U)$. This immediately gives us

$$y \in \Omega(x_0) \Longrightarrow y \in \bigcap_{t>0} X_t(cl(U)) = A.$$

Question III. (a) Prove that the system

$$\frac{dx}{dt} = x(1-x^2-y^2)-y+\frac{1}{2}xy, \qquad \frac{dy}{dt} = y(1-x^2-y^2)+x+y^2+x^22, \qquad (x,y) \in R^2,$$

has at least one periodic orbit. (Hint: use polar coordinates.)

(b) Prove that every orbit of the system

$$\begin{cases} \dfrac{dx}{dt} = 2x - y - 4x^3, \\ \dfrac{dy}{dt} = -x - 2y - z, \\ \dfrac{dz}{dt} = -y - 2z, \qquad\qquad (x, y, z) \in R^3, \end{cases}$$

tends to an equilibrium as $t \to +\infty$. How many orbits does the attractor of this system contain?

Solutions (10 points each; a - unseen, b - seen similar). III(a). Introduce polar coordinates: $x = r\cos\phi$, $y = r\sin\phi$.

$$\frac{dr}{dt} = \cos\phi\frac{dx}{dt} + \sin\phi\frac{dy}{dt} = r - r^3 + r^2\sin\phi$$

$$\frac{d\phi}{dt} = \frac{1}{r}(\cos\phi\frac{dy}{dt} - \sin\phi\frac{dx}{dt}) = 1 + \frac{1}{2}r\cos\phi.$$

As we see, $r'(t) > 0$ at small $r > 0$ and $r'(t) < 0$ at all large r, so the ω-limit set of any non-zero point must be finite and lie at non-zero r. There can be no equilibria at $r \neq 0$: if $\dot\phi = 0$, then $r \geq 2$, then $\dot r \leq r + r^2 - r^3 \leq -2$, i.e. $\dot\phi$ and $\dot r$ cannot vanish simultaneously. Now, by the Poincare-Bendixson theorem, the ω-limit set of any non-zero initial condition is a periodic orbit.

III(b). This is a gradient system defined by the potential $V(x, y, z) = x^4 - x^2 + xy + y^2 + yz + z^2$. As $V \to +\infty$ as $(x, y, z) \to \infty$, the potential V is a Lyapunov function. Therefore, the global attractor exists and consists of equilibria and the orbits that connect them. The equilibria are found as follows: $\dot z = 0 \Longrightarrow y = -2z$, $\dot y = 0 \Longrightarrow x = -2y - z = 3z$, $\dot x = 0 \Longrightarrow 8z - 108z^3 = 0$, which gives us 3 equilibria:

$$O(0, 0, 0), \quad O_+(3z_0, -2z_0, z_0), \quad O_-(-3z_0, 2z_0, -z_0)$$

where $z_0^2 = 2/27$. The linearisation matrix of the system at O is $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$, the characteristic equation

$$P(\lambda) = -(2-\lambda)((\lambda+2)^2 - 1) - 2 - \lambda = \lambda^3 + 2\lambda^2 - 6\lambda - 8 = 0.$$

This equation has one positive and two negative roots (as $P(-\infty) = -\infty < 0$, $P(3) = 19 > 0$, $P(0) = -8 < 0$, $P(+\infty) = +\infty > 0$), so O is a saddle with one-dimensional unstable manifold and two-dimensional stable manifolds. The system is symmetric with respect to $(x, y, z) \to (-x, -y, -z)$, the points O_+ and O_- are symmetric to each other, so they have the same stability type. The potential must have at least one minimum which corresponds to a stable equilibrium, O is not stable, so both the points O_+ and O_- are stable. It follows that the attractor consists of the three equilibria and the two unstable separatrices of O, this makes 5 orbits.

Question IV. Draw the phase portrait for the system on the plane

$$\frac{dx}{dt} = 1 - 6y + x^2, \qquad \frac{dy}{dt} = 1 - 2y - x^2,$$

in the following steps.

(a) Find the equilibria and determine their types.

(b) Draw null-clines. They divide the plane into 5 regions. Determine which of these regions are forward-invariant (i.e. the orbits cannot leave them as time grows) and which are backward-invariant (the orbits cannot leave them as time decreases).

(c) Prove that this system has no periodic orbits.

(d) Finish the phase portrait by drawing the separatrices of the saddle.

Solutions (5 points each; seen similar). IV(a): The equilibria are found from the equation

$$1 = 6y - x^2, \qquad 1 = x^2 + 2y,$$

which gives $x = \pm\frac{\sqrt{2}}{2}, y = \frac{1}{4}$. The linearisation matrix at $O_1(\frac{\sqrt{2}}{2}, \frac{1}{4})$ is $A_1 = \begin{pmatrix} \sqrt{2} & -6 \\ -\sqrt{2} & -2 \end{pmatrix}$. The determinant of A_1 is negative, so O_1 is a saddle. The linearisation matrix at O_2 is $A_2 = \begin{pmatrix} -\sqrt{2} & -6 \\ \sqrt{2} & -2 \end{pmatrix}$. We have $det(A_1) = 8\sqrt{2} > 0$, $tr(A_2) = -2 - \sqrt{2} < 0$, so O_2 is a stable point.

IV(b): Null-clines are two parabolas, $L_1 : y = \frac{x^2}{6} + \frac{1}{6}$, $L_2 : y = \frac{1}{2} - \frac{x^2}{2}$. They intersect at the equilibria, and divide the phase plane into 5 regions (see the figure). The region I bounded by the arcs of L_1 and L_2 to the right of the saddle O_1 is forward invariant, as the vector field on its boundary ($\dot{x} = 0, \dot{y} < 0$ on the arc of L_1 and $\dot{x} > 0, \dot{y} = 0$ on the arc of L_2) looks inside the region. None of these regions is backward-invariant.

IV(c): By Dulac criterion, a periodic orbit (if exists) must intersect the line where the divergence of the vector field vanishes. In our case this is the line $x = 1$. There must be at least 2 such intersections, one corresponds to the orbit going from $x < 1$ to $x > 1$, another corresponds to the orbit going backwards. To proceed from $x < 1$ to $x > 1$, we must have $\dot{x} \geq 0$ at $x = 1$, which gives $1 - 6y + 1 \geq 0 \Longrightarrow y \leq 1/3$. The point $(x = 1, y = 1/3)$ lies at the intersection with the arc of L_1 that bounds the forward-invariant region I. Thus, for the orbit to return to the line $x = 1$, it must, first, enter region I, but the latter is forward-invariant, so the orbit will never leave it, hence it can never close up.

IV(d): The saddle O_1 has two stable separatrices and two unstable separatrices. The stable separatrices must tend to infinity as $t \to -\infty$. Indeed, there are no periodic orbits (by IV(c)), nor unstable equilibria, so no point can be an α-limit point to them by virtue of Poincare-Bendixson theorem (the separatrices cannot form homoclinic loops as well, by the same Dulac criterion as in IV(c)). The are also two unstable separatrices, which leave it at $t = -\infty$ in opposite directions. One of the separatrices must enter region I (it separates the orbits which enter this region by crossing L_1 from the orbits which enter the region by crossing L_2), so it will stay in this region forever, hence it must tend to infinity (as there are no equilibria or periodic orbits there, hence there are no suitable candidates for an ω-limit set for it, by Poincare-Bendixson theorem). The other separatrix leaves in the opposite direction, i.e. it enters the bounded region III between L_1 and L_2. Now, one proves that it tends to the stable point O_2. If not, it must leave region III by crossing the upper arc of L_1 and entering region IV above this arc. In this region $\dot{y} < 0, \dot{x} < 0$, so the orbit must leave this region across the left arc of L_1 and enter region V. In this region $\dot{y} < 0, \dot{x} > 0$, so the orbit must cross the left arc of L_2 and enter region II. In this region $\dot{x} > 0$, and the separatrix has two choices: it either hits L_2 at some point P, enters the forward-invariant region I and never leaves, or hits L_1 and enters region III again. In the first case the region bounded by the arc of the separatrix between O_1 and P and the arc of L_2 between P and O_1 would be backward-invariant, it would contain a stable separatrix of O_1, which is impossible as the stable separatrices must be unbounded, as was shown above. Thus, the unstable separatrices must enter region III again, by intersecting the lower arc of L_1 again. In this case the region bounded by the arc of the separatrix from O_1 till this intersection point and the arc of L_2 from this point to O_1 is forward invariant, so the unstable separatrix remains there forever. The only possible ω-limit point of it is the stable point O_2.

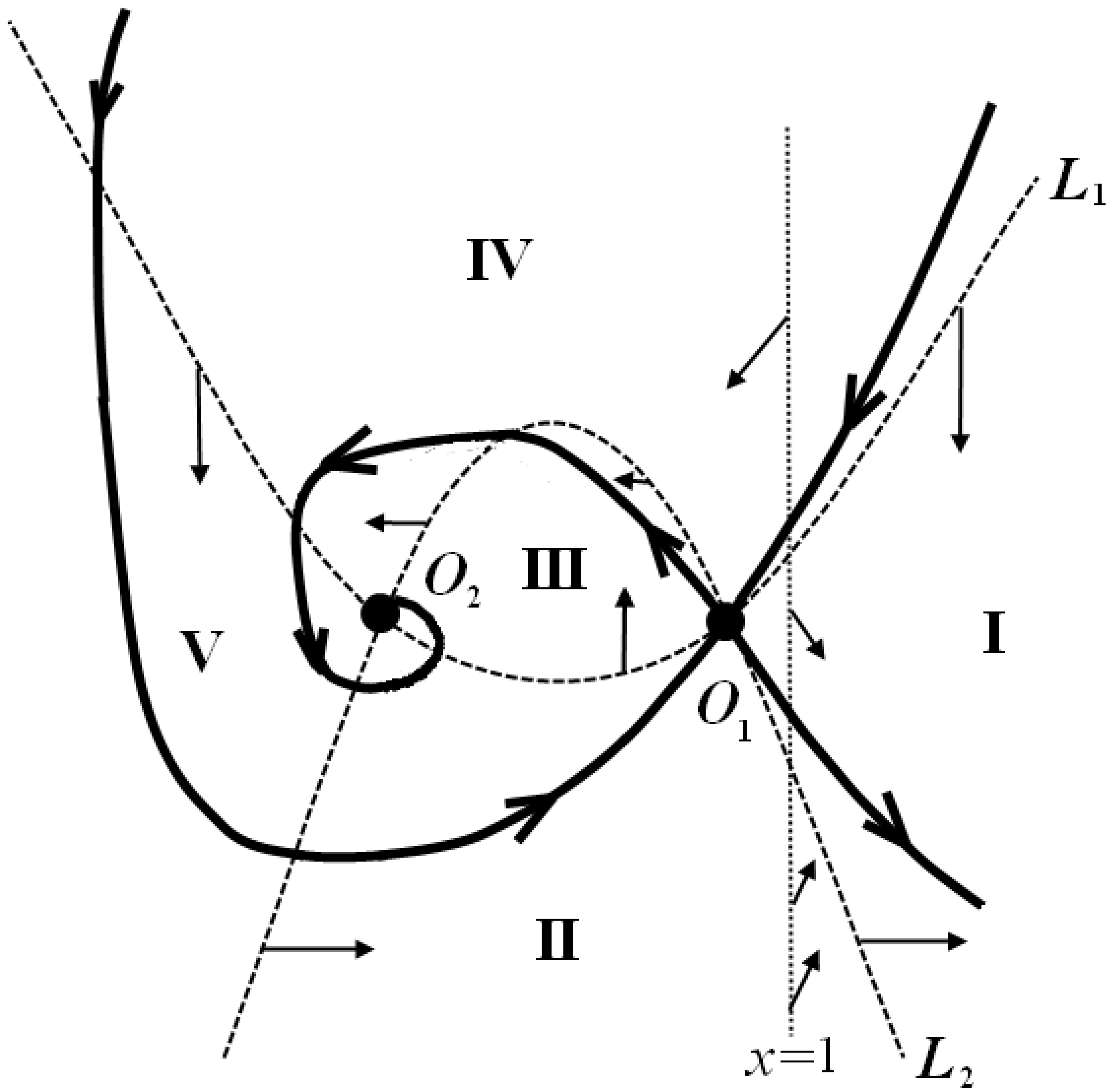

Figure 1: The phase potrtrait.

Mastery Question. Prove the Poincare-Bendixson theorem: for any smooth system of differential equations on a plane, the ω-limit set of a bounded orbit is either a periodic orbit, or an equilibrium, or a union of equilibria and orbits asymptotic to equilibria.

Solution. (20 points, seen). Take any bounded orbit $X = \{x_t\}$ in R^2, let y_0 be some its ω-limit point. Let $Y = \{y_t\}$ be the orbit of y_0. As X is bounded, its ω-limit set is bounded, i.e. y_t stays bounded for all t. Therefore, it has at least one α-limit point and at least one ω-limit point. Let z be any α-limit or ω-limit point of Y. It is enough to prove that if any such point z is not an equilibrium state, then y_t is periodic. Assume z is not an equilibrium. Then the phase velocity vector is non-zero at z, so any small arc γ transverse to this vector at the point z is a local cross-section: it divides a small neighbourhood U of z into two halves, U_- and U_+, such that for every point in U_- its orbit must intersect γ, cross to U_+ as t grows, and then leave U. For every point in U_+, its orbit must cross γ to U_- and leave U as time decreases. Since z is a limit point for y_t, there must be two moments of time, $t_1 < t_2$ such that $y_{t_1} \in \gamma$, $y_{t_2} \in \gamma$. If $y_{t_1} = y_{t_2}$, then y_t is a periodic orbit. If $y_{t_1} \neq y_{t_2}$, consider the curve $\mathcal{L}$ formed by the union of the invariant curve $\{y_t | t \in [t_1, t_2]\}$ and by the arc γ' of γ between y_{t_1} and y_{t_2}. By Jordan lemma, the curve $\mathcal{L}$ divides the plane into two open regions, D_+ and D_- (the orbits that start at γ' go from D_- to D_+ as time grows). The region D_+ is forward-invariant, D_- is backward-invariant, so y_t lies in D_+ for all $t > t_2$ and y_t lies in D_- for all $t < t_1$. This leads to a contradiction. Indeed, every point of Y is an ω-limit point of x_t. This means that x_t visits every open neighbourhood of every point of the orbit Y at a sequence of values of time which tends to $+\infty$. The open sets D_+ and D_- are neighbourhoods of some points of Y, so x_t must come both to D_+ and D_- at some tending to infinity sequence of time moments, i.e. it must come to D_+ then leave it to D_-, then come back, and so on, but this contradicts to the forward-invariance of D_+.