Памяти:

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"There is nothing more practical than a good theory."
James C. Maxwell

"... le souci du beau nous conduit aux mêmes choix que celui de l'utile."
Henri Poincaré

Preface

Many phenomena in science and technology are dynamical in nature. Stationary regimes, periodic motions and beats from modulations have long been believed to be the only possible observable states. However, discoveries in the second half of the 20th century have dramatically changed our traditional view of the character of dynamical processes. The breakthrough came with the discovery of a new type of oscillations called *dynamical chaos*. A deepening of our understanding of dynamical phenomena has since led us to a clear recognition that ours is a *nonlinear* world. This has resulted in the emergence of *nonlinear dynamics* as a scientific discipline whose aim is to study the common laws (regularities) of nonlinear dynamical processes.

A typical scheme for investigating a new phenomenon usually proceeds as follow: the relevant experiment or observation is studied by first constructing an adequate mathematical model in the form of dynamical equations. This model is analyzed and the result is compared with the experimental phenomenon.

This approach was first suggested by Newton. The laws that Newton discovered have provided a foundation for the mathematical modeling of numerous problems, including Celestial mechanics. The solution of the restricted two-body problem gives a brilliant explanation of the experimental Kepler's laws. In fact, starting with Newton, this method for modeling nature has dominated the field for many years. However, even such a purely scientific approach must be validated by questioning the correspondence between a real phenomenon and its phenomenological model, which had been aptly put by Brillouin: "A mathematical model differs from reality just as a globe differs from the earth".

A mathematical model in nonlinear dynamics usually consists of a system of equations with analytically given nonlinearities, and a finite number of parameters. The system may be described by ordinary differential equations, partial differential equations, equations with a delay, integro-differential equations, etc. In this book we will deal only with lumped (discrete-space) systems described by ordinary differential equations. Furthermore, we will restrict ourselves to a study of *non-conservative* systems thereby leaving aside the "ideal" dynamics of Hamiltonian systems (which Klein, at the end of the 19th century, had characterized as being the most "attractive mechanics without friction").

A system of differential equations is written in the form

$$\frac{dx}{dt} = X(x)\,,$$

where the independent variable t is called the time. One of the postulates of nonlinear dynamics which dates back to Aristotle and is based on common sense is that all observable states must be stable. This implies that in any comprehensive study of systems of differential equations, our attention must be focused on the character of the solutions over an *infinite* time interval. The systems considered from this point of view are called *dynamical*. Although the notion of a dynamical system is a mathematical abstraction — indeed we know from cosmology that even our Universe has only a *finite* life time — nevertheless, many phenomena of the real world have been successfully explained *via* the theory of dynamical systems. In the language of this theory the mathematical image of a stationary state is an equilibrium state, that of self-oscillations is a limit cycle, that of modulation is an invariant torus with a quasi-periodic trajectory, and the image of dynamical chaos is a strange attractor; namely, an attracting limit set composed of unstable trajectories.

In principle, the first three types of motions cited may be explained by a linear theory. That was the approach of the 19th century, which concerned mainly various practical applications modeled in terms of linear ordinary or partial differential equations. The most famous example is the problem of controlling steam engines whose investigation had led to the solution of the problem of stability of equilibrium states; namely to the classic Routh–Hurwitz criterion.

The most remarkable events in nonlinear dynamics can be traced to the twenties and the thirties of the 20th century. This period is characterized by the rapid development of radio-engineering. A common feature of many nonlinear radio-engineering problems is that the associated transient processes are typically very fast, thereby making it less time-consuming to carry out complicated experiments. The fact that the associated mathematical models in those days are usually simple systems of quasi-linear equations also plays an important role. This has in turn allowed researchers to conduct rather complete investigations of the models using methods based on Poincaré's theory of limit cycles and Lyapunov's stability theory.

Another significant event from that period is the creation of a mathematical theory of oscillations in two-dimensional systems. In particular, Andronov and Pontryagin identified a large class of rough (structurally stable) systems which admit a rather simple mathematical description. Moreover, all principal bifurcations of limit cycles were studied (Andronov, Leontovich) and complete topological invariants for both rough systems (Andronov, Pontryagin) and generic systems (Leontovich, Mayer) were described. Shortly after that, specialists from various areas of research applied these mathematically transparent and geometrically comprehensive methods to investigate concrete two-dimensional systems. This stage of the development is documented in the classic treatise "Theory of oscillations" by Andronov, Vitt and Khaikin.[1]

Further development in this subject included the attempt at a straightforward generalization of the concepts of planar systems, namely, the aim of extending the conditions of structural stability and bifurcations to the high-dimensional case. In no way does this approach indicate narrow visions. On the contrary, this was a mathematically sound strategy. Indeed, it was understood that entrance into space must bring new types of motions which may become crucial in nonlinear dynamics. As was mentioned previously, the mathematical image of modulation is a torus with quasi-periodic trajectories. Quasi-periodic trajectories are a particular case of almost-periodic trajectories which, by definition, are unclosed trajectories whose main feature is that they have *almost-periods* — the time intervals over which the trajectory returns close to its initial state. The quasi- and almost-periodic trajectories are self-limiting. A broader class of self-limiting trajectories consists of Poisson-stable trajectories. This kind of trajectory was discovered by Poincaré while studying the stability of the restricted three-body problem. A Poisson-stable trajectory also returns arbitrarily close to its initial state, but for an arbitrary but fixed small neighborhood of the initial state, the sequence of the associated return

[1]This book was first published in 1937 but without the name of Vitt, who had already been repressed.

times may be unbounded, *i.e.* the motion is unpredictable. In accordance with Birkhoff's classification, stationary, periodic, quasi-periodic, almost-periodic and Poisson-stable trajectories exhaust all types of motions associated with non-transient behaviors.

In the early thirties Andronov posed the following basic question in connection with the mathematical theory of oscillations: Can a Poisson-stable trajectory be Lyapunov stable? The answer was given by Markov: If a Poisson-stable trajectory is stable in the sense of Lyapunov (to be more precise, uniformly stable), then it must be almost-periodic. It seemed therefore that no other motions, apart from those which are almost-periodic, exist in nonlinear dynamics. Therefore, despite new discoveries in the qualitative theory of high-dimensional systems in the sixties it was not clear whether this theory had any value beyond pure mathematics. But this did not last long.

For within a relatively short period of time Smale had established the foundation for a theory of structurally stable systems with complex behavior in the trajectories, a theory that is generally referred to nowadays as *the hyperbolic theory*. In essence, a new mathematical discipline with its own terminology, notions and problems has been created. Its achievements have led to one of the most amazing fundamental discoveries of the 20th century — *dynamical chaos*.[2] Hyperbolic theory had provided examples of strange attractors which might be the mathematical image of chaotic oscillations, such as the well-known turbulent flows in hydrodynamics.

Nevertheless, the significance of strange attractors in nonlinear dynamics were not widely appreciated, especially not by specialists in turbulence. There were a few reasons for their reluctance. By mathematical construction, known hyperbolic attractors possess such a complex topologically structure that it did not allow one to conceive of any reasonable scenarios for their emergence. This has led one to regard hyperbolic attractors as being the result of a pure abstract scheme irrelevant to real dynamical processes.[3] Moreover, the phenomenon of *chaos* which has been observed in many concrete models could scarcely be associated with hyperbolic attractors because of the appearance of stable periodic orbits of long periods, either for the given parameter values, or for nearby ones. This enabled skeptics to argue that any observable chaotic

[2]Chronologically, this discovery came after the creation of "relativity theory" and "quantum mechanics".

[3]The possibility of applying hyperbolic attractors to nonlinear dynamics remains problematic even today.

behavior represents a transient process only. In this regard, we must emphasize that the persistence of the unstable behavior of trajectories of a strange attractor with respect to sufficiently small changes in control parameters is the essence of the problem: In order for a phenomenon to be observable it must be stable with respect to external perturbations.

The breakthrough in this controversy came in the mid seventies with the appearance of a simple low-order model

$$\begin{aligned}\dot{x} &= -\sigma(x-y)\,,\\ \dot{y} &= rx - y - xz\,,\\ \dot{z} &= -bz + xy\,,\end{aligned}$$

where chaotic behavior in its solutions was discovered numerically by E. Lorenz in 1962. A detailed analysis carried out by mathematicians revealed the existence of a strange attractor which is not hyperbolic but structurally unstable. Nevertheless, the main feature persisted, namely, the attractor preserved the instability behavior of the trajectories under small smooth perturbations of the system. Such attractors, which contain a single equilibrium state of the saddle type, are called *Lorenz attractors*. The second remarkable fact related to these attractors is that the Lorenz attractor may be generated via a finite number of easily observable bifurcations from systems endowed with only trivial dynamics.

Since then, dynamical chaos has been almost universally accepted as a legitimate and fundamental phenomenon of nature. The Lorenz model has since become a *de facto* proof of the existence of chaos, even though the model itself, despite its hydrodynamical origin, contains "too little water".[4] More recently, a much more realistic mathematical model of a real physical system called *Chua's Circuit* has also been proved rigorously to exhibit dynamical chaos, and whose experimental results agree remarkably well with both mathematical analysis and computer simulations [76–79].

We will not discuss further the relevance of the theory of strange attractors but note only that the theory of nonlinear oscillations created in the thirties had been so clear and understandable that generations of nonlinear researchers were able to apply it successfully to solve problems from many scientific disciplines. A different situation occurred in the seventies. Limit cycles and tori

[4]The Lorenz system represents the simplest Galerkin approximation of the problem of the convection of a planar layer of fluid.

which exhibit a unified character were replaced by strange attractors which possess a much more complex mathematical structure. They include smooth or non-smooth surfaces and manifolds, sets with a local structure represented as a direct product of an interval and a Cantor set, or even more sophisticated sets. Today, a specialist in complex nonlinear dynamics must either have a strong mathematical background in the qualitative theory of high-dimensional dynamical systems, or at least a sufficiently deep understanding of its main statements and results. We wish to remark that just as nonlinear equations cannot usually be integrated by quadratures, the majority of concrete dynamical models do not admit "a qualitative integration" by a purely mathematical analysis. This inevitably leads to the use of computer analysis as well. Hence, an ultimate requirement for any formal statement in the qualitative theory of differential equations is that it must have a complete and concrete character. It must also be free of unnecessary restrictions which, paraphrasing Hadamard, are not dictated by the needs of science but by the abilities of the human mind.

In most cases, the parameter space of a high-dimensional model may be partitioned into two regions according to whether the model exhibits simple or complex behaviors in its trajectories. The primary indication or sign of the presence of complex behavior will be associated in this book with the presence of a Poincaré homoclinic trajectory. Although Poincaré had discovered these trajectories in the restricted three-body problem, *i.e.* in a Hamiltonian system, such trajectories are essential objects of study in all fields of nonlinear dynamics as well. In general, the presence of Poincaré homoclinic trajectories leads to rather important conclusions. It was simultaneously established by Smale and L. Shilnikov (from opposite locations on the globe) that systems with a Poincaré homoclinic trajectory possess infinitely many co-existing periodic trajectories and a continuum of Poisson-stable trajectories. All of them are unstable. In essence, these homoclinic structures are the elementary bricks of dynamical chaos.

As for high-dimensional systems with simple behavior of trajectories, they are quite similar to planar systems [80]. In principle, the only new feature is the possibility of the existence in the phase space of an invariant torus with a quasi-periodic trajectory covering the torus. So, any concrete model may be completely analyzed in this region of the parameter space.

The situation is fundamentally different in the case of systems with complex trajectory behavior. Indeed, it has been established recently by Gonchenko,

L. Shilnikov and Turaev that a complete analysis of most models of nonlinear dynamics is unrealistic [28].

This book is concerned only with the qualitative theory of high-dimensional systems of differential equation with simple dynamics. For an extremely rich variety of such systems which arise in practical applications, the reader is referred to the very large systems of nonlinear differential equations (typically with dimensions greater than 10,000 state variables) associated with *Cellular Neural Networks* [81], which include lattice dynamical systems and cellular automata as special cases. We have partitioned this book into two parts. The first part is mainly introductory and technical in nature. In it we consider the behavior of trajectories close to simple equilibrium states and periodic trajectories, as well as discuss some problems related to the existence of an invariant torus. It is quite natural that we first present the classical results concerning the stability problem. Of special concern are the unstable equilibria and periodic trajectories of the saddle type. Such trajectories play a crucial role in the contemporary qualitative theory. For example, saddle equilibrium states may form unseparated parts of strange attractors. Saddles are also related to some principally important problems of a nonlocal character, etc. Our technique for investigating the behavior of systems near saddle trajectories in this book is based on the method suggested by L. Shilnikov in the sixties. The main feature of this method is that the solution near a saddle is sought not as a solution of the Cauchy problem but as a solution of a special boundary-value problem. Since this method has not yet been clearly presented in the literature, but is known only to a small circle of specialists, it is discussed in detail in this book.

In the second part of this book we analyze the principal bifurcations of equilibrium states, as well as of periodic, homoclinic and heteroclinic trajectories. The theory of bifurcations has a key role in nonlinear dynamics. Its roots go back to the pioneering works of Poincaré and Lyapunov on the study of the form of a rotating fluid. A bifurcation theory based on the notion of roughness, or structural stability, has since been developed. Whereas in the rough (robust) case small changes do not induce significant changes in the states of a system, the bifurcation theory explains what happens in the non-rough case, including many possible qualitative transformations. Some of these transitions may be dangerous, possibly leading to catastrophic and irreversible situations. The bifurcation theory allows one to predict many real-world phenomena. In particular, notions such as the soft and the rigid (severe) regimes of excitation of oscillations, the safe and dangerous boundaries of the stability regions of steady states and periodic motions, hysteresis, phase-locking, etc., have all been formulated and analyzed via bifurcation theory.

In this book we give special attention to the boundaries of stability of equilibria and periodic trajectories in the parameter space. Along with standard bifurcations, both local and global, we also examine a bifurcation phenomenon discovered recently by L. Shilnikov and Turaev [66], the so-called "blue sky catastrophe". The essence of this phenomenon is that in the parameter space there may exist stability boundaries of a periodic trajectory such that upon approaching the boundary both the length and the period of the periodic trajectory tend to infinity, whereas the periodic orbit resides at a finite distance from any equilibrium state in a bounded region of the phase space. This bifurcation has not yet been observed in models of physical systems, although a three-dimensional two-parameter model with a polynomial right-hand side is known [25].

This book is essentially self-contained. All necessary facts are supplied with complete proofs except for some well-known classical results such as the Poincaré–Denjoy theory on the behavior of trajectories on an invariant torus.

The basis of this book is a special course in which the first author gave at the Nizhny Novgorod (formerly, Gorky) University over the last thirty years. This course usually proceeds with a one-year lecture on the qualitative theory of two-dimensional systems, which was delivered by Prof. E. A. Leontovich-Andronova for many years. Besides that, discussions on certain aspects of this course had formed the subject of student seminars, and weekly scientific seminars at the Department of Differential Equations of the Institute for Applied Mathematics & Cybernetics. This book will appeal to beginners who have chosen the qualitative theory and the theory of bifurcations and strange attractors as their majors. Undoubtedly, this book will also be useful for specialists in the above subjects and in related mathematical disciplines, as well as for a broad audience of interdisciplinary researchers on nonlinear dynamics and chaos, who are interested in the analysis of concrete dynamical systems.

Part I of this book consists of six chapters and two appendices.

In Chap. 1 we describe the principal properties of an autonomous system, give the notion of an abstract dynamical system and select the principal types of trajectories and invariant sets necessary for further presentation. In addition, we discuss some problems of qualitative integration of differential equations which is based on the notion of topological equivalence. The material of this chapter also has reference value, beginners may call on it when needed.

In Chap. 2 we examine the behavior of trajectories in a neighborhood of a structurally-stable equilibrium state. Our approach here goes back to Poincaré. Using this approach we classify the main types of equilibrium states. Special attention is given to equilibria of the saddle types, and, in particular, to leading and nonleading (strongly stable) invariant manifolds. We also give sufficient attention to the asymptotic representation of solutions near a saddle point. As mentioned, our methods are based on Shilnikov's boundary-value problem. In addition, we prove some theorems on invariant manifolds. We would like to stress that along with well-known theorems on stable and unstable manifolds of a saddle, some rather important results which we will need later are given here. In the last section of the chapter some useful information concerning Poincare's theory of resonances for local bifurcation problems are presented.

In Chap. 3 we discuss structurally-stable periodic trajectories. Our consideration is focused on the behavior of trajectories of the Poincaré map in a neighborhood of the fixed point. As in the case of equilibria we investigate an associated boundary-value problem near a saddle fixed point and prove a theorem on the existence of its invariant manifolds. Sections 3.10–3.12 and 3.14 are concerned only with the properties of periodic trajectories in continuous time.

Invariant tori are considered in Chap. 4. More specifically, we study a non-autonomous system which depends periodically, as well as quasi-periodically, on time. This class of non-autonomous system can be extended to higher dimensions by adding some equations having a specific form with respect to cyclic variables. To prove the existence of an invariant torus in such a system, we use a universal criterion, the so-called annulus principle which is applicable for systems with small perturbations. In the case of a periodic external force, the behavior of the trajectories on a two-dimensional invariant torus may be modeled by an orientable diffeomorphism of a circle. In relation to this we present a brief review of some related results from the Poincaré–Denjoy theory. We complete this chapter with a discussion of an important problem of nonlinear dynamics, namely, the synchronization problem associated with the phenomenon of "beats" in modulations.

The final two chapters, Chap. 5 and 6, are dedicated to local and global center manifolds, respectively. We re-prove in Chap. 5 a well-known result that in a small neighborhood of a structurally unstable equilibrium state, or near a bifurcating periodic trajectory of a $\mathbb{C}^r$-smooth dynamical system, there exists locally an invariant $\mathbb{C}^r$-smooth center manifold whose dimension is equal to the number of characteristic exponents with a zero real part in the case of equilibrium states, or to the number of multipliers lying on a unit circle in the case of periodic trajectories. Our proof of the center manifold theorem relates it to the study of a specific boundary value problem and covers all basic local invariant manifolds (strongly stable and unstable, extended stable and unstable, and strongly stable and unstable invariant foliations). We discuss how the existence of the center manifold and the invariant foliation allows one to reduce the problem of investigating the local bifurcations of a system to that of a corresponding sub-system on the center manifold, thereby significantly decreasing the dimension of the problem.

In Chap. 6 the proof of the analog of the theorem on the center manifold for the case of global bifurcations is presented. Unlike the local case, the dimension of the non-local center manifold does not depend on the degree of degeneracy of the Jacobian matrix, but is equal to some integer which can be estimated in terms of the numbers of negative and positive characteristic exponents of saddle trajectories comprising a heteroclinic cycle. Another characteristic of the non-local center manifold is that it is only $\mathbb{C}^1$-smooth in general. The restriction on such center manifolds may only be used for studying those bifurcation problems which admit the solution within the framework of $\mathbb{C}^1$-smoothness. Therefore, in contrast to the local bifurcation theory, one cannot directly apply non-local center manifolds to study various delicate bifurcation phenomena which require more smoothness. Hence, the theorem contains, in essence, certain qualitative results which only allow us to anticipate some possible dynamics of the trajectories in a small neighborhood of a homoclinic cycle, as well as to estimate the dimensions of the stable and unstable manifolds of trajectories lying in its neighborhood, and, consequently, to evaluate the number of positive and negative Lyapunov exponents of these trajectories. We consider in detail only the class of systems possessing the simplest cycle; namely, a bi-asymptotic trajectory (a homoclinic loop) which begins and ends at the same saddle equilibrium state. We then extend this result to general heteroclinic cycles.

In the Appendix we prove a theorem on the reduction of a system to a special form which is quite suitable for analysis of the trajectories near a saddle point. This theorem is especially important because an often postulated assumption on a straight-forward linearization of the system near a saddle may sometimes lead to subsequent confusion when more subtle details of the behavior of the trajectories are desired. The essence of our proof is a technique (based on the reduction of the problem to a theorem on strong stable invariant manifold) for making a series of coordinate transformations which are robust to small, smooth perturbations of the system. We will use this special form in the second part of this book when we study homoclinic bifurcations.

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Contents

Chapter 1

BASIC CONCEPTS

1.1. Necessary background from the theory of ordinary differential equations

The main objects of our study are autonomous systems of ordinary differential equations written in the form

$$\dot{x} \stackrel{\text{def}}{=} \frac{dx}{dt} = X(x)\,, \tag{1.1.1}$$

where $x = (x_1, \ldots, x_n)$, $X(x) = (X_1, \ldots, X_n)$. We assume that $X_1, \ldots, X_n$ are $\mathbb{C}^r$-smooth ($r \geq 1$) functions defined in a certain region $D \subseteq \mathbb{R}^n$. In the theory of dynamical systems it is customary to regard the variable t as time and the region D as the phase space, which may be bounded or unbounded, or may coincide with the Euclidean space $\mathbb{R}^n$. A differentiable mapping $\varphi\colon \tau \mapsto D$, where τ is an interval of the t-axis, is called a solution $x = \varphi(t)$ of system (1.1.1) if

$$\dot{\varphi}(t) = X(\varphi(t))\,, \qquad \text{for any } t \in \tau\,. \tag{1.1.2}$$

Since by assumption the conditions of Cauchy's theorem hold, it follows that for any $x_0 \in D$ and any $t_0 \in \mathbb{R}^1$ there exists a unique solution φ satisfying the initial condition

$$x_0 = \varphi(t_0)\,. \tag{1.1.3}$$

The solution is defined on some interval (t^-, t^+) containing $t = t_0$. In general, the endpoints t^- and t^+ may be finite, or infinite.

The solutions of system (1.1.1) possess the following properties:

1. If $x = \varphi(t)$ is a solution of (1.1.1), then obviously $x = \varphi(t + C)$ is also a solution defined on the interval $(t^- - C, t^+ - C)$.

2. The solutions $x = \varphi(t)$ and $x = \varphi(t + C)$ may be considered as solutions corresponding to the same initial point x_0 but at different initial time t_0.

3. A solution satisfying (1.1.3) may be written in the form $x = \varphi(t - t_0, x_0)$, where $\varphi(0, x_0) = x_0$.

4. If $x_1 = \varphi(t_1 - t_0, x_0)$ then $\varphi(t - t_0, x_0) = \varphi(t - t_1, x_1)$. Denoting $t_1 - t_0$ as a new t_1 and $t - t_1$ as t_2, we get the so-called group property of solutions:

$$\varphi(t_2, \varphi(t_1, x_0)) = \varphi(t_1 + t_2, x_0)\,. \tag{1.1.4}$$

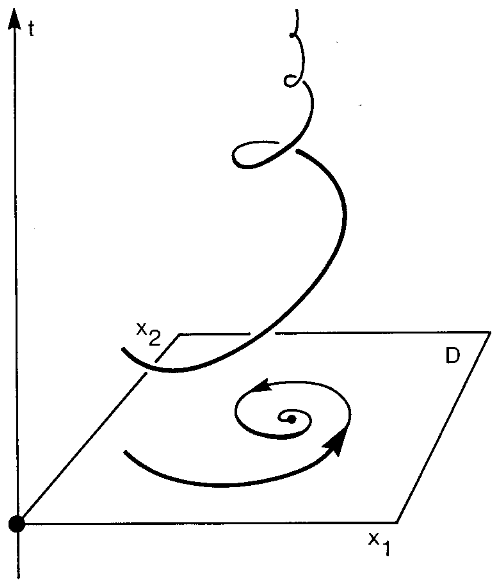
It is well known that the solution $x = \varphi(t - t_0, x_0)$ of the Cauchy problem (1.1.3) for a $\mathbb{C}^r$-smooth system (1.1.1) is smooth ($\mathbb{C}^r$) with respect to time and initial data x_0. The first derivative $\xi(t - t_0, x_0) \equiv \frac{\partial \varphi}{\partial x_0}$ satisfies the so-called *variational equation* $\dot{\xi} = X'(\varphi(t - t_0, x_0))\xi$ with the initial condition $\xi(0; x_0) = I$ (the identity matrix). The variational equation is a linear non-autonomous system obtained by formal differentation of (1.1.1). Further differentation gives equations for higher derivatives.

There are two geometrical interpretations of the solutions of system (1.1.1). The first interpretation relates to the phase space D, the second to the so-called *extended phase space* $D \times \mathbb{R}^1$. In the first interpretation we may consider any solution which satisfies the given initial condition (1.1.3) as a parametric equation (with parameter t) of some curve. This curve is traced out by the points $\varphi(t, x_0)$ in phase space D as t varies. In standard terminology such curves are called *phase trajectories*, or simply, trajectories (or orbits or, occasionally, phase curves). A system of differential equations (1.1.1) defines the right-hand side of a vector field in the phase space, where Eq. (1.1.2) means that the velocity vector $X(x)$ is tangent to the phase trajectory at the point x. By uniqueness of the solution of Cauchy problem (1.1.3) for a smooth vector field X, there is only one trajectory passing through each point in the phase space.

In the second interpretation, the solution of system (1.1.1) is considered as a curve in the extended phase space $D \times \mathbb{R}^1$. Such a curve is called *an integral curve*. There is an explicit link between trajectories and integral curves. Each phase trajectory is the projection of a corresponding integral curve onto the phase space along the t-axis, as depicted in Fig. 1.1.1. However, in contrast to integral curves which are curves in the strict sense of the term, their projections onto the phase space may no longer be curves but points. Such points are called *equilibrium states*. They correspond to the constant solutions $x = x^*$. By (1.1.2) $X(x^*) = 0$, *i.e.* equilibrium states are singular points of the vector field. It is natural to pose the following question: can phase trajectories intersect each other? This question is resolved by the following theorem.

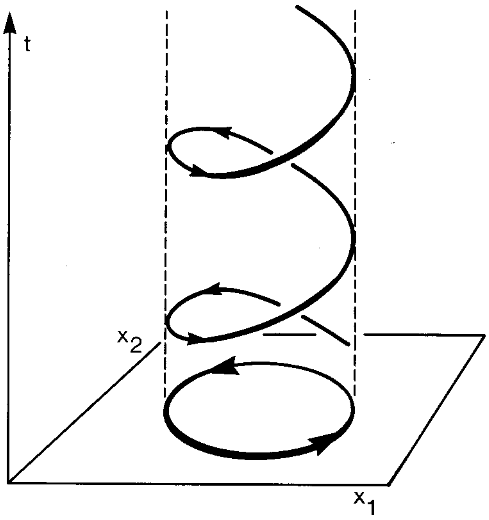
Theorem 1.1. *Let a trajectory L, other than an equilibrium state, correspond to a solution $\varphi(t)$ of system (1.1.1) such that $\varphi(t_1) = \varphi(t_2)$ for $t_1 \neq t_2$. Then $\varphi(t)$ is defined for all t and is periodic, and L is a simple smooth closed curve.*

If τ is the least period of $\varphi(t)$, then the parametric equation of L assumes the form $x = \varphi(t)$, $t_0 \leq t \leq t_0 + \tau$, where inside this interval distinct values of t correspond to distinct points of L.



(a)

Fig. 1.1.1. The projection of an integral curve onto the phase space D may be an unclosed trajectory (a) or, for example, a periodic trajectory (b).



(b)

Fig. 1.1.1. (*Continued*)

For a proof of this theorem we refer the reader to the book *Theory of Dynamical Systems on a Plane* by Andronov, Leontovich, Gordon and Maier [6].

The trajectory L corresponding to a periodic solution $\varphi(t)$ is called *a periodic trajectory.*

Any other trajectory which is neither an equilibrium state nor a periodic trajectory is an unclosed curve. It follows from Theorem 1.1 that an unclosed trajectory has no points of self-intersection.

Note that any two solutions which differ from each other only in the choice of the initial time t_0 correspond to the same trajectory. Vice versa: any two distinct solutions corresponding to the same trajectory are identical up to a time shift $t \to t + C$. It follows that all solutions corresponding to the same periodic trajectory are periodic of the same period.

In the case where the solution corresponding to a given trajectory L is defined for all $t \in (-\infty, +\infty)$ we will say that L is *an entire trajectory.* Any trajectory which lies in a bounded region is an entire trajectory.

From the view point of kinematics, the point $\varphi(t)$ is called *a representative point* and its trajectory is called *the associated motion.* Moreover, for any trajectories other than equilibrium states, one can introduce a positive direction of the motion which points in the direction of increasing t. At each point of such a trajectory this direction is determined by the associated tangent vector. To emphasize this we will label all trajectories with arrowheads.

Along with system (1.1.1) let us consider an associated "time-reverse" system

$$\dot{x} = -X(x)\,. \tag{1.1.5}$$

The vector field of system (1.1.5) is obtained from that of (1.1.1) by reversing the direction of each tangent vector. It is easy to see that each solution $x = \varphi(t)$ of system (1.1.1) corresponds to a solution $x = \varphi(-t)$ of system (1.1.5) and vice versa. It is clear also that systems (1.1.1) and (1.1.5) have the same phase curves up to a change of time $t \to -t$. Thus, the time-oriented trajectories of one system are obtained from the corresponding trajectories of the other by reversing the direction of the arrowheads.

Consider next the system

$$\dot{x} = X(x)f(x)\,, \tag{1.1.6}$$

where the $\mathbb{C}^r$-smooth function $f(x)\colon D \mapsto \mathbb{R}^1$ does not vanish in D. Observe that systems (1.1.1) and (1.1.6) have the same phase curves which differ only by time parametrization. Moreover, the trajectories of both systems have the same directions if $f(x) > 0$, and opposite if $f(x) < 0$. If $x = \varphi(t - t_0, x_0)$ is a trajectory of (1.1.1) passing through x_0 at $x = x_0$, then parametrization of time along this trajectory by the rule $d\tilde{t} = \frac{dt}{f(\varphi(t-t_0,x_0))}$ or $\tilde{t} = t_0 + \int_{t_0}^{t} \frac{ds}{f(\varphi(s-t_0,x_0))}$ gives a trajectory of (1.1.6). We will call a transformation of such kind *rescaling of time* or *change of time*.

Observe that in the case of system (1.1.1) we are interested only in the form of the trajectory, there is no need to involve the independent variable t. In this case we can consider the following more symmetric system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n}\,.$$

If, for example, X_n is non-zero in a certain sub-region $G \subset D$, the form of the trajectories in G may be found by solving the system

$$\frac{dx_i}{dx_n} = X_i X_n^{-1}\,.$$

This method is especially effective for studying two-dimensional systems.

Generally speaking, not all trajectories may be continued over the infinite interval $\tau = (-\infty, +\infty)$. In other words, not all trajectories are entire trajectories.[1] Examples of entire trajectories are equilibrium states and periodic trajectories. From the point of view of dynamics, the entire trajectories, or those which may be defined at least for all positive t over an infinite interval of time, are of special interest. The reason is that, despite the importance of the information revealed by transient solutions over a finite interval of time, the most interesting phenomena observed in natural science and engineering obtain an adequate explanation only if time t increases without bounds. Systems whose solutions can be continued over an infinite period of time were named *dynamical systems* by Birkhoff. An abstract definition of such systems which takes into account their group properties, will be presented in the following section.

1.2. Dynamical systems. Basic notions

Three components are used in the definition of a dynamical system. (1) A metric space D called the phase space. (2) A time variable t which may be either continuous, *i.e.* $t \in \mathbb{R}^1$, or discrete, *i.e.* $t \in \mathbb{Z}$. (3) An evolution law, *i.e.* a mapping of any given point x in D and any t to a uniquely defined state $\varphi(t, x) \in D$ which satisfies the following group-theoretic properties:

$$\begin{aligned} &1. \quad \varphi(0, x) = x\,. \\ &2. \quad \varphi(t_1, \varphi(t_2, x)) = \varphi(t_1 + t_2, x)\,. \\ &3. \quad \varphi(t, x) \text{ is continuous with respect to } (x, t)\,. \end{aligned} \tag{1.2.1}$$

In the case where t is continuous the above conditions define a continuous dynamical system, or flow. In other words, a flow is a one-parameter group of *homeomorphisms*[2] of the phase space D. Fixing x and varying t from $-\infty$ to $+\infty$ we obtain an orientable curve[3] as before, called a phase trajectory. The following classification of phase trajectories is natural: equilibrium states, periodic trajectories and unclosed trajectories. We will call $\{x \colon \varphi(t, x), t \geq 0\}$ *a positive semi-trajectory* and $\{x : \varphi(t, x), t \leq 0\}$ a *negative semi-trajectory.*

[1] There are systems whose solutions tend to infinity at some *finite* time. Such systems are not *dynamically defined systems*.

[2] *i.e.* one-to-one, continuous mappings with continuous inverse. This follows directly from the group property (1.2.1) that $\varphi(-t, \cdot)$ is inverse to $\varphi(t, \cdot)$.

[3] The orientation is induced by the direction of motion.

Observe that in the case of an unclosed trajectory any point of the trajectory partitions the trajectory into two parts: a positive semi-trajectory and a negative semi-trajectory.

In the case where the mapping $\varphi(t,x)$ is a *diffeomorphism*[4] the flow is a smooth dynamical system. In this case, the phase space D is endowed with some additional smooth structures. The phase space D is usually chosen to be either $\mathbb{R}^n$, or $\mathbb{R}^{n-k} \times \mathbb{T}^k$, where $\mathbb{T}^k$ may be a k-dimensional torus $\underbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k \text{ times}}$, a smooth surface, or a manifold. This allows us to set up a correspondence between a smooth flow and its associated vector field by defining a velocity field

$$X(x) = \left.\frac{d\varphi(t,x)}{dt}\right|_{t=0}. \tag{1.2.2}$$

By definition, the trajectories of the smooth flow are the trajectories of the system $\dot{x} = X(x)$. In this book we will study mainly the properties of smooth dynamical systems.

Discrete dynamical systems are often called *cascades* for simplicity. A cascade possesses the following remarkable feature. Let us select a homeomorphism $\varphi(1,x)$ and denote it by $\psi(x)$. It is obvious that $\varphi(t,x) = \psi^t(x)$, where

$$\psi^t = \psi\,(\underbrace{\psi(\cdots\psi}_{t-1 \text{ times}}(x)))\,.$$

Hence, in order to define a cascade it is sufficient to specify only the homeomorphism $\psi\colon D \mapsto D$.

In the case of a discrete dynamical system the sequence $\{x_k\}_{k=-\infty}^{+\infty}$ where $x_{k+1} = \psi(x_k)$, is called *a trajectory* of the point x_0. Trajectories may be of three types:

1. A point x_0. The point is a fixed point of the homeomorphism $\psi(x)$, *i.e.* it is mapped by $\psi(x)$ into itself.

2. A cycle $(x_0,\ldots,x_{k-1})$, where $x_i = \psi^i(x_0)$, $i = (0,\ldots,k-1)$ and $x_0 = \psi^k(x_0)$ moreover, $x_i \neq x_j$ for $i \neq j$. The number k is called *the period*, and each point x_i is called *a periodic point of period k*. Observe that a fixed point is a periodic point of period 1.

[4] A one-to-one, differentiable mapping with a differentiable inverse.

3. A bi-infinite trajectory, *i.e.* a sequence $\{x_k\}_{-\infty}^{+\infty}$, where $k \to \pm\infty$, $x_i \neq x_j$ for $i \neq j$. In this case, as in the case of flows, we will say that such a trajectory is unclosed.

When $\psi(x)$ is a diffeomorphism, the cascade is a smooth dynamical system. Examples of cascades of this type appear in the study of non-autonomous periodic systems in the form

$$\dot{x} = X(x, t)\,,$$

where $X(x,t)$ is continuous with respect to all variables in $\mathbb{R}^n \times \mathbb{R}^1$, is smooth with respect to x and periodic of period τ with respect to t. It is assumed that the system has solutions which may be continued over the interval $t_0 \leq t \leq t_0 + \tau$. Given a solution $x = \varphi(t, x_0)$, where $\varphi(0, x_0) = x_0$ we may define a mapping

$$x_1 = \varphi(\tau, x) \tag{1.2.3}$$

of the hyper-plane $t = 0$ into the hyper-plane $t = \tau$. It follows from the periodicity of $X(x,t)$ that (X, t_1) and (X, t_2) must be identified if $(t_2 - t_1)$ is divisible by τ. Thus, (1.2.3) may be regarded as a diffeomorphism $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$.[5]

Before proceeding further, we need to introduce some notions.

A set A is said to be *invariant* with respect to a dynamical system if $A = \varphi(t, A)$ for any t. Here, $\varphi(t, A)$ denotes the set $\bigcup\limits_{x \in A} \varphi(t, x)$. It follows from this definition that if $x \in A$, then the trajectory $\varphi(t, x)$ lies in A.

We call a point x_0 *wandering* if there exists an open neighborhood $U(x_0)$ of x_0 and a positive T such that

$$U(x_0) \cap \varphi(t, U(x_0)) = \emptyset \quad \text{ for } t > T\,. \tag{1.2.4}$$

Applying the transformation $\varphi(-t, \cdot)$ to (1.2.4) we obtain

$$\varphi(-t, U(x_0)) \cap U(x_0) = \emptyset \quad \text{ for } t < T\,.$$

Hence, the definition of a wandering point is symmetric with respect to reversion of time.

[5]Observe that system (1.2.3) may be written as an autonomous system

$$\dot{x} = X(x, \theta)\,, \quad \dot{\theta} = 1\,,$$

where θ is taken in modulo τ.

Let us denote by $\mathcal{W}$ the set of wandering points. The set $\mathcal{W}$ is open and invariant. Openness follows from the fact that together with x_0 any point in $U(x_0)$ is wandering. The invariance of $\mathcal{W}$ follows from the fact that if x_0 is a wandering point, then the point $\varphi(t_0, x_0)$ is also a wandering point for any t_0. To show this let us choose $\varphi(t_0, U(x_0))$ to be a neighborhood of the point $\varphi(t_0, x_0)$. Then

$$\varphi(t_0, U(x_0)) \cap \varphi(t, \varphi(t_0, U(x_0)) = \emptyset \quad \text{for } t > T\,.$$

Hence, the set of *non-wandering points* $\mathcal{M} = D \backslash \mathcal{W}$ is closed and invariant. The set of non-wandering points may be empty. To illustrate the latter consider a dynamical system defined by the autonomous system

$$\begin{aligned}\dot{x} &= X(x, \theta)\,,\\ \dot{\theta} &= 1\end{aligned}$$

in phase space $\mathbb{R}^{n+1}$, $x = (x_1, \ldots, x_n)$. Observe that (1.2.4) holds here since $\theta(t) = \theta_0 + t$ increases monotonically with t. Hence, every point in $\mathbb{R}^{n+1}$ is a wandering point.

It is clear that equilibrium states, as well as all points on periodic trajectories, are non-wandering. All points on bi-asymptotic trajectories which tend to equilibrium states and periodic trajectories as $t \to \pm\infty$ are also non-wandering. Such a bi-asymptotic trajectory is unclosed and called *a homoclinic trajectory.* The points on Poisson-stable trajectories are also non-wandering points.

Definition 1.1. *A point x_0 is said to be positive Poisson-stable if given any neighborhood $U(x_0)$ and any $T > 0$ there exists $t > T$ such that*

$$\varphi(t, x_0) \subset U(x_0)\,. \tag{1.2.5}$$

If for any $T > 0$ there exists t such that $t < -T$ and (1.2.5) *holds, then the point x_0 is called a negative Poisson-stable point. If a point is positive and negative Poisson stable it is said to be Poisson-stable.*

Observe that if a point x_0 is positive (negative) Poisson-stable, then any point on the trajectory $\varphi(t, x_0)$ is also positive (negative) Poisson stable. Thus, we may introduce the notion of a P^+-trajectory (positive Poisson-stable), a P^--trajectory (negative Poisson-stable) and merely a P-trajectory (Poisson-stable). It follows directly from (1.2.5) that P^+, P^- and P-trajectories consist of non-wandering points.

It is obvious that equilibrium states and periodic trajectories are closed P-trajectories.

Theorem 1.2. (Birkhoff)[6] *If a P^+ (P^-, P)-trajectory is unclosed, then its closure Σ contains a continuum of unclosed P-trajectories.*

Let us choose a positive sequence $\{T_n\}$ where $T_n \to +\infty$ as $n \to +\infty$. It follows from the definition of a P^+-trajectory that there exists a sequence $\{t_n\} \to +\infty$ as $n \to +\infty$ such that $\varphi(t_n, x_0) \subset U(x_0)$. An analogous statement holds in the case of a P^--trajectory. This implies that a P-trajectory successively intersects any ε-neighborhood $U_\varepsilon(x_0)$ of the point x_0 infinitely many times.[7] Let $\{t_n(\varepsilon)\}_{-\infty}^{+\infty}$ be chosen such that $t_n(\varepsilon) < t_{n+1}(\varepsilon)$ and let $\varphi(t_n(\varepsilon), x_0) \subset U_\varepsilon(x_0)$. The values

$$\tau_n(\varepsilon) = t_{n+1}(\varepsilon) - t_n(\varepsilon)$$

are called *Poincaré return times.* Two essentially different cases are possible for an unclosed P-trajectory:

1. The sequence $\{\tau_n(\varepsilon)\}$ is bounded for any finite ε, *i.e.* there exists a number $L(\varepsilon)$ such that $\tau_n(\varepsilon) < L(\varepsilon)$ for any n. Observe that $L(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

2. The sequence $\{\tau_n(\varepsilon)\}$ is unbounded for any sufficiently small ε.

In the first case the P-trajectory is called *recurrent.* For such a trajectory all trajectories in its closure Σ are also recurrent, and the closure itself is *a minimal set.*[8] The principal property of a recurrent trajectory is that it returns to an ε-neighborhood of the point x_0 within a time not greater than $L(\varepsilon)$. However, in contrast to periodic trajectories, whose return times are fixed, the return time for a recurrent trajectory is not constrained.

In the second case, the closure Σ of the P-trajectory is called *a quasi-minimal set.* In this case, there always exist in Σ other invariant closed subsets which may be equilibrium states, periodic trajectories, or invariant tori,

[6] See the proof in [14].

[7] In the case of flows the set of times during which a P-trajectory passes through $U_\varepsilon(x_0)$ consists of infinitely many time intervals $I_n(\varepsilon)$, where $t_n(\varepsilon)$ is chosen to be one of the values in $I_n(\varepsilon)$.

[8] A set is called *minimal* if it is non-empty, invariant, closed and contains no proper subsets possessing these three properties.

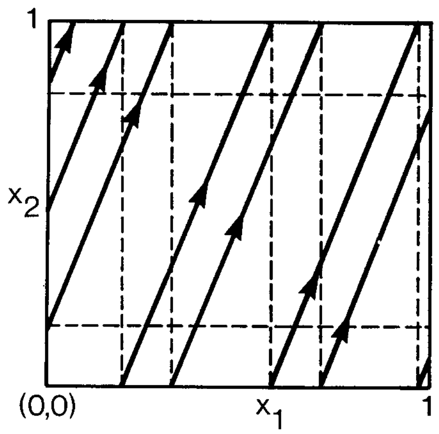


Fig. 1.2.1. The flow on a torus can be represented as a flow on the unit square. The slope of all parallel trajectories is equal to ω_2/ω_1. Gluing the opposite sides of the square gives a two-dimensional torus.

etc. Since a P-trajectory may approach such subsets arbitrarily closely, the Poincaré return times can therefore be arbitrarily large.

The simplest example of a flow all of whose trajectories are Poisson stable is *a quasi-periodic flow* on a two-dimensional torus $\mathbb{T}^2$ defined by the equations

$$\begin{aligned} \dot{x}_1 &= \omega_1 \,, \\ \dot{x}_2 &= \omega_2 \,, \end{aligned} \tag{1.2.6}$$

where ω_1/ω_2 is irrational. This flow may be represented as a flow defined on a unit rectangle with the points $(x_1, 0)$ and $(x_1, 1)$, and $(0, x_2)$ and $(1, x_2)$ identified, as shown in Fig. 1.2.1. In this case, $\Sigma = \mathbb{T}^2$ is a minimal set, and the flow possesses an unclosed trajectory which is everywhere dense on the torus.[9] When ω_1/ω_2 is rational, all trajectories of (1.2.6) on $\mathbb{T}^2$ are periodic.

Let $f(x_1, x_2)$ be a function defined on the torus $\mathbb{T}^2$, *i.e.* $f(x_1+1, x_2+1) = f(x_1+1, x_2) = f(x_1, x_2)$. Assume also that f is smooth and vanishes at one point (x_1^0, x_2^0) only. The flow defined by the system

$$\begin{aligned} \dot{x}_1 &= \omega_1 f_1(x_1, x_2) \,, \\ \dot{x}_2 &= \omega_2 f_2(x_1, x_2) \,, \end{aligned}$$

[9]This is called *a quasi-periodic trajectory.*

on $\mathbb{T}^2$ is quasi-minimal. In this case Σ also coincides with $\mathbb{T}^2$. However, Σ contains an invariant subset which is the point (x_1^0, x_2^0). All trajectories of the flow on the torus are Poisson stable except for two trajectories: one tends to (x_1^0, x_2^0) as $t \to +\infty$, whereas another as $t \to -\infty$, respectively. We will meet other examples of quasi-minimal sets in multi-dimensional autonomous systems.

Let us introduce next the notion of an attractor.

Definition 1.2. *An attractor $\mathcal{A}$ is a closed invariant set which possesses a neighborhood* (*an absorbing domain*) $U(\mathcal{A})$ *such that the trajectory $\varphi(t,x)$ of any point x in $U(\mathcal{A})$ satisfies the condition*

$$\rho((\varphi(t,x), \mathcal{A}) \to 0 \quad \textit{as } t \to +\infty\,, \tag{1.2.7}$$

where

$$\rho(x, \mathcal{A}) = \inf_{x_0 \in \mathcal{A}} \|x - x_0\|\,.$$

The simplest examples of attractors are stable equilibrium states, stable periodic trajectories and stable invariant tori containing quasi-periodic trajectories.

This definition of an attractor does not preclude the possibility that it may contain other attractors. It is reasonable to restrict the notion of an attractor by imposing a quasi-minimality condition. There exist a variety of attractors which meet this condition. Of special interest among them are the so-called *strange attractors* which are invariant closed sets comprised of only unstable trajectories.

To conclude this section we remark that there are also systems in which $t \in \mathbb{R}^+$ where $\mathbb{R}^+$ denotes the non-negative half-line, or those in which $t \in \mathbb{Z}^+$ where $\mathbb{Z}^+$ denotes the set of non-negative integers. In the former case a dynamical system is defined by a semi-flow (semi-group), or by a non-invertible mapping in the latter.

1.3. Qualitative integration of dynamical systems

The study of any phenomenon which exhibits dynamical behavior usually begins with the construction of an associated mathematical model of a dynamical system in the form (1.2.1). Having a model in an explicit form allows us to follow the evolution of its state as time t varies, since the initial data defines a unique solution of (1.2.1). To undertake a complete study of the model we must find this solution, *i.e.* "to integrate" the original system. "Integrating a system" means obtaining an analytical expression for its solution. However, this goal can be achieved only for a very small class of dynamical systems; namely, for systems of linear equations with constant coefficients, and for some very special equations which might be integrated in *quadratures*. Moreover, even if the solution is given in analytical form, the component functions which define the solution may be so complicated that a straightforward analysis becomes practically impossible. Besides that, the problem of finding an analytical form of a solution is not the primary goal of nonlinear dynamics, which is concerned mainly with such "qualitative" properties as the number of equilibrium states, stability, the existence of periodic trajectories, etc. Thus, following Poincaré's approach, instead of attempting a direct integration of the differential equations, we try to extract information concerning the character and form of the functions determined by these equations from the equations themselves.[10] More specifically, we seek to describe the important qualitative features of these functions via a geometrical representation of the phase trajectories. This is the reason why this method is called "qualitative integration".

The first step in our qualitative study is to identify all possible types of trajectories having distinct behaviors and "forms". The second step is to give a description for each group of qualitatively similar trajectories. To achieve a complete description it is necessary to identify certain more essential or "special" trajectories. But here we run into a formidable problem: What properties of the trajectories must we find in order to characterize the qualitative structure of the partition of the phase space into trajectories?

The first step is simple. In fact, it can be reformulated as follows: we must find where a trajectory tends to as $t \to +\infty$ ($t \to -\infty$). Here, we must assume that the trajectory L defined by $x = \varphi(t)$ remains in some bounded region of the phase space for $t \geqslant t_0$ ($t \leqslant t_0$). The following concepts are essential in this study.

Definition 1.3. *A point x^* is called an ω-limit point of the trajectory L if*

$$\lim_{k\to\infty} \varphi(t_k) = x^* ,$$

for some sequence $\{t_k\}$ where $t_k \to +\infty$ as $k \to \infty$.

[10] "Analyse des travans de Henri Poincaré faite par lui-même" [54].

A similar definition of an α-*limit* point applies to $t_k \to -\infty$ as $k \to \infty$. We denote the set of all ω-limit points of a trajectory L by Ω_L, and that of α-limit points by $\mathcal{A}_L$. Observe that an equilibrium state is the unique limit point of itself. In the case where a trajectory L is periodic, all of its points are α and ω-limit points, *i.e.* $L = \Omega_L = \mathcal{A}_L$. In the case where L is an unclosed Poisson-stable trajectory, the sets Ω_L and $\mathcal{A}_L$ coincide with its closure $\bar{L}$. The set $\bar{L}$ is either a minimal set (if L is a recurrent trajectory) or a quasi-minimal set if the Poincaré return times of L are unbounded. All equilibrium states, periodic trajectories, and Poisson-stable trajectories are said to be *self-limit* trajectories.

The structure of the sets Ω_L and $\mathcal{A}_L$ has been more completely studied in the case of *two-dimensional* dynamical systems where all trajectories remain in some bounded domain of the plane as $t \to \pm\infty$. In this case, Poincaré and Bendixson [13] had established that the set Ω_L can only be of one of the following three topological types:

I. Equilibrium states.

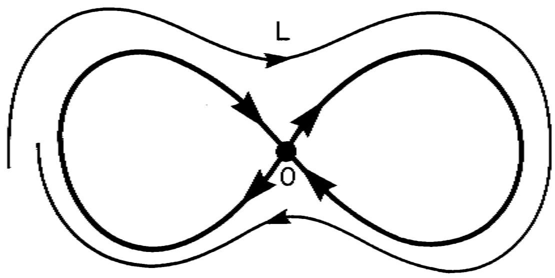
II. Periodic trajectories.

III. Cycles composed of equilibrium states and of connecting trajectories which tend to these equilibrium states as $t \to \pm\infty$.

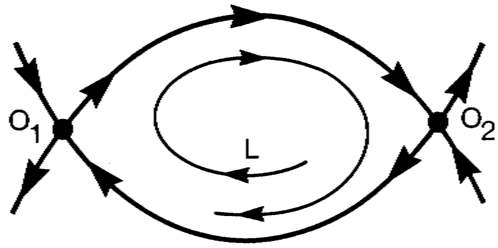
Figure 1.3.1 shows examples of limit sets of type III where the equilibrium states are labeled by O. Using the general classification above, we may enumerate *all* types of positive semi-trajectories in planar systems:

1. equilibrium states;
2. periodic trajectories;
3. semi-trajectories tending to an equilibrium state;
4. semi-trajectories tending to a periodic trajectory;
5. semi-trajectories tending to a limit set of type III.

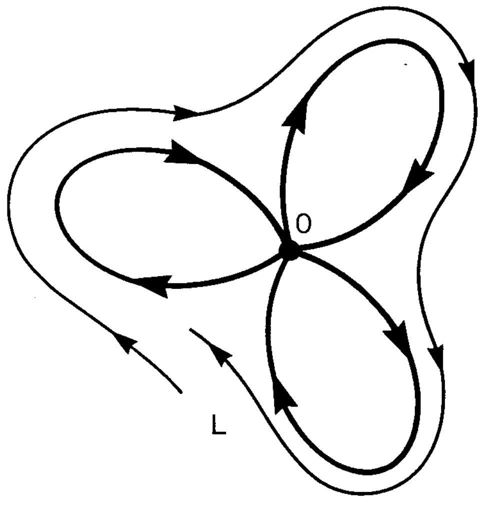
An analogous situation occurs in the case of negative semi-trajectories. Among periodic trajectories in the two-dimensional case a special role is assumed by those which are either the ω-limit set, or the α-limit set of unclosed trajectories located in the inner, or the outer domain of a periodic trajectory, as shown in



(a)

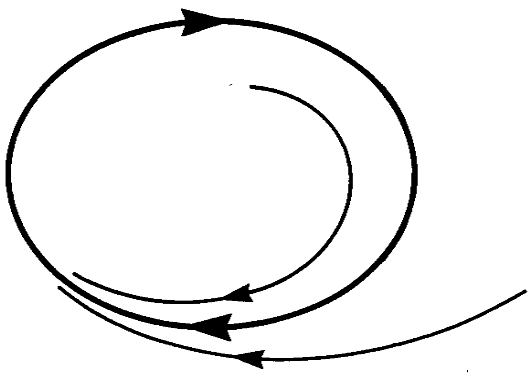


(b)

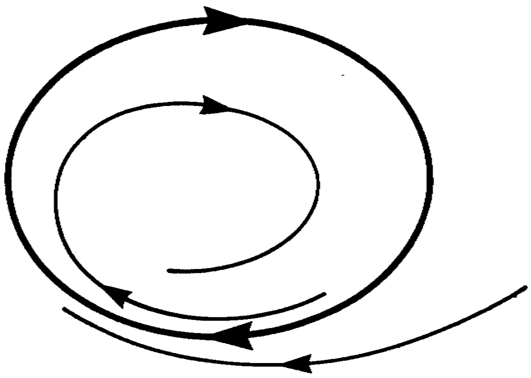


(c)

Fig. 1.3.1. Examples of two ω-limit homoclinic cycles in (a) and (c), and of a heteroclinic cycle in (b) formed by two trajectories going from one equilibrium state to another.



(a)



(b)

Fig. 1.3.2. (a) An ω-limit cycle. (b) A limit cycle which is both ω-limit and α-limit for unclosed trajectories in its neighborhood.

Fig. 1.3.2. Such a periodic trajectory is called *a limit cycle* in the theory of two-dimensional systems.

The corresponding situation in the case of higher dimensions is much more complicated. In this case, in addition to equilibrium states and periodic trajectories, the limit set may be a minimal, or a quasi-minimal set of various topological types, such as a strange attractor in the form of a smooth, or a non-smooth manifold, or a fractal set with a local structure represented as a direct product of a disk and a Cantor set and ever more exotic sets.

Let us now turn to the problem concerning the study of the totality of trajectories. In fact, characterizing a dynamical system means topologically (or qualitatively) partitioning the phase space into the region of the existence of trajectories of different topological types. We usually refer to this problem as "constructing the phase portrait". This problem poses the question: When are two phase portraits similar? In terms of the qualitative theory of dynamical systems we can answer this question by introducing the notion of topological equivalence.

Definition 1.4. *Two systems are said to be topologically equivalent if there exists a homeomorphism of the respective phase spaces which maps the trajectories of one system into the trajectories of the second.*[11]

This definition implies that equilibrium states, as well as periodic and unclosed trajectories of one system, are respectively mapped into equilibrium states, as well as periodic and unclosed trajectories of the other system. The topological equivalence of two systems in some sub-regions of the phase space is defined in a similar manner. The latter is usually used for studying local problems, for example, in a neighborhood of an equilibrium state, or near periodic or homoclinic trajectories. The definition of topological equivalence of two dynamical systems gives an indirect definition of the qualitative structure of partition of the phase space into the regions of the existence of trajectories of topologically different types. Such structures must be invariant with respect to all possible homeomorphisms of the phase space.

Let G be a bounded sub-region of the phase space and let $H = \{h_i\}$ be a set of homeomorphisms defined on G. We can introduce a metric as follows

$$\operatorname{dist}(h_1, h_2) = \sup_{x \in G} \|h_1 x - h_2 x\| .$$

Definition 1.5. *We call a trajectory L, $L \in G$, special if for a sufficiently small $\varepsilon > 0$, for all homeomorphisms h_i satisfying $\operatorname{dist}(h_i, I) < \varepsilon$, where I is the identity homeomorphism, the following condition holds*

$$h_i L = L .$$

It is clear that all equilibrium states and periodic trajectories are special trajectories. Unclosed trajectories may also be special. For example, all trajectories of a two-dimensional system which tend to saddle equilibrium states

[11] See Sec. 2.5 for details.

both as $t \to +\infty$ and as $t \to -\infty$ are also special trajectories. Since such trajectories separate certain regions in the plane they are called *separatrices* (see examples of separatrices in Fig. 1.3.1). A definition for special semi-trajectories may be introduced in an analogous way.

Definition 1.6. *Two trajectories L_1 and L_2 are said to be equivalent if given $\varepsilon > 0$, there exist homeomorphisms $h_1, h_2, \ldots, h_{m(\varepsilon)}$ such that*

$$L_2 = h_{m(\varepsilon)} \cdots h_1 L_1 \, .$$

where $\mathrm{dist}(h_k, I) < \varepsilon$ $(k = 1, 2, \ldots, m(\varepsilon))$.

We will call each set of equivalent trajectories *a cell*. Observe that all trajectories in a cell are of the same topological type. In particular, if a cell is composed of unclosed trajectories, then all of them have the same ω-limit set and the same α-limit set.

Special trajectories and cells are especially important for two-dimensional systems. In this case we may identify some set S by selecting a single trajectory from each cell (all special trajectories belong to S by definition). We will call this set S *a scheme*.[12]

Let us assume that S consists of a finite number of trajectories.[13]

Theorem 1.3. *The scheme is a complete topological invariant.*

This theorem, together with its proof, occupies a significant part of the book *Theory of Dynamical Systems on the Plane* by Andronov, Leontovich, Gordon and Maier [6]. This theory provides not only a mathematical foundation for a theory of oscillations of two-dimensional systems but also gives a recipe for the investigation of concrete systems. In particular, the investigation proceeds in the following order: First, classify the equilibrium states, and then all special trajectories such as separatrices tending to saddle equilibria and trajectories approaching limit sets of type III, either as $t \to +\infty$, or as $t \to -\infty$. This entire collection of special trajectories determines a schematic portrait called *a skeleton* which allows one to partition the phase space into cells, as well as to study the behavior of the trajectories within each cell.

Unfortunately, this does not work when we examine systems of higher dimensions. The set of special trajectories in a three-dimensional system may

[12]The set S can be considered as a factor-system with respect to the above equivalence relation.

[13]The finiteness condition of S is rather general, holding for a wide class of planar systems.

already be infinite, or may even form a continuum. The same situation applies to cells. Thus, the problem of finding a complete topological invariant in this case seems to be quite unrealistic. This is the reason why we must reconcile to the concept of a relatively-incomplete classification based on some topological invariants which apply only to certain cases. Nevertheless, the basic approach for studying concrete high-dimensional systems remains the same as in the two-dimensional case; namely, it begins by examining the equilibrium states and the periodic trajectories. We will consider this "comprehensive" local theory in Chaps. 2 and 3, respectively.

Chapter 2

STRUCTURALLY STABLE EQUILIBRIUM STATES OF DYNAMICAL SYSTEMS

2.1. Notion of an equilibrium state. A linearized system

Let us consider a system of differential equations

$$\dot{x} = X(x) \tag{2.1.1}$$

where $x \in \mathbb{R}^n$ and X is a smooth function in some region $D \subset \mathbb{R}^n$.

Definition 2.1. *A trajectory $x(t)$ of system* (2.1.1) *is called an equilibrium state if it does not depend on time*, *i.e.* $x(t) \equiv x_0 = const$.

It follows from the definition that the coordinates of the equilibrium state can be found as the solution of the system:

$$X(x_0) = 0\,. \tag{2.1.2}$$

If the Jacobian matrix $\partial X/\partial x$ is non-singular at the point x_0, then, by virtue of the implicit function theorem, there are no other solutions of Eq. (2.1.2) nearby x_0. This means that the equilibrium state is *isolated*. However, even when the Jacobian matrix is singular the equilibrium state is usually isolated (excluding the case where the right-hand side of $X(x)$ is of a very special type). Thus, in the general case system (2.1.1) has only a finite number of equilibrium states in any bounded subregion of $\mathbb{R}^n$. Furthermore, when the right-hand side of (2.1.1) is polynomial there are standard algebraic methods for the evaluation of the number of equilibrium states.

From the point of view of numerical simulations the determination of all isolated solutions of system (2.1.2) (or equivalently of all stationary states of (2.1.1)) in a bounded subregion of $\mathbb{R}^n$ is a relatively simple task for small n. However, the number of equilibrium states of a system of higher dimension may be very large and therefore searching for all of them becomes problematic.

The study of system (2.1.1) near an equilibrium state is based on a standard linearization procedure.

Let a point $O(x = x_0)$ be an equilibrium state of system (2.1.1). The substitution

$$x = x_0 + y \tag{2.1.3}$$

shifts the origin to O. With the new variables the system may be written as

$$\dot{y} = X(x_0 + y)\,, \tag{2.1.4}$$

or, by Taylor expansion near $x = x_0$, as

$$\dot{y} = X(x_0) + \frac{\partial X(x_0)}{\partial x} y + o(y)\,. \tag{2.1.5}$$

Since $X(x_0) = 0$ system (2.1.5) becomes

$$\dot{y} = Ay + g(y)\,, \tag{2.1.6}$$

where

$$A = \frac{\partial X(x_0)}{\partial x}\,;$$

A is a constant $(n \times n)$-matrix and $g(y)$ satisfies the condition

$$g(0) = \frac{\partial g(0)}{\partial y} = 0\,. \tag{2.1.7}$$

In the general case, the last term in (2.1.6) is of a higher order of smallness (with respect to the usual norm) than the first term. It is apparent that the behavior of the trajectories of system (2.1.6) in a small neighborhood of the origin is governed primarily by the *linearized system*

$$\dot{y} = Ay\,. \tag{2.1.8}$$

The study of linear systems was the major paradigm of non-conservative dynamics in the 19th century and at the beginning of the 20th century. The main source of such systems was the theory of automatic control, in particular, the control theory of steam engines. The central problem of linear dynamics in that period was the search for the most effective criteria of stability for stationary states.[1]

The stability of an equilibrium state is determined by the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of the Jacobian matrix A which are the roots of the characteristic equation

$$\det |A - \lambda I| = 0 \tag{2.1.9}$$

where I is the identity matrix. The roots of the characteristic equation are also called *the characteristic exponents* of the equilibrium state. The equilibrium state is *stable* when all of its characteristic exponents lie in the left half-plane (LHP) on the complex plane. Moreover, any deviations from equilibrium decay exponentially with decrements of damping proportional to the values $\mathrm{Re}\,\lambda_i$, $(i = 1, \ldots, n)$. Thus, the primary problem of constructing a simple and effective criterion of the stability of an equilibrium state was in finding some explicit conditions in terms of the entries of the matrix A such that it would allow one to determine, without having to solve the characteristic equation, when all of its eigenvalues lie in open LHP.

Here, we present the most popular algorithm called a Routh–Hurwitz criterion. Let $(a_0, \ldots, a_n)$ be the coefficients of the polynomial $\det |\lambda I - A|$:

$$\det |\lambda I - A| = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n \,.$$

We construct an $(n \times n)$-matrix:

$$\tilde{A} = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 & 0 \\ 0 & a_1 & a_3 & \cdots & 0 & 0 \\ 0 & a_0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-2} & a_n \end{vmatrix} \tag{2.1.10}$$

and find the minors $\Delta_1 = a_1$, $\Delta_2 = a_1 a_2 - a_0 a_3, \ldots,$ $\Delta_n = \det \tilde{A}$. Here, Δ_i is the determinant of the matrix whose entries lie on the intersection of the first i rows and the first i columns of the matrix $\tilde{A}$.

[1] The necessity for studying nonlinear nonconservative systems emerged only in the first part of the 20th century, in the context of the investigation of the phenomenon of sustained oscillations in vacuum-tube oscillators.

Routh–Hurwitz criterion. *All characteristic exponents have negative real parts if and only if each Δ_i is positive.*

The *mathematical* question of the correspondence between the properties of the nonlinear system near the equilibrium state and those of the associated linearized system was first posed in papers by Poincaré and Lyapunov. This problem has now been resolved to a considerable extent. In the following sections we will study it in detail and describe the behavior of trajectories of nonlinear systems in a neighborhood of their structurally stable (*equiv.* rough) equilibrium states, *i.e.* those which have no characteristic exponents with zero real part. We note that the presentation below differs from the usual treatment in the sense that we will focus on those features of the system which one needs for the study of strange attractors containing saddle equilibrium states, for example, the Lorenz attractor, the spiral attractors, double-scroll attractors in the Chua circuit, etc.

2.2. Qualitative investigation of 2- and 3-dimensional linear systems

In this and in the following two sections we will study the behavior of solutions of the linearized system. Moreover, we will restrict ourselves to structurally stable equilibrium states only.

Let us begin with low dimensional cases $n = 2$ and $n = 3$.

When $n = 2$ the system assumes this general form:

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y\,, \\ \dot{y} &= a_{21}x + a_{22}y\,. \end{aligned} \tag{2.2.1}$$

The corresponding characteristic equation is

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \tag{2.2.2}$$

and its roots are

$$\lambda_{1,2} = (a_{11} + a_{22})/2 \pm \sqrt{(a_{11} + a_{22})^2/4 - (a_{11}a_{22} - a_{12}a_{21})}\,.$$

The names of the basic equilibrium states of two-dimensional systems were first given by Poincaré. They depend on the values of the characteristic exponents $\lambda_{1,2}$ as follows:

1. Both λ_1 and λ_2 are real and negative: $\lambda_1 < 0$ and $\lambda_2 < 0$. Such an equilibrium state O is called *a stable node*. When $\lambda_1 \neq \lambda_2$ system (2.2.1) can be reduced to

$$\begin{aligned} \dot{\xi} &= \lambda_1 \xi\,, \\ \dot{\eta} &= \lambda_2 \eta \end{aligned} \tag{2.2.3}$$

by a non-singular linear transformation of the space variables, where $\xi(t)$ and $\eta(t)$ are the projections of the phase point $(x(t), y(t))$ onto the eigenvectors of the matrix $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ corresponding to the eigenvalues λ_1 and λ_2, respectively. The general solution of system (2.2.3) is

$$\xi = e^{\lambda_1 t} \xi_0\,, \quad \eta = e^{\lambda_2 t} \eta_0\,. \tag{2.2.4}$$

Since both $\lambda_{1,2}$ are negative, all trajectories are attracted to the origin as $t \to +\infty$. Furthermore, every trajectory approaches the origin O tangentially either to the ξ-axis or to the η-axis. In order to verify this, let us examine the following equation of the integral curves of the system (2.2.3)

$$\eta \xi_0^\nu = \xi^\nu \eta_0 \tag{2.2.5}$$

where $\nu = |\lambda_2|/|\lambda_1|$. For definiteness, let $|\lambda_2|$ be greater than $|\lambda_1|$. Then $\nu > 1$ and, by virtue of (2.2.5), all trajectories approach O tangentially to the ξ-axis except for the two trajectories which lie on the η-axis, see Fig. 2.2.1. The ξ- and η-axes are respectively called *the leading* and *the non-leading directions*.

When $\lambda_1 = \lambda_2 = -\lambda < 0$ system (2.2.1) can be written in one of the following forms below:

$$\begin{aligned} \dot{\xi} &= -\lambda \xi + \eta\,, \\ \dot{\eta} &= -\lambda \eta \end{aligned} \tag{2.2.6}$$

(the non-trivial Jordan block), or

$$\begin{aligned} \dot{\xi} &= -\lambda \xi\,, \\ \dot{\eta} &= -\lambda \eta\,. \end{aligned} \tag{2.2.7}$$

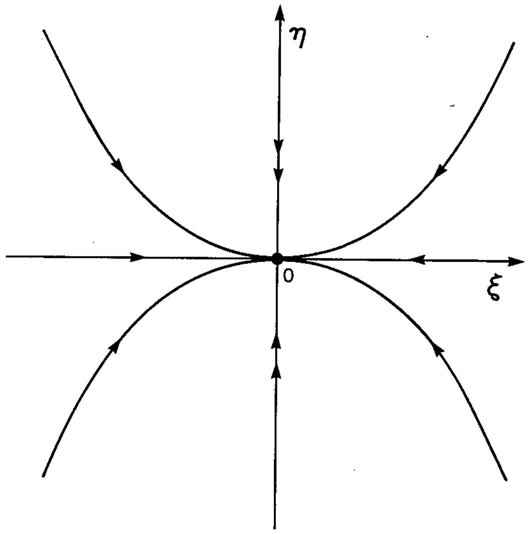


Fig. 2.2.1. A stable node. Double arrows label the strongly stable (non-leading) direction which coincides with the η-axis.

The general solution of the system (2.2.6) is given by

$$\xi = e^{-\lambda t}\xi_0 + te^{-\lambda t}\eta_0\,, \quad \eta = e^{-\lambda t}\eta_0 \tag{2.2.8}$$

and that of the system (2.2.7) is given by

$$\xi = e^{-\lambda t}\xi_0\,, \quad \eta = e^{-\lambda t}\eta_0\,. \tag{2.2.9}$$

Figure 2.2.2 shows the phase portrait in the first case. All trajectories tend to O tangentially to the unique eigenvector, namely, the ξ-axis. In the second case any trajectory approaches O along its own eigen-direction as shown in Fig. 2.2.3. Such a node is called *a dicritical node*.

2. A pair of complex-conjugate roots: $\lambda_{1,2} = -\rho \pm i\omega$, $\rho > 0$, $\omega > 0$. In this case the equilibrium state O is called *a stable focus*. By a non-singular linear change of coordinates the system (2.2.1) can be transformed into:

$$\begin{aligned} \dot{\xi} &= -\rho\xi - \omega\eta\,, \\ \dot{\eta} &= \omega\xi - \rho\eta\,. \end{aligned} \tag{2.2.10}$$

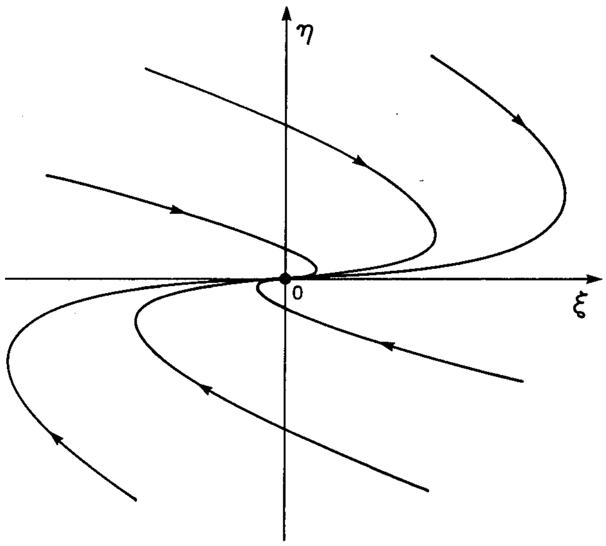


Fig. 2.2.2. Another stable node. Every trajectory enters the origin along the only leading direction which is the ξ-axis.

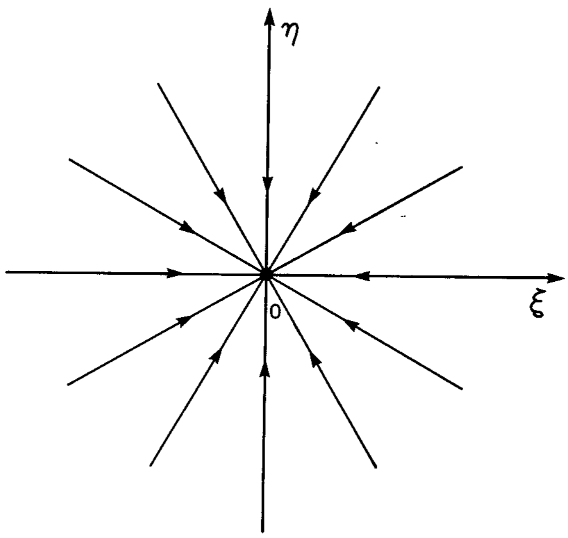


Fig. 2.2.3. A dicritical node. Every trajectory tends to O along its own direction.

In polar coordinates $\xi = r\cos\varphi$, $\eta = r\sin\varphi$, (2.2.10) can be recast as

$$\begin{aligned} \dot{r} &= -\rho r\,, \\ \dot{\varphi} &= \omega\,. \end{aligned} \tag{2.2.11}$$

The general solution of system (2.2.11) is given by

$$\begin{aligned} r(t) &= e^{-\rho t} r_0\,, \\ \varphi(t) &= \omega t + \varphi_0\,, \end{aligned} \tag{2.2.12}$$

or, having returned to the Cartesian coordinates, by

$$\begin{aligned} \xi(t) &= e^{-\rho t}(\xi_0\cos(\omega t) - \eta_0\sin(\omega t))\,, \\ \eta(t) &= e^{-\rho t}(\xi_0\sin(\omega t) + \eta_0\cos(\omega t))\,. \end{aligned} \tag{2.2.13}$$

The phase portrait is represented in Fig. 2.2.4. Any trajectory (with the exception of O) has the form of a "counter-clockwise" spiral tending towards to the origin O as $t \to +\infty$.

3. Both λ_1 and λ_2 are real but of opposite signs: $\lambda_1 = \gamma > 0$, $\lambda_2 = -\lambda < 0$. Such an equilibrium point is called *a saddle*. A linear change of variables

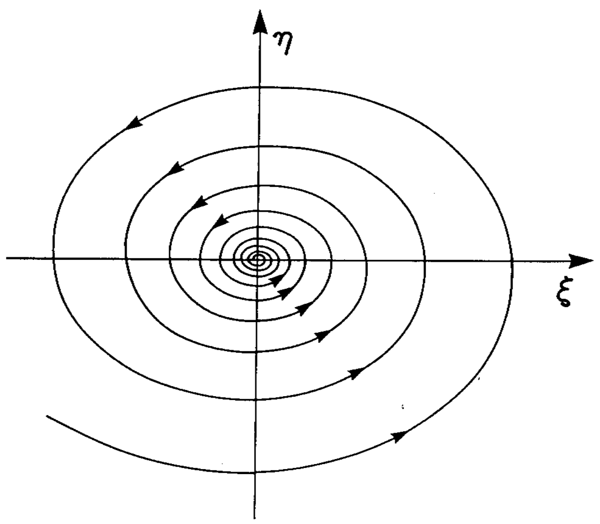


Fig. 2.2.4. A stable focus on a plane.

brings the system (2.2.1) to the form

$$\begin{aligned} \dot{\xi} &= \gamma\xi\,, \\ \dot{\eta} &= -\lambda\eta\,. \end{aligned} \tag{2.2.14}$$

The general solution of the system (2.2.14) is given by

$$\xi = e^{\gamma t}\xi_0\,, \quad \eta = e^{-\lambda t}\eta_0\,. \tag{2.2.15}$$

The corresponding equation of integral curves is given by

$$\eta\xi^{\nu} = \xi_0^{\nu}\eta_0\,, \tag{2.2.16}$$

where $\nu = \lambda/\gamma$. The portrait of the phase space (or just "the phase space") near the saddle is shown in Fig. 2.2.5. There are four exclusive trajectories called *the separatrices*, two *stable* and two *unstable*, which tend to the saddle O as $t \to +\infty$ and as $t \to -\infty$ respectively. All other trajectories pass by the saddle. The pair of the stable separatrices together with the saddle O compose *the stable invariant subspace* of the

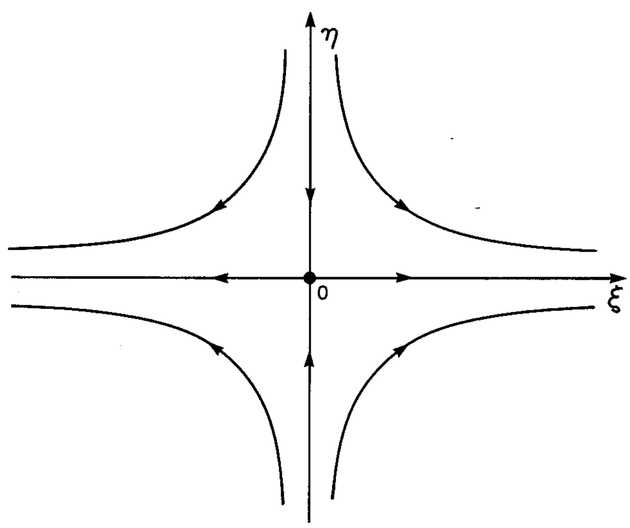


Fig. 2.2.5. A planar saddle.

saddle (the η-axis). *The unstable invariant subspace* of the saddle (the ξ-axis) consists of the unstable separatrices and of the saddle point.

4. The case where the real parts of both characteristic exponents are positive can be simply reduced to the cases (1) and (2) above by the change of time $t \to -t$, so that the directions of the arrows in the respective phase portraits are reversed. When the characteristic exponents are real, the associated equilibrium state is called *an unstable node*. In the case of complex characteristic exponents it is called *an unstable focus* (see Figs. 2.2.6 and 2.2.7).

5. Let us now consider the equilibrium states of three-dimensional systems. Consider first the case where the characteristic exponents λ_i $(i = 1, 2, 3)$ are real and $\lambda_3 < \lambda_2 < \lambda_1 < 0$. Then, the associated three-dimensional system may be reduced to the form

$$\dot{x} = \lambda_1 x\,, \quad \dot{y} = \lambda_2 y\,, \quad \dot{z} = \lambda_3 z\,. \tag{2.2.17}$$

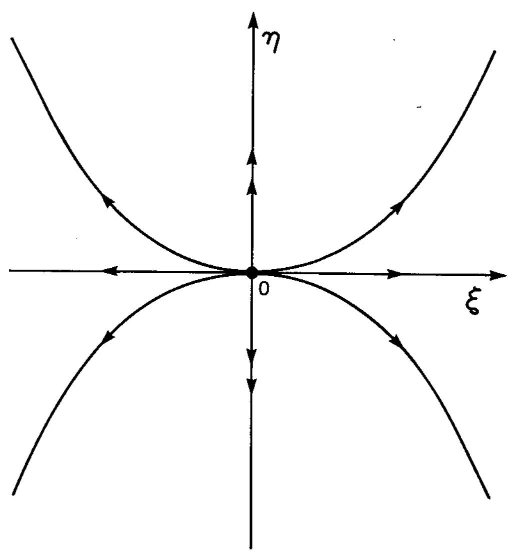


Fig. 2.2.6. An unstable node. The picture is obtained from Fig. 2.2.1 by reversing the time.

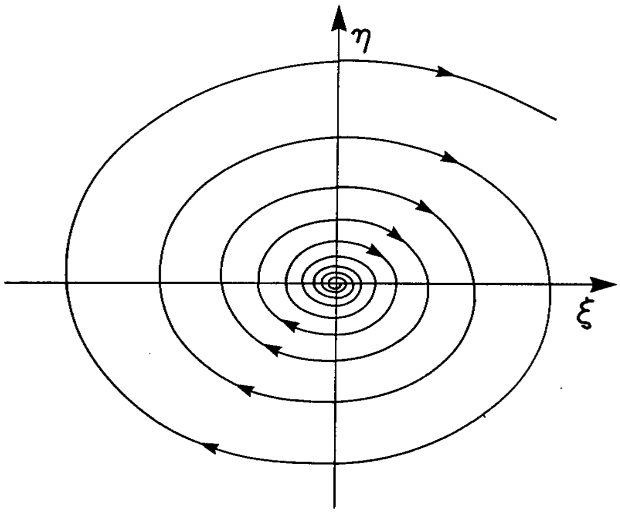


Fig. 2.2.7. An unstable focus. A trajectory traces out a "clockwise" spiral on the plane.

Its general solution is given by

$$x = e^{\lambda_1 t} x_0\,, \quad y = e^{\lambda_2 t} y_0\,, \quad z = e^{\lambda_3 t} z_0\,. \tag{2.2.18}$$

Since all λ_i's are negative, the point O is a stable equilibrium state, *i.e.* all trajectories tend to O as $t \to +\infty$. Furthermore, all trajectories outside of *the non-leading plane* (y, z) approach O along *the leading direction* that coincides with the x-axis, see Fig. 2.2.8. Such an equilibrium state is called *a stable node.*

Let us now consider the case where among the characteristic exponents there is a pair of complex-conjugate $\lambda_{2,3} = -\rho \pm i\omega$. The equilibrium state of the system

$$\dot{x} = \lambda_1 x\,, \quad \dot{y} = -\rho y - \omega z\,, \quad \dot{z} = \omega y - \rho z \tag{2.2.19}$$

in the case where $-\rho < \lambda_1 < 0$, O is also called *a stable node.* The general solution has the form

$$\begin{aligned} x(t) &= e^{\lambda_1 t} x_0\,, \\ y(t) &= e^{-\rho t}(y_0 \cos(\omega t) - z_0 \sin(\omega t))\,, \\ z(t) &= e^{-\rho t}(y_0 \sin(\omega t) + z_0 \cos(\omega t))\,. \end{aligned} \tag{2.2.20}$$

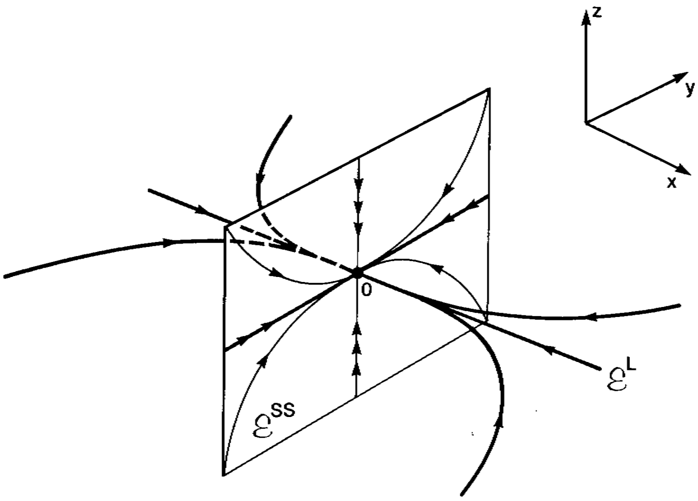


Fig. 2.2.8. A stable node in $\mathbb{R}^3$. The fewer the arrows the weaker is the rate of contraction. The leading subspace $\mathcal{E}^L$ is one-dimensional, two-dimensional subspace $\mathcal{E}^{ss}$ is non-leading.

(see (2.2.13)). The phase portrait of this system is shown in Fig. 2.2.9. It follows from (2.2.20) that

$$\sqrt{y(t)^2 + z(t)^2} = e^{-\rho t}\sqrt{y_0^2 + z_0^2}\,.$$

Moreover, for any trajectory whose initial point does not lie in the non-leading plane (y, z), we obtain

$$\sqrt{y(t)^2 + z(t)^2} = C|x(t)|^{\nu} \tag{2.2.21}$$

where $\nu = \rho/|\lambda_1|$ and $C = \sqrt{y_0^2 + z_0^2}/|x_0|^{\nu}$. Since $\nu > 1$, all such trajectories approach O along the leading x-axis.

6. When $\lambda_1 < -\rho < 0$ the equilibrium state of system (2.2.19) is called *a stable focus*. Relation (2.2.21) still holds, but as $\nu < 1$, all trajectories for which $C \neq 0$ (*i.e.* having initial points which are not on the x-axis) tend to O tangentially to the plane (y, z) as shown in Fig. 2.2.10. In this case, the x-axis is called *the non-leading direction* and the (y, z)-plane is *the leading plane*, respectively.

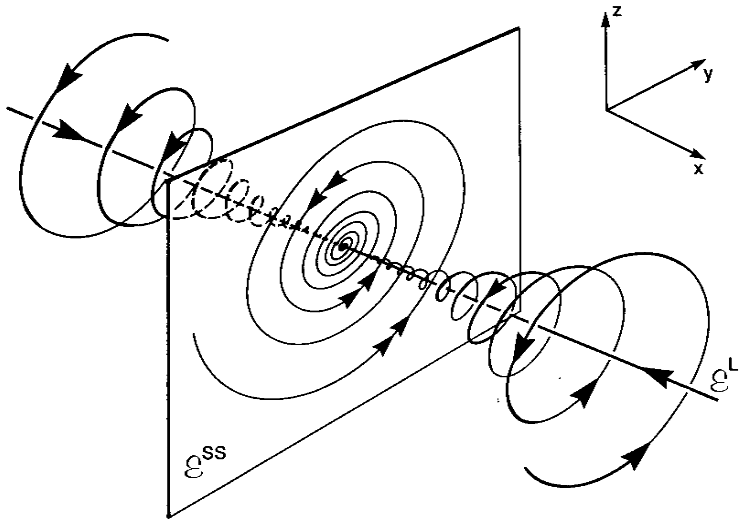


Fig. 2.2.9. Another possible stable node in $\mathbb{R}^3$. Although the point O is a stable focus on $\mathcal{E}^{ss}$, all trajectories outside $\mathcal{E}^{ss}$ go to O along the one-dimensional leading subspace $\mathcal{E}^L$.

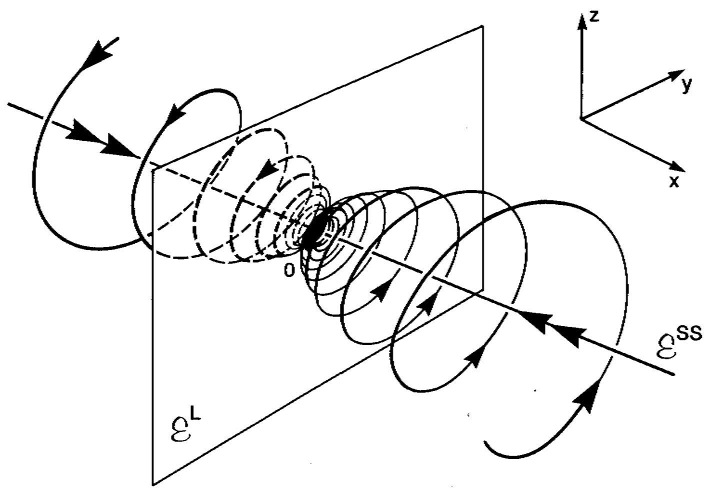


Fig. 2.2.10. A stable focus in $\mathbb{R}^3$. In contrast to Fig. 2.2.9, all trajectories outside of the one-dimensional subspace $\mathcal{E}^{ss}$ tend to O tangentially to the two-dimensional leading subspace $\mathcal{E}^L$.

7. When all characteristic exponents lie to the right of the imaginary axis (*i.e.* in the open right-half plane (RHP)), by reversion of time $t \to -t$, we reduce the problem to the cases considered above. Here, all trajectories tend to the equilibrium state as $t \to -\infty$. As before, there exist two kinds of equilibrium states, namely: *an unstable node*, if the characteristic exponent nearest to the imaginary axis is real, and *an unstable focus*, when the characteristic exponents nearest to the imaginary axis comprise a complex-conjugate pair. The corresponding phase portraits are similar to those shown in Figs. 2.2.8–2.2.10 but with all the arrows pointing in the opposite direction.

8. If there are characteristic exponents both to the left and to the right of the imaginary axis, the equilibrium state is either *a saddle* or *a saddle-focus* (this name was also given by Poincaré), see Figs. 2.2.11–2.2.14.

   Let us assume that $\lambda_1 > 0$ and $\lambda_s < 0$, $(s = 2, 3)$ in (2.2.17). Then, the equilibrium state of system (2.2.17) is a saddle, see Fig. 2.2.11. The general solution is also given by (2.2.18). Because $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, the coordinates y and z decay exponentially to zero as $t \to +\infty$

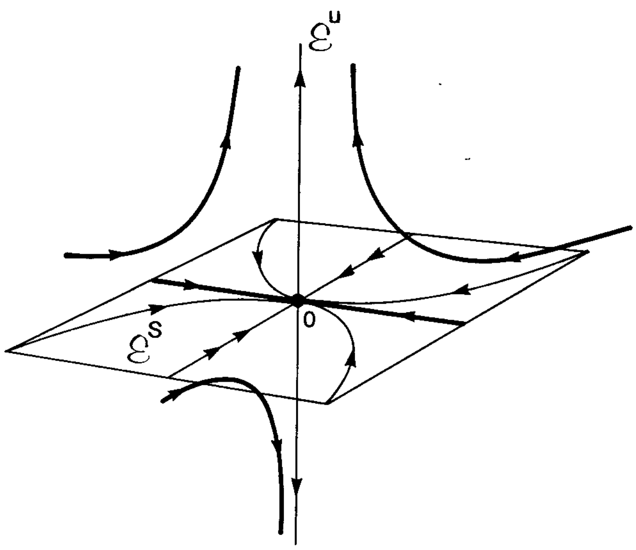


Fig. 2.2.11. A saddle O with the two-dimensional stable subspace $\mathcal{E}^s$ and the one-dimensional unstable subspace $\mathcal{E}^u$.

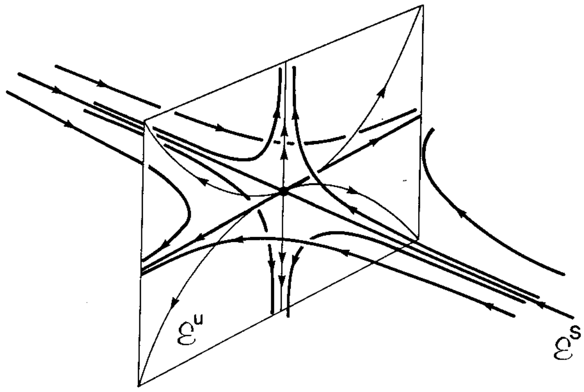


Fig. 2.2.12. A saddle (1,2).

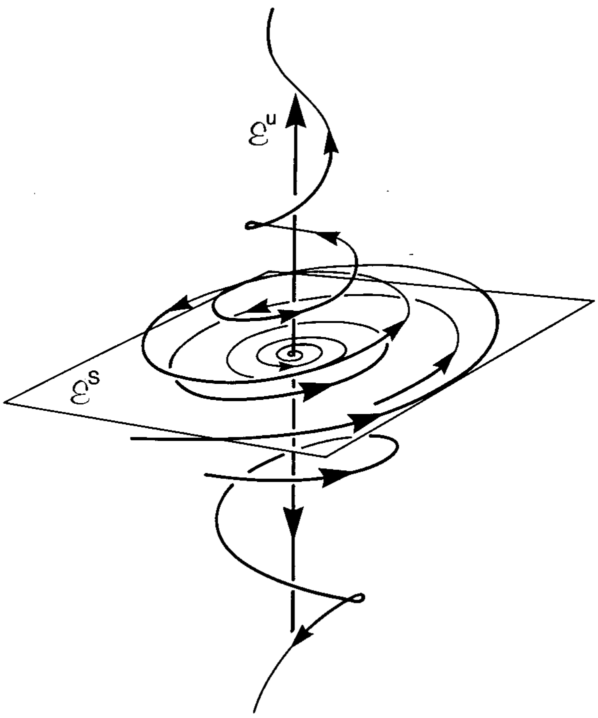


Fig. 2.2.13. A saddle-focus (2,1). It has a two-dimensional stable subspace $\mathcal{E}^s$ and a one-dimensional unstable subspace $\mathcal{E}^u$.

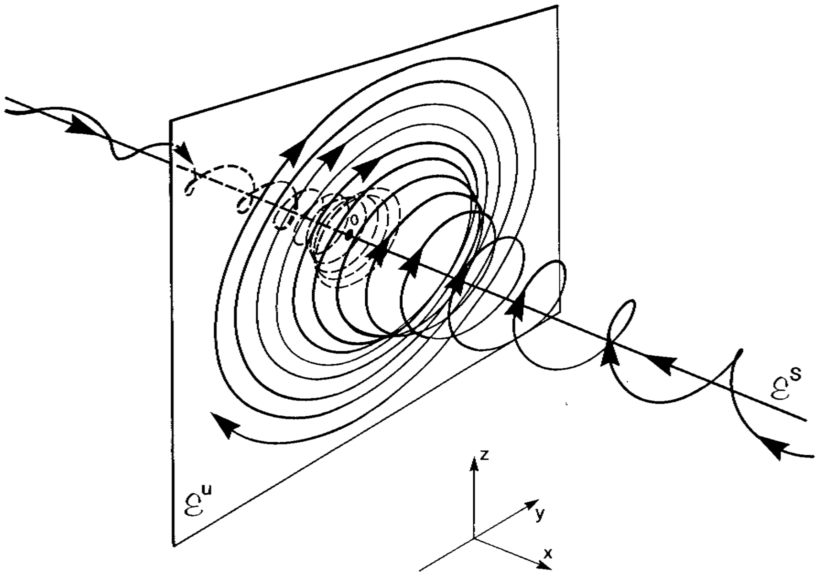


Fig. 2.2.14. A saddle-focus (1,2). Vice versa Fig. 2.2.13.

while the x-coordinate tends to infinity. On the other hand, the x-coordinate decreases to zero as $t \to -\infty$. Therefore, all trajectories which lie entirely in *the stable subspace* $\mathcal{E}^s$: $x = 0$, tend to the saddle O as $t \to +\infty$, while all trajectories which lie in *the unstable subspace* $\mathcal{E}^u$: $(y = 0, z = 0)$, tend to the saddle as $t \to -\infty$. The trajectories outside of $\mathcal{E}^s \bigcup \mathcal{E}^u$ pass nearby but away from the saddle.

The trajectories of the system (2.2.19) near a saddle-focus behave similarly. Now, $\lambda_1 > 0$ and $\lambda_{2,3} = -\rho \pm i\omega$, where $\rho > 0$. The only difference is that in the case of a saddle, the point O is a node on the stable subspace, whereas it is a stable focus on the stable subspace in the case of a saddle-focus.

The case where $\lambda_1 < 0$, $\mathrm{Re}\,\lambda_2 > 0$, $\mathrm{Re}\,\lambda_3 > 0$ is reduced to two previous cases by changing the time variable $t \to -t$, see Figs. 2.2.12 and 2.2.14.

2.3. High-dimensional linear systems. Invariant subspaces

Let us consider the system

$$\dot{y} = Ay\,, \tag{2.3.1}$$

where $y \in \mathbb{R}^n$. The general solution is given by

$$y(t) = e^{At} y_0\,. \tag{2.3.2}$$

Recall that the exponential e^B of a matrix B is defined as the sum of the matrix series

$$e^B = I + B + B^2/2 + \cdots + B^k/k! + \cdots\,,$$

here and below I denotes the identity matrix. Thus, the general solution of the system (2.3.1) can be rewritten as

$$y(t) = (I + At + A^2t^2/2 + \cdots + A^kt^k/k! + \cdots)y_0\,. \tag{2.3.3}$$

The convergence of the series (2.3.3) for any t is obvious since $\|A^kt^k/k!\|$ does not exceed $\|A\|^k|t|^k/k!$ which decays very fast as $k \to \infty$. To verify that (2.3.3) is the general solution, we should note from (2.3.3) that

$$y(0) = y_0$$

and

$$\begin{aligned}\dot{y}(t) &= (A + A^2t + \cdots + A^kt^{k-1}/(k-1)! + \cdots)y_0 \\ &= A(I + At + A^2t^2/2 + \cdots + A^{k-1}t^{k-1}/(k-1)! + \cdots)y_0 = Ay(t)\,.\end{aligned}$$

Let us elaborate the expression (2.3.2). If all eigenvalues of the matrix A are real and different, then one may choose the eigen-basis as a coordinate frame such that matrix A becomes diagonal, *i.e.*

$$A = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{pmatrix},$$

where we have abused our notation by using the same symbol A to denote the original matrix in the new basis to avoid clutter. In this basis we have

$$A^k = \begin{pmatrix} \lambda_1^k & & & \mathbf{0} \\ & \lambda_2^k & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n^k \end{pmatrix}$$

and, therefore

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & & \mathbf{0} \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\lambda_n t} \end{pmatrix}.$$

Thus, if we denote the components of the vector $y \in \mathbb{R}^n$ by $(y_1, \ldots, y_n)$ in the given basis, then the solution of the system can be rewritten in the form:

$$y_s(t) = e^{\lambda_s t} y_{s0}\,, \quad (s = 1, \ldots, n)\,. \tag{2.3.4}$$

If all eigenvalues (we will also call them *the characteristic exponents*) are different as before, but some of them are complex, then there exists a basis in which A attains a block-diagonal form:

$$\begin{pmatrix} A_1 & & & \mathbf{0} \\ & A_2 & & \\ & & \ddots & \\ \mathbf{0} & & & A_m \end{pmatrix}, \tag{2.3.5}$$

where each block A_j corresponds to either a real eigenvalue, or to a pair of complex-conjugate eigenvalues (recall that if A is the real matrix, then the complex-conjugate λ_i^* of any complex eigenvalue λ_i is also an eigenvalue). If λ_j is real, its corresponding block is a (1×1)-matrix:

$$A_j = (\lambda_j)\,. \tag{2.3.6}$$

If $\lambda = \rho + i\omega$ and $\lambda^* = \rho - i\omega$ are a pair of complex-conjugate eigenvalues, then the corresponding block is a (2×2)-matrix:

$$A_j = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} = \begin{pmatrix} \operatorname{Re}\lambda & -\operatorname{Im}\lambda \\ \operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}. \tag{2.3.7}$$

In this basis

$$A^k = \begin{pmatrix} A_1^k & & & \mathbf{0} \\ & A_2^k & & \\ & & \ddots & \\ \mathbf{0} & & & A_m^k \end{pmatrix}.$$

Furthermore, for the complex λ we have

$$A_j^k = \begin{pmatrix} \operatorname{Re}(\lambda^k) & -\operatorname{Im}(\lambda^k) \\ \operatorname{Im}(\lambda^k) & \operatorname{Re}(\lambda^k) \end{pmatrix},$$

hence we obtain

$$e^{At} = \begin{pmatrix} e^{A_1 t} & & & \mathbf{0} \\ & e^{A_2 t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{A_m t} \end{pmatrix},$$

where

$$e^{A_j t} = \begin{cases} e^{\lambda t} \quad \text{for} \quad A_j = (\lambda) \\ \begin{cases} \begin{pmatrix} \operatorname{Re} e^{\lambda t} & -\operatorname{Im} e^{\lambda t} \\ \operatorname{Im} e^{\lambda t} & \operatorname{Re} e^{\lambda t} \end{pmatrix} = e^{\rho t} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \\ \text{for} \quad A_j = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}. \end{cases} \end{cases}$$

So, the general solution (2.3.2) has the form

$$y_s(t) = e^{\lambda_s t} y_{s0} \tag{2.3.8}$$

for real λ_s and

$$\begin{cases} y_s(t) = e^{\rho t}(y_{s0} \cos(\omega t) - y_{s+1,0} \sin(\omega t)) & (2.3.9) \\ y_{s+1}(t) = e^{\rho t}(y_{s0} \sin(\omega t) + y_{s+1,0} \cos(\omega t)) & (2.3.10) \end{cases}$$

for complex $\lambda_s = \lambda_{s+1}^* = \rho + i\omega$.

If A has some multiple eigenvalues one may make a linear transformation so that A becomes block-diagonal in a Jordan basis. Blocks corresponding to simple eigenvalues remain unchanged from the previous case, but for each real eigenvalue λ of multiplicity k, the corresponding block is a $(k \times k)$-matrix of the form

$$\begin{pmatrix} \lambda & \delta_1 & & & & \mathbf{0} \\ & \lambda & \delta_2 & & & \\ & & \lambda & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \delta_{k-1} \\ \mathbf{0} & & & & & \lambda \end{pmatrix}, \tag{2.3.11}$$

where δ_i is either 0 or 1. For each pair of complex-conjugate eigenvalues of multiplicity k, the corresponding block is a $(2k \times 2k)$-matrix of the form

$$\begin{pmatrix} \Lambda & \delta_1 I & & & & \mathbf{0} \\ & \Lambda & \delta_2 I & & & \\ & & \Lambda & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \delta_{k-1} I \\ \mathbf{0} & & & & & \Lambda \end{pmatrix}, \tag{2.3.12}$$

where the matrix Λ is given by (2.3.7), I is the (2×2) identity matrix and δ_i is either 0 or 1. In this case $y(t)$ can also be easily found from formula (2.3.3). For the coordinates y_s corresponding to simple eigenvalues, formulae (2.3.12) and (2.3.13) and (2.3.10) remain unchanged.

For coordinates $(y_{i+1}, \dots, y_{i+k})$ corresponding to a real eigenvalue λ of multiplicity k in the case of the complete Jordan block (*i.e.* when all δ's in (2.3.15) are equal to 1) the following formulae are valid:

$$\begin{aligned} y_{i+k}(t) &= e^{\lambda t} y_{i+k,0}\,, \\ y_{i+k-1}(t) &= e^{\lambda t}(y_{i+k-1,0} + t y_{i+k,0})\,, \\ \vdots \qquad & \qquad \vdots \qquad \qquad \vdots \\ y_{i+j}(t) &= e^{\lambda t} \sum_{s=j}^{s=k} \frac{y_{i+s,0} t^{s-j}}{(s-j)!}\,, \\ \vdots \qquad & \qquad \vdots \qquad \qquad \vdots \end{aligned} \tag{2.3.13}$$

or, equivalently,

$$(y_{i+1}(t), \dots, y_{i+k}(t)) = e^{\lambda t}(y_{i+1,0}, \dots, y_{i+k,0}) e^{J_k t}\,, \tag{2.3.14}$$

where J_k denotes the $(k \times k)$-matrix

$$\begin{pmatrix} 0 & & & & & \mathbf{0} \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}$$

(this is the transposed non-diagonal part of (2.3.15)).

For coordinates $(y_{i+1}, \dots, y_{i+2k})$, corresponding to a pair of the complex-conjugate eigenvalues of multiplicity k in the case of the complete Jordan block, we have

$$
\begin{aligned}
y_{i+2j-1}(t) &= e^{\rho t} \sum_{s=j}^{s=k} (y_{i+2s-1,0}\ \cos(\omega t) - y_{i+2s,0}\ \sin(\omega t)) t^{s-j}/(s-j)!\,, \\
y_{i+2j}(t) &= e^{\rho t} \sum_{s=j}^{s=k} (y_{i+2s-1,0}\ \sin(\omega t) + y_{i+2s,0}\ \cos(\omega t)) t^{s-j}/(s-j)!
\end{aligned}
\tag{2.3.15}
$$

or, equivalently

$$
\begin{pmatrix} y_{i+1}(t) & \cdots & y_{i+2k-1}(t) \\ y_{i+2}(t) & \cdots & y_{i+2k}(t) \end{pmatrix} = e^{\Lambda t} \begin{pmatrix} y_{i+1,0} & \cdots & y_{i+2k-1,0} \\ y_{i+2,0} & \cdots & y_{i+2k,0} \end{pmatrix} e^{J_k t}\,,
\tag{2.3.16}
$$

where Λ is matrix (2.3.7), J_k is the same as in (2.3.14).

If λ is real and the Jordan block is not complete, *i.e.* in (2.3.15) some δ_j vanish, then the block corresponding to λ may be partitioned into two sub-blocks as follows

$$
\left(\begin{array}{cccc|cccc}
\lambda & \delta_1 & & 0 & & & & \\
 & \lambda & \ddots & & & & & \\
 & & \ddots & \delta_{j-1} & & \mathbf{0} & & \\
0 & & & \lambda & & & & \\
\hline
 & & & & \lambda & \delta_{j+1} & & 0 \\
 & & \mathbf{0} & & & \lambda & \ddots & \\
 & & & & & & \ddots & \delta_k \\
 & & & & 0 & & & \lambda
\end{array}\right)
$$

and the exponential is separately found for each sub-block. Analogous calculations are carried out for each complex eigenvalue λ of multiplicity k with a non-complete Jordan block.

We can now prove the following lemma which gives a standard estimate for the norm of the exponential matrix. This estimate will be frequently used throughout the book.

Lemma 2.1. *For given an arbitrarily small $\varepsilon > 0$, one can choose an appropriate basis in $\mathbb{R}^n$ such that the solution $y(t) = e^{At}y_0$ of the linear system*

$$\dot{y} = Ay$$

satisfies the following inequalities

$$\|y(t)\| \le \|e^{At}\|\,\|y_0\| \le e^{(\max\,\mathrm{Re}\,\lambda_i+\varepsilon)t}\,\|y_0\| \qquad \text{for } \ t \ge 0\,; \tag{2.3.17}$$

$$\|y(t)\| \ge \|e^{-At}\|^{-1}\,\|y_0\| \ge e^{(\min\,\mathrm{Re}\,\lambda_i-\varepsilon)t}\,\|y_0\| \qquad \text{for } \ t \ge 0\,; \tag{2.3.18}$$

$$\|y(t)\| \le \|e^{At}\|\,\|y_0\| \le e^{(\min\,\mathrm{Re}\,\lambda_i-\varepsilon)t}\,\|y_0\| \qquad \text{for } \ t \le 0\,; \tag{2.3.19}$$

$$\|y(t)\| \ge \|e^{-At}\|^{-1}\,\|y_0\| \ge e^{(\max\,\mathrm{Re}\,\lambda_i+\varepsilon)t}\,\|y_0\| \qquad \text{for } \ t \le 0\,, \tag{2.3.20}$$

where λ_i $(i = 1, \ldots, n)$ are the characteristic exponents of the matrix A and the norm $\|\cdot\|$ of the vector $y \in \mathbb{R}^n$ denotes the Euclidean norm: $\sqrt{y_1^2 + \cdots + y_n^2}$.

Proof. The proof is analogous for all four inequalities, so let us consider the first one. To prove (2.3.35) in the case when all characteristic exponents are simple, we choose the basis such that equalities (2.3.12), (2.3.13) and (2.3.10) hold whence (2.3.35) follows immediately.

In the case of multiple characteristic exponents, after choosing the Jordan basis the formulae for $y(t)$ (see (2.3.29) and (2.3.31)) have power factors t^k which give the following estimate for the norm of $y(t)$:

$$\|y(t)\| \le e^{\max(\mathrm{Re}\,\lambda_i)t}\|y_0\|\; Q(|t|)\,,$$

where Q is a polynomial of degree less than the largest multiplicity of the characteristic exponents. Since for any arbitrarily small $\varepsilon > 0$ there is some $C(\varepsilon)$ such that

$$Q(|t|) \le Ce^{\varepsilon|t|}\,,$$

we obtain the estimate

$$\|y(t)\| \le Ce^{(\max\,\mathrm{Re}\,\lambda_i+\varepsilon)t}\,\|y_0\| \quad \text{ for } \ t \ge 0\,.$$

To make the constant C equal to 1 we note that the Jordan basis can be chosen such that non-zero values δ_j's in (2.3.15) and (2.3.22) are equal to the given arbitrarily small ε. To do this, instead of the coordinates

$$(y_{i+1}, \ldots, y_{i+k})\,,$$

which correspond to the Jordan block (2.3.15) for real eigenvalues, we must choose the coordinates

$$(y_{i+1}/\varepsilon^{k-1}, y_{i+2}/\varepsilon^{k-2}, \dots, y_{i+k})\,.$$

Similarly, for complex eigenvalues (see formula (2.3.22)), we must replace the coordinates

$$(y_{i+1}, \dots, y_{i+2k})$$

by the coordinates

$$(y_{i+1}/\varepsilon^{k-1},\ y_{i+2}/\varepsilon^{k-1},\ y_{i+3}/\varepsilon^{k-2},\ y_{i+4}/\varepsilon^{k-2}, \dots, y_{i+2k-1},\ y_{i+2k})\,.$$

In this basis, the factor ε^{s-j} appears in front of t^{s-j} in formulae (2.3.29) and (2.3.31) or, equivalently, the coefficient ε appears in front of $J_k t$ in (2.3.30) and (2.3.32). As a result we obtain the following estimate for $y(t)$

$$\|y(t)\| \le e^{\max(\operatorname{Re}\lambda_i)t}\|y_0\|\ \|e^{\varepsilon J_k t}\|\,. \tag{2.3.21}$$

Since $\|J_k\| < 1$, the following estimate is valid:

$$\begin{aligned}\|e^{\varepsilon_k t}\| \le (1 + \varepsilon\|J_k\|t + \varepsilon^2\|J_k\|^2 t^2/2 + \cdots +\\ +\varepsilon^m\|J_k\|^m t^m/m! + \cdots) = e^{\varepsilon\|J_k\|t} \le e^{\varepsilon t}\,,\end{aligned}$$

whence (2.3.39) implies (2.3.35).

It is easily seen from the proof that, when the exponents with the maximal real part are simple, we may then assume $\varepsilon = 0$ in inequalities (2.3.35), (2.3.38). If the exponents with the minimal real part are simple, then we may assume $\varepsilon = 0$ in inequalities (2.3.36) and (2.3.37).

We note also that any arbitrary basis alters inequalities (2.3.35)–(2.3.38) so that an additional coefficient may appear in the right-hand side (the coefficient is, generally, greater than 1 in (2.3.35) and (2.3.37) and less than 1 in (2.3.36) and (2.3.38). Indeed, the transformation from one basis to another is merely a linear change of variables

$$x = Py$$

with some non-singular matrix P. In the new variables x we have

$$\|x\| \le \|P\|\ \|y\|\,, \quad \|y\| \le \|P^{-1}\|\ \|x\|\,.$$

Thus, instead of (2.3.35), for example, we obtain the following inequality for $x(t)$:

$$\|x(t)\| \le Ce^{(\max\ \operatorname{Re}\lambda_i + \varepsilon)t}\|x_0\| \quad \text{for} \ \ t \ge 0\,, \tag{2.3.22}$$

where

$$C = \|P\| \, \|P^{-1}\| \geq 1 \,. \tag{2.3.23}$$

In the particular case where all of the characteristic exponents λ_i lie to the left of the imaginary axis, inequality (2.3.35) becomes the following

$$\|y(t)\| \leq e^{-\lambda t} \|y_0\| \quad \text{for} \quad t \geq 0 \,, \tag{2.3.24}$$

where $\lambda > 0$ is such that $\operatorname{Re} \lambda_i < -\lambda$ at all i (and if the characteristic exponents nearest to the imaginary axis are simple, one may choose $\lambda = \min |\operatorname{Re} \lambda_i|$). Thus, in this case, every trajectory of the linear system (2.3.1) tends exponentially to O as $t \to +\infty$. Such an equilibrium state is called *an exponentially asymptotically stable equilibrium state.*

Let us reorder the characteristic exponents of the stable equilibrium state so that $\operatorname{Re} \lambda_1 \geq \operatorname{Re} \lambda_2 \geq \cdots \geq \operatorname{Re} \lambda_n$. We assume also that first m exponents have the same real parts $\operatorname{Re} \lambda_i = \operatorname{Re} \lambda_1 \, (i = 1, \ldots, m)$ and $\operatorname{Re} \lambda_i < \operatorname{Re} \lambda_1 \, (i = m+1, \ldots, n)$. Let us denote by $\mathcal{E}^L$ and $\mathcal{E}^{ss}$ the m-dimensional and the $(n-m)$-dimensional eigen-subspaces of the matrix A, which correspond to the characteristic exponents $(\lambda_1, \ldots, \lambda_m)$ and $(\lambda_{m+1}, \ldots, \lambda_n)$, respectively. The subspace $\mathcal{E}^L$ is called *the leading invariant subspace* and $\mathcal{E}^{ss}$ is called *the non-leading* or *the strongly stable invariant subspace.*

These names are derived from the fact that as $t \to +\infty$ all trajectories, except for those lying in $\mathcal{E}^{ss}$, tend to the equilibrium state O tangentially to the subspace $\mathcal{E}^L$. Moreover, the trajectories from $\mathcal{E}^{ss}$ tend to O faster than $e^{(\operatorname{Re} \lambda_{m+1} + \varepsilon)t}$, whereas the convergence velocity of the other trajectories does not exceed $e^{(\operatorname{Re} \lambda_1 - \varepsilon)t}$, where the constant $\varepsilon > 0$ may be chosen arbitrarily small.

To prove this statement we note that each vector $y \in \mathbb{R}^n$ is uniquely represented in the form $y = u + v$, where $u \in \mathcal{E}^L$ and $v \in \mathcal{E}^{ss}$. In the (u, v) coordinates system (2.3.1) is written as

$$\begin{aligned} \dot{u} &= A_1 u \,, \\ \dot{v} &= A_2 v \,, \end{aligned}$$

where $\operatorname{spectr} A_1 = \{\lambda_1, \ldots, \lambda_m\}$ and $\operatorname{spectr} A_2 = \{\lambda_{m+1}, \ldots, \lambda_n\}$. The general solution is given by

$$u(t) = e^{A_1 t} u_0 \,, \quad v(t) = e^{A_2 t} v_0 \,. \tag{2.3.25}$$

According to Lemma 2.1 (see (2.3.36), (2.3.35)), it follows from (2.3.43) that

$$\|u(t)\| \geq e^{(\operatorname{Re}\lambda_1-\varepsilon)t}\|u_0\|\,,$$
$$\|v(t)\| \leq e^{(\operatorname{Re}\lambda_{m+1}+\varepsilon)t}\|v_0\|$$

for positive t, where ε may be made arbitrarily small by a suitable choice of the bases in $\mathcal{E}^L$ and $\mathcal{E}^{ss}$. Hence we can obtain the following inequality

$$\|v(t)\|\;\|u_0\|^{\nu} \leq \|v_0\|\;\|u(t)\|^{\nu} \tag{2.3.26}$$

where $\nu > 1$. It is seen from (2.3.44) that if $\|u_0\| \neq 0$, then any trajectory approaches O tangentially to the leading subspace $v = 0$.

In the case $m = 1$, *i.e.* where λ_1 is real and $\operatorname{Re}\lambda_i < \lambda_1$, $(i = 2, \ldots, n)$, the leading subspace is a straight line. Such an equilibrium state is called *a stable node* (see Figs. 2.2.8, 2.2.9).

If $m = 2$ and $\lambda_{1,2} = -\rho \pm i\omega$, $\rho > 0$, $\omega \neq 0$, then the corresponding equilibrium state is called *a stable focus*. The leading subspace here is two-dimensional and all trajectories which do not belong to $\mathcal{E}^{ss}$ have the shape of spirals winding around towards O, see Fig. 2.2.10.

The unstable case, where $\operatorname{Re}\lambda_i > 0$, $(i = 1, \ldots, n)$, is reduced to the previous one by reversion of time $t \to -t$. Therefore, the solution can be estimated as:

$$\|y(t)\| \leq e^{-\lambda|t|}\|y_0\|\,, \quad \text{for} \quad t \leq 0\,, \tag{2.3.27}$$

where $\lambda > 0$ is an arbitrary constant satisfying $\operatorname{Re}\lambda_i > \lambda$. By virtue of (2.3.45), all trajectories tend exponentially to O as $t \to -\infty$. Such equilibrium states are *exponentially completely unstable*.

The leading and the non-leading subspaces are defined here in the same way as in case of stable equilibrium states (but for $t \to -\infty$). When the leading subspace is one-dimensional, the equilibrium state is called *an unstable node*. When the leading subspace is two-dimensional and a pair of complex-conjugate exponents is nearest to the imaginary axis, then such an equilibrium state is called *an unstable focus*.

Now, let k characteristic exponents lie to the left of the imaginary axis and $(n - k)$ to the right of it, *i.e.* $\operatorname{Re}\lambda_i < 0\,(i = 1, \ldots, k)$ and $\operatorname{Re}\lambda_j > 0\,(j = k+1, \ldots, n)$, where $k \neq 0, n$. Such an equilibrium state is *an equilibrium state of the saddle type*.

A linear non-singular change of variables transforms the system (2.3.1) into

$$\begin{aligned} \dot{u} &= A^- u\,, \\ \dot{v} &= A^+ v \end{aligned} \tag{2.3.28}$$

where $\operatorname{spectr} A^- = \{\lambda_1, \ldots, \lambda_k\}$ and $\operatorname{spectr} A^+ = \{\lambda_{k+1}, \ldots, \lambda_n\}$, and $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$. The general solution is given by

$$u(t) = e^{A^- t} u_0\,, \quad v(t) = e^{A^+ t} v_0\,. \tag{2.3.29}$$

According to Lemma 2.1, for the variables u and v the estimates analogous, respectively, to (2.3.42) and (2.3.45) are valid, *i.e.* any trajectory from *the stable invariant subspace* $\mathcal{E}^s$: $v = 0$ tends exponentially to O as $t \to +\infty$, and any trajectory from *the unstable invariant subspace* $\mathcal{E}^u$: $u = 0$ tends exponentially to O as $t \to -\infty$; the neighboring trajectories pass nearby but away from the saddle.

Thus, the saddle is the stable equilibrium for the system on $\mathcal{E}^s$ and is completely unstable on $\mathcal{E}^u$. Furthermore, *stable and unstable leading and non-leading subspaces*, respectively, $\mathcal{E}^{sL}, \mathcal{E}^{uL}, \mathcal{E}^{ss}$ and $\mathcal{E}^{uu}$ can be defined in the subspaces $\mathcal{E}^s$ and $\mathcal{E}^u$. We will call a direct sum $\mathcal{E}^{sE} = \mathcal{E}^s \oplus \mathcal{E}^{uL}$ *the extended stable invariant subspace*, and $\mathcal{E}^{uE} = \mathcal{E}^u \oplus \mathcal{E}^{sL}$ *the extended unstable invariant subspace*. The invariant subspace $\mathcal{E}^L = \mathcal{E}^{sE} \cap \mathcal{E}^{uE}$ is called *the leading saddle subspace*.

If the point O is a node in both $\mathcal{E}^s$ and $\mathcal{E}^u$, such an equilibrium state is called *a saddle*. Therefore, the dimensions of both $\mathcal{E}^{sL}$ and $\mathcal{E}^{uL}$ are equal to 1.

When the point O is a focus on at least one of two subspaces $\mathcal{E}^s$ and $\mathcal{E}^u$, then O is called *a saddle-focus*. Depending on the dimensions of the stable and the unstable leading subspaces, we may define three types of saddle-foci; namely:

- *saddle-focus* (2,1) — a focus on $\mathcal{E}^s$ and a node on $\mathcal{E}^u$;
- *saddle-focus* (1,2) — a node on $\mathcal{E}^s$ and a focus on $\mathcal{E}^u$;
- *saddle-focus* (2,2) — a focus on both $\mathcal{E}^s$ and $\mathcal{E}^u$;

The phase portraits for two types of three-dimensional saddles and saddle-foci $(2,1)$ and $(1,2)$ are shown in Figs. 2.2.11–2.2.14; a four-dimensional saddle-focus $(2,2)$ is schematically represented in Fig. 2.3.1.

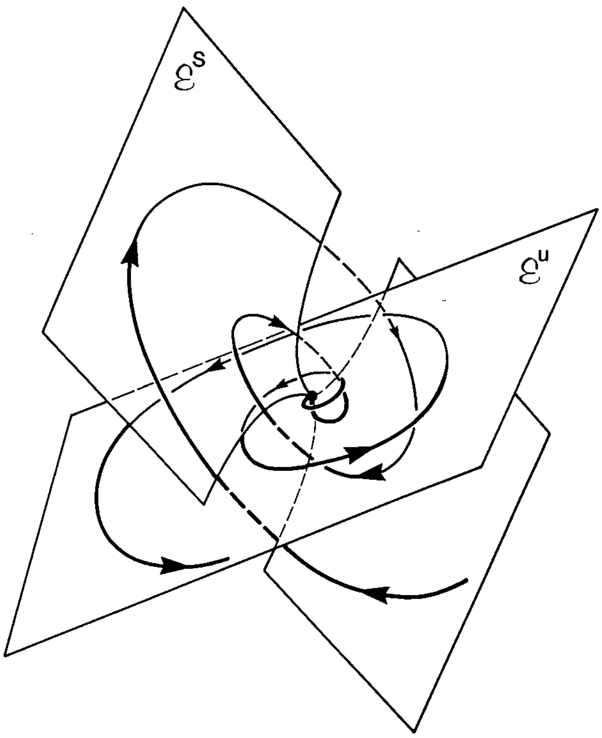


Fig. 2.3.1. The pseudo-projection of a saddle-focus (2,2) into $\mathbb{R}^3$. Both stable and unstable invariant subspaces are of dimension two.

2.4. Behavior of trajectories of a linear system near saddle equilibrium states

The theory considered in the previous sections is sufficient to resolve the following important question. Let a linear system have a structurally stable equilibrium state O of the saddle type. Choose a point M^+ in the stable invariant subspace $\mathcal{E}^s$ of O and a point M^- in its unstable subspace $\mathcal{E}^u$. Surround the point M^+ by a small neighborhood V^+, and the point M^- by a neighborhood V^-. We ask if there are any points within V^+ whose trajectories reach V^-. How is the set of such points organized and which properties does the map defined by the trajectories that connect V^+ and V^- possess?

This problem is purely geometrical for the linearized system. However, we notice that this setting is almost identical to that of the problem of the behavior of trajectories near saddle equilibria in chaotic systems.

Let us first consider three-dimensional examples. Let the point O be a saddle, *i.e.* both of its leading exponents are real. To be specific, let us suppose that the stable subspace is two-dimensional and the unstable subspace is one-dimensional. The system can then be written as

$$\begin{aligned}\dot{x} &= -\lambda_1 x\,,\\ \dot{u} &= -\lambda_2 u\,,\\ \dot{y} &= \gamma y\,,\end{aligned}$$

where $0 < \lambda_1 < \lambda_2$, $\gamma > 0$. The unstable subspace $\mathcal{E}^u$ here coincide with the y-axis, and the stable subspace $\mathcal{E}^s$ is the (x, u)-plane. The u-axis is the non-leading subspace $\mathcal{E}^{ss}$ and the x-axis is the leading subspace $\mathcal{E}^{sL}$. The extended unstable subspace $\mathcal{E}^{uE}$ is the (x, y)-plane and the extended stable subspace $\mathcal{E}^{sE}$ is the entire space $\mathbb{R}^3$.

The general solution of the system is

$$\begin{aligned}x(t) &= e^{-\lambda_1 t}\, x_0\,,\\ u(t) &= e^{-\lambda_2 t}\, u_0\,,\\ y(t) &= e^{\gamma t}\, y_0\,.\end{aligned}$$

In $\mathcal{E}^s$ we choose a point $M^+(x^+, u^+, y = 0)$ not lying in $\mathcal{E}^{ss}$, *i.e.* $x^+ \neq 0$. Without loss of generality, we suppose $x^+ > 0$. Let us choose a small $\varepsilon > 0$ and at the point M^+ build a small rectangle $\Pi^+ = \{x = x^+, |u - u^+| < \varepsilon, |y| < \varepsilon\}$. The coordinates on Π^+ are (u, y). The intersection $y = 0$ of the stable subspace with Π^+ partitions Π^+ into two sub-components. Choose the component (shaded area in Fig. 2.4.1) where $y > 0$ and follow trajectories starting with each point on it.

Consider a neighborhood U_δ: $\{|x| \leq \delta,\ |u| \leq \delta,\ |y| \leq \delta\}$ of the saddle for some $\delta > 0$. Starting with any point $M(x = x^+,\ u,\ y > 0) \in \Pi^+$, the trajectory leaves U_δ as $t \to +\infty$, crossing Π^-: $\{y = \delta\}$ at the point $\bar{M}(\bar{x}, \bar{u}, y = \delta)$ whose coordinates are given by the following formulae

$$\bar{x} = e^{-\lambda_1 t}\, x^+\,, \quad \bar{u} = e^{-\lambda_2 t}\, u\,, \tag{2.4.1}$$

$$\delta = e^{\gamma t}\, y\,. \tag{2.4.2}$$

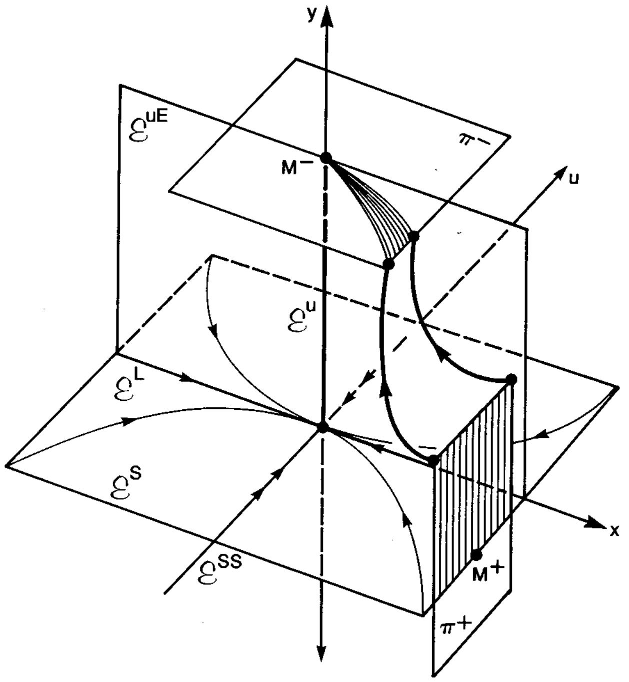


Fig. 2.4.1. Trajectories near a saddle. The image of the rectangle Π^+ is a curvilinear triangle on Π^- which is tangential to the extended unstable subspace $\mathcal{E}^{uE}$. The intersection of the stable subspace $\mathcal{E}^s$ on Π^+ is mapped into the point M^-.

By resolving (2.4.2) we obtain the flight time from Π^+ to Π^-

$$t = \frac{1}{\gamma} \ln \frac{\delta}{y} \,.$$

Substituting the latter into (2.4.1) gives us the expression for the coordinates of the points M in terms of those of $\bar{M}$

$$\begin{aligned} \bar{x} &= x^+ \left(\frac{y}{\delta}\right)^{\nu} , \\ \bar{u} &= u \left(\frac{y}{\delta}\right)^{\alpha\nu} , \end{aligned} \tag{2.4.3}$$

where $\nu = \lambda_1/\gamma$, $\alpha = \lambda_2/\lambda_1 > 1$. It is clear from this formula that the map $M \mapsto \bar{M}$ is contracting with respect to the non-leading coordinate u provided that y is small enough. Moreover, as $y \to +0$ (*i.e.* the point M tending to $\mathcal{E}^S$) the contraction becomes infinitely strong.

The map (2.4.3) along the trajectories of the system takes the upper part of the rectangle Π^+ onto a curvilinear wedge on Π^-:

$$C_2\,\bar{x}^\alpha \le \bar{u} \le C_1\,\bar{x}^\alpha\,, \quad C_{1,2} = (u^+ \pm \varepsilon)/(x^+)^\alpha\,. \tag{2.4.4}$$

The wedge adjoins to the point $M^-(\bar{x} = 0,\ \bar{u} = 0,\ \bar{y} = \delta) = \Pi^- \cap \mathcal{E}^u$. Since $\alpha > 1$ and $C_{1,2} \neq \infty$ (as $x^+ \neq 0$) the wedge touches the extended unstable subspace $\mathcal{E}^{uE}$: $\bar{u} = 0$ at the point M^-, as shown in Fig. 2.4.1.

The case when $\mathcal{E}^s$ is one-dimensional and $\mathcal{E}^u$ is two-dimensional is reduced to that considered above by means of the reversion of time. Therefore, if we choose the points $M^+ \in \mathcal{E}^s$ and $M^- \in \mathcal{E}^u \backslash \mathcal{E}^{uu}$ and construct two transverse rectangles Π^+ and Π^-, the set of points on Π^+ whose trajectories reach Π^- has also the form a curvilinear wedge as in (2.4.4), and its image on Π^- is one of the two components of $\Pi^- \backslash \mathcal{E}^u$ as shown in Fig. 2.4.2.

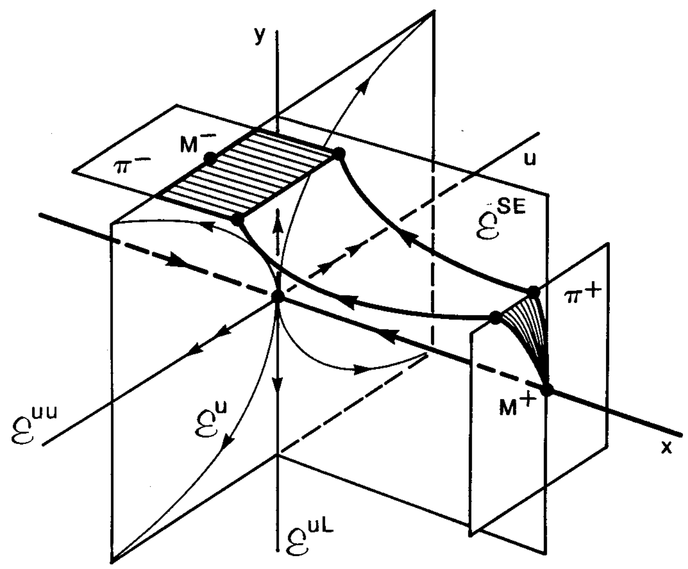


Fig. 2.4.2. The behavior of trajectories near this saddle is the reverse situation depicted in Fig. 2.4.1.

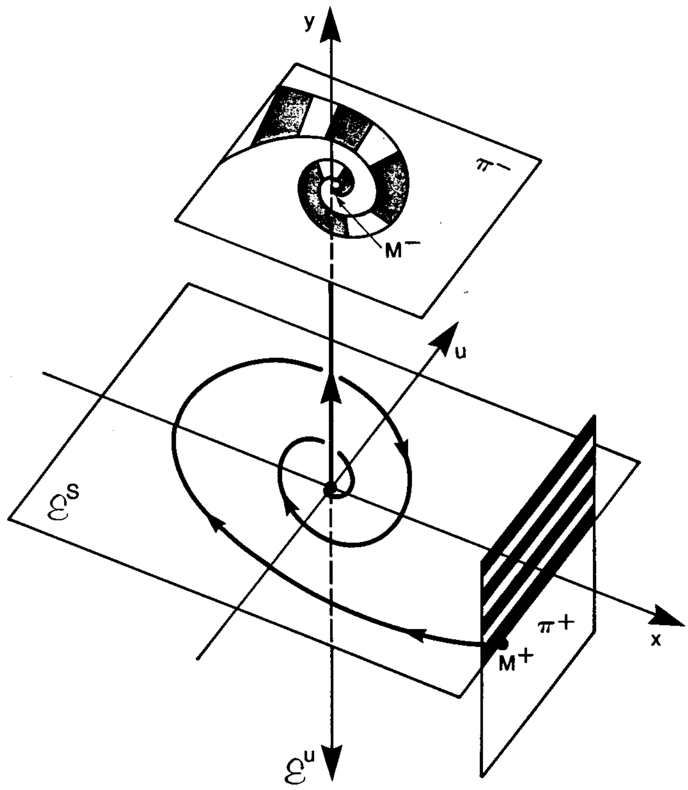


Fig. 2.4.3. The map near a saddle-focus $(2,1)$. The zebra pattern on Π^+ is mapped along the trajectories inside two spirals around the point M^- which is the image of the intersection of $\mathcal{E}^s$ and Π^+.[2]

When O is a saddle-focus (2,1) (see Fig. 2.4.3) the system can be represented in the form

$$\begin{aligned}\dot{x} &= -\rho x - \omega u\,,\\ \dot{u} &= \omega x - \rho u\,,\\ \dot{y} &= \gamma y\,,\end{aligned}\qquad (2.4.5)$$

[2]Remark. We must notice that the formulae below are derived for the case where the cross-section Π^+ is oriented along but not transversally to the x-axis.

where $\rho > 0$, $\omega > 0$, $\gamma > 0$. The general solution here is given by

$$\begin{aligned} x(t) &= e^{-\rho t}(x_0 \cos(\omega t) - u_0 \sin(\omega t)) \,, \\ u(t) &= e^{-\rho t}(x_0 \sin(\omega t) + u_0 \cos(\omega t)) \,, \\ y(t) &= e^{\gamma t} y_0 \,. \end{aligned} \tag{2.4.6}$$

Let us select an arbitrary point $M^+(x^+, u^+, y = 0)$ on $\mathcal{E}^s \backslash O$. We can always assume $u^+ = 0$ due to a rotation of the coordinate frame, so that formulae (2.4.5) and (2.4.6) remain unchanged. Through the point M^+ we construct the rectangle $\Pi^+ = \{u = 0, |x - x^+| < \varepsilon, |y| < \varepsilon\}$. Since the derivative $\dot{u}$ does not vanish at M^+, it does not vanish in a small neighborhood of M^+ by virtue of continuity. Therefore, the trajectories starting with Π^+ must cross it transversely.

The trajectories starting from $\Pi^+ \cap \{y > 0\}$ leave the neighborhood of the saddle-focus and pass through the plane Π^-: $y = \delta$. In this case the map from $\Pi^+ \cap \{y > 0\}$ into Π^- is represented by the formulae:

$$\begin{aligned} \bar{x} &= x e^{-\rho t} \cos(\omega t) \,, \\ \bar{u} &= x e^{-\rho t} \sin(\omega t) \,, \\ \delta &= y e^{\gamma t} \,, \end{aligned}$$

or

$$\begin{aligned} \bar{x} &= x \left(\frac{y}{\delta}\right)^{\nu} \cos\left(\frac{\omega}{\gamma} \ln \frac{y}{\delta}\right) , \\ \bar{u} &= -x \left(\frac{y}{\delta}\right)^{\nu} \sin\left(\frac{\omega}{\gamma} \ln \frac{y}{\delta}\right) , \end{aligned}$$

where $\nu = \rho/\gamma$. Having introduced the polar coordinates $\bar{x} = \bar{r}\cos(\bar{\varphi})$ and $\bar{u} = \bar{r}\sin(\bar{\varphi})$ on Π^-, the map may be written as

$$\begin{aligned} \bar{r} &= |x| \left(\frac{y}{\delta}\right)^{\nu} , \\ \bar{\varphi} &= -\frac{\omega}{\gamma} \ln\left(\frac{y}{\delta}\right) + \varphi_0 \,, \end{aligned}$$

where

$$\varphi_0 = \begin{cases} 0 \,, & \text{if} \quad x^+ > 0 \\ \pi \,, & \text{if} \quad x^+ < 0 \,. \end{cases}$$

The rectangle Π^+ is bounded by the segments of the straight lines $x = x^+ + \varepsilon$ and $x = x^+ - \varepsilon$. The image of Π^+ on Π^- is therefore bounded by a pair of logarithmic spirals defined by:

$$(|x^+| - \varepsilon)\, e^{\frac{\rho}{\omega}(\varphi_0 - \bar{\varphi})} \leq \bar{r} \leq (|x^+| + \varepsilon)\, e^{\frac{\rho}{\omega}(\varphi_0 - \bar{\varphi})}$$

which wind around the point $M^- = \Pi^- \cap \mathcal{E}^u$ as drawn in Fig. 2.4.3.

The above result can easily be transformed to apply to a saddle-focus (1,2) in backward time.

Let us consider next a saddle-focus (2,2). The system is written in the following form:

$$\begin{aligned}
\dot{x}_1 &= -\rho_1 x_1 - \omega_1 x_2\,, \\
\dot{x}_2 &= -\rho_1 x_2 + \omega_1 x_1\,, \\
\dot{y}_1 &= \rho_2 y_1 - \omega_2 y_2\,, \\
\dot{y}_2 &= \rho_2 y_2 + \omega_2 y_1
\end{aligned}$$

where $\rho_1 > 0$, $\omega_1 > 0$, $\rho_2 > 0$, $\omega_2 > 0$. The general solution is given by

$$\begin{aligned}
x_1 &= e^{-\rho_1 t}\left(x_{10}\cos(\omega_1 t) - x_{20}\sin(\omega_1 t)\right), \\
x_2 &= e^{-\rho_1 t}\left(x_{10}\sin(\omega_1 t) + x_{20}\cos(\omega_1 t)\right), \\
y_1 &= e^{\rho_2 t}\left(y_{10}\cos(\omega_2 t) - y_{20}\sin(\omega_2 t)\right), \\
y_2 &= e^{\rho_2 t}\left(y_{10}\sin(\omega_2 t) + y_{20}\cos(\omega_2 t)\right).
\end{aligned} \tag{2.4.7}$$

The map T from $\Pi^+ = \{x_2 = 0, |x_1 - x_1^+| < \varepsilon, |y_1| < \varepsilon, |y_2| < \varepsilon\}$ into $\Pi^- = \{\bar{y}_2 = 0, |\bar{y}_1 - y_1^-| < \varepsilon, |\bar{x}_1| < \varepsilon, |\bar{x}_2| < \varepsilon\}$ along the trajectories of the system is given by the following formulae:

$$\begin{aligned}
\bar{x}_1 &= x_1 e^{-\rho_1 t}\cos(\omega_1 t)\,, \\
\bar{x}_2 &= x_1 e^{-\rho_1 t}\sin(\omega_1 t)\,,
\end{aligned} \tag{2.4.8}$$

$$\begin{aligned}
y_1 &= \bar{y}_1 e^{-\rho_2 t}\cos(\omega_2 t)\,, \\
y_2 &= -\bar{y}_1 e^{-\rho_2 t}\sin(\omega_2 t)\,.
\end{aligned} \tag{2.4.9}$$

In order to find the domain D of the map T it is more convenient to use (2.4.8) and (2.4.9) directly rather than to express the flight time t through (y_1, y_2). If we choose an arbitrary x_1 satisfying $|x_1 - x_1^+| < \varepsilon$, and $\bar{y}_1$ satisfying $|\bar{y}_1 - y_1^-| < \varepsilon$, and a sufficiently large t, then formulae (2.4.8) and (2.4.9) give us the values of $\bar{x}_1, \bar{x}_2, y_1, y_2$ such that the trajectory starting with the point $M = (x_1, 0, y_1, y_2)$ of Π^+ intersects Π^- at the point $\bar{M} = (\bar{x}_1, \bar{x}_2, \bar{y}_1, 0)$. The domain D is composed of all points M whose x_1-coordinate lies in the interval $|x_1 - x_1^+| < \varepsilon$ and whose $y_{1,2}$-coordinates are found from (2.4.9) for some appropriate choice of $\bar{y}_1$ and t.

By virtue of (2.4.9), the point (y_1, y_2) traces out a logarithmic spiral as t varies while x_1 and $\bar{y}_1$ are kept fixed. This means that the set D has the shape of a roulette stretched along the x_1-direction, and twisted in the (y_1, y_2)-coordinates. The image of D in Π^- has a similar shape, see Fig. 2.4.4.

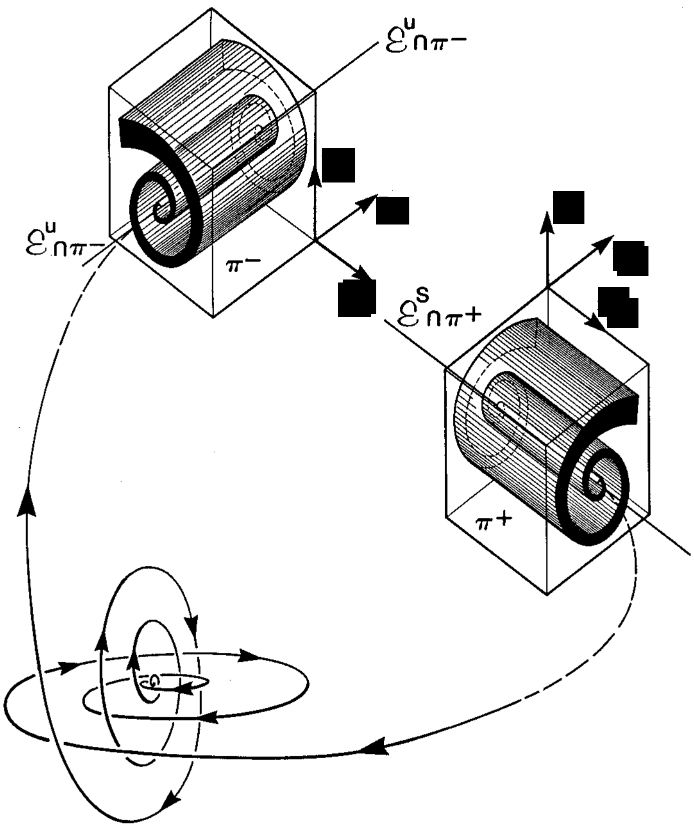


Fig. 2.4.4. The map near a saddle-focus $(2, 2)$. See comments in the text.

In a high-dimensional case the system near a saddle is represented in the following form:

$$\dot{x} = A^- x\,, \qquad \dot{y} = A^+ y\,,$$
$$\dot{u} = B^- u\,, \qquad \dot{v} = B^+ v\,,$$

where x and y are the leading variables, u and v are the non-leading ones. The spectrum of the matrix A^- lies on a straight line $\operatorname{Re} z = -\lambda < 0$ in the complex plane and the spectrum of A^+ lies on a straight line $\operatorname{Re} z = \gamma > 0$. The real parts of the eigenvalues of the matrix B^- are strictly less than some $-\hat{\lambda} < -\lambda$, whereas those of the matrix B^+ are strictly greater than some $\hat{\gamma} > \gamma$.

Thus, $\mathcal{E}^s$ is the subspace $(y = 0, v = 0)$;

$\mathcal{E}^u$: $(x = 0, u = 0)$;
$\mathcal{E}^{ss}$: $(x = 0, y = 0, v = 0)$;
$\mathcal{E}^{sL}$: $(u = 0, y = 0, v = 0)$;
$\mathcal{E}^{uu}$: $(x = 0, u = 0, y = 0)$;
$\mathcal{E}^{uL}$: $(x = 0, u = 0, v = 0)$;
$\mathcal{E}^{sE}$: $(v = 0)$;
$\mathcal{E}^{uE}$: $(u = 0)$;
$\mathcal{E}^{L}$: $(u = 0, v = 0)$.

Let us select some points $M^+(x = x^+, u = u^+, y = 0, v = 0) \in \mathcal{E}^s \backslash \mathcal{E}^{ss}$ and $M^-(x = 0, u = 0, y = y^-, v = v^-) \in \mathcal{E}^u \backslash \mathcal{E}^{uu}$, $\|x^+\| \neq 0$, $\|y^-\| \neq 0$. The map from an ε-neighborhood of M^+ into an ε-neighborhood of M^- along trajectories of the system is given by

$$\bar{x} = e^{A^- t} x\,, \qquad \bar{y} = e^{A^+ t} y\,, \tag{2.4.10}$$

$$\bar{u} = e^{B^- t} u\,, \qquad \bar{v} = e^{B^+ t} v\,, \tag{2.4.11}$$

where t is the flight time.

Because Eq. (2.4.10) for the leading coordinates is independent of Eq. (2.4.11) for the non-leading coordinates, the action of the map in the leading coordinates is the same as in the examples above. By Lemma 2.1, it follows from (2.4.10) that

$$\|y\| = \|e^{-A^+ t} \bar{y}\| \geq e^{-(\gamma + \cdots)t} \left(\|y^-\| - \cdots \right)$$

whence

$$t \geq \frac{1}{\gamma + \cdots} \ln \frac{\left(\|y^-\| - \cdots \right)}{\|y\|}\,.$$

We observe that (by fixing the size of both neighborhoods) the flight time t grows to infinity as $y \to 0$, proportionally to $\ln \|y\|$. Moreover, by Lemma 2.1, the following estimates

$$\|\bar{u}\| \leq \|e^{B^+ t}\| \, \|u\| \leq e^{-\hat{\lambda} t} \, \|u\| \, ,$$

$$\|v\| \leq \|e^{-B^- t}\| \, \|\bar{v}\| \leq e^{-\hat{\gamma} t} \, \|\bar{v}\| \, ,$$

hold, which implies that the map is strongly contracting in the non-leading stable directions and strongly expanding in the non-leading unstable directions, provided $\|y\|$ is sufficiently small. In fact, it follows from formulae (2.4.10) and (2.4.11) that:

$$\|\bar{u}\| \leq C_1 \|\bar{x}\|^{\alpha_1} \, , \quad C_1 = (\|u^+\| + \varepsilon)/(\|x^+\| - \varepsilon)^{\alpha_1} \, , \quad \alpha_1 = \hat{\lambda}/\lambda > 1 \, ,$$

$$\|v\| \leq C_2 \|y\|^{\alpha_2} \, , \quad C_2 = (\|v^-\| + \varepsilon)/(\|y^-\| - \varepsilon)^{\alpha_2} \, , \quad \alpha_2 = \hat{\gamma}/\gamma > 1 \, .$$

Since $\|x^+\| \neq 0$ and $\|y^-\| \neq 0$, it follows that $C_{1,2} \neq \infty$, and therefore the domain of the map $(x, y, u, v) \mapsto (\bar{x}, \bar{y}, \bar{u}, \bar{v})$ is located inside a wedge tangential to the extended stable subspace $\mathcal{E}^{sE}$ at the point M^+, and the range of the map lies inside a wedge tangential to the extended unstable subspace $\mathcal{E}^{uE}$ at the point M^-. Figures 2.4.5–2.4.8 illustrate the action of the map for four-dimensional saddles and saddle-focus (2,1).

2.5. Topological classification of structurally stable equilibrium states

The contrast between the linearized system

$$\dot{y} = Ay \tag{2.5.1}$$

and the original nonlinear system

$$\dot{y} = Ay + g(y) \tag{2.5.2}$$

is that integrating the latter is, generally speaking, an unrealistic problem. This leads us to a very natural question first posed by Poincaré and Lyapunov: Under what conditions do trajectories of the system (2.5.2) near the equilibrium state behave similarly to the trajectories of the linearized system (2.5.1)?

The answer presumably depends on what we understand by "similar behavior". The reader should be aware that the classical understanding of this question differs significantly from the contemporary view.

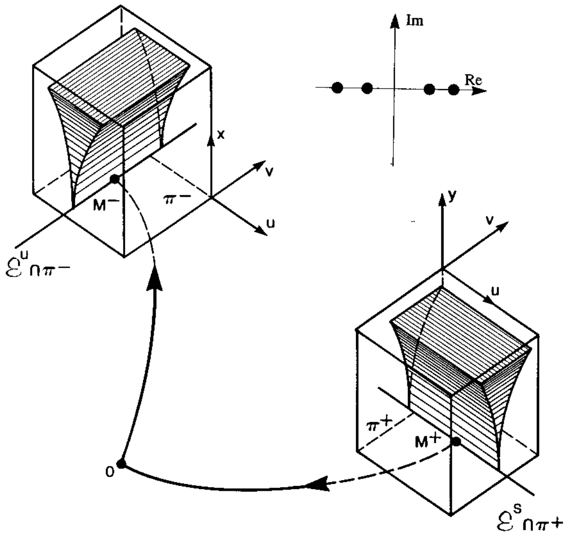


Fig. 2.4.5. The map near a saddle in $\mathbb{R}^4$. The point O has two eigenvalues with positive real parts and two eigenvalues with negative real parts, *i.e.* the saddle has a two-dimensional stable and a two-dimensional unstable subspace.

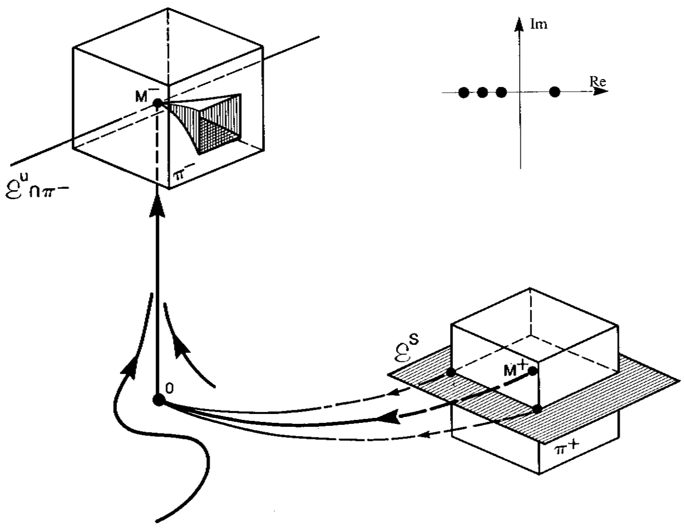


Fig. 2.4.6. The map near a saddle in $\mathbb{R}^4$. The point O is a saddle with a three-dimensional stable subspace $\mathcal{E}^s$ and a one-dimensional unstable subspace $\mathcal{E}^u$.

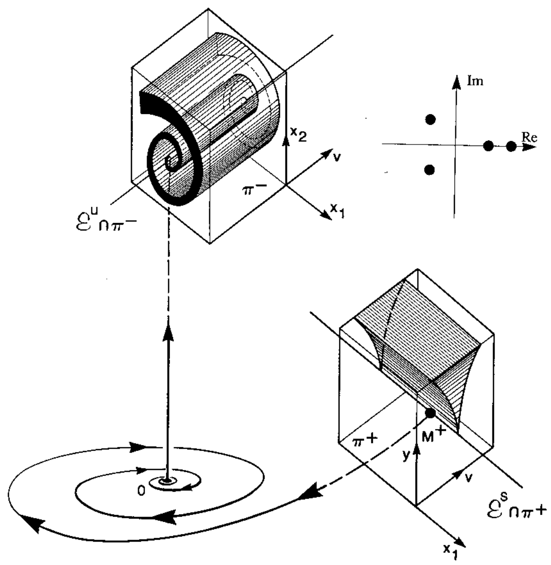


Fig. 2.4.7. The map along trajectories passing by a saddle-focus (2,1). The difference between the maps is in the dimension of the stable and the unstable subspaces. See the location of eigenvalues.

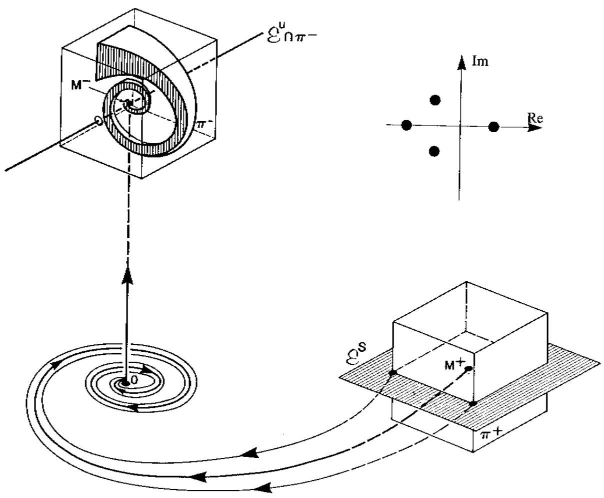


Fig. 2.4.8. The map along trajectories passing by a saddle-focus (2,1). The difference between the maps is in the dimension of the stable and the unstable subspaces. See the location of eigenvalues.

In contemporary terminology, two systems are said to behave identically if they are *topologically equivalent.*

Definition 2.2. *Two n-dimensional systems*

$$\dot{y} = Y_1(y) \quad \textit{and} \quad \dot{y} = Y_2(y)$$

defined in regions D_1 and D_2 respectively, are topologically equivalent in subregions $U_1 \subseteq D_1$ and $U_2 \subseteq D_2$ if there exists a homeomorphism

$$\eta\colon U_1 \to U_2\,,$$

which maps a trajectory (*a semi-trajectory, an interval of a trajectory*) *of the first system into a trajectory* (*a semi-trajectory, an interval of a trajectory*) *of the second one while preserving the orientation* (*direction of motion*).

We also emphasize that the question of equivalence of the original nonlinear system and its linearization at an equilibrium state is senseless if there is at least one characteristic exponent on the imaginary axis. That is, one may not expect topological equivalence between both systems if the equilibrium state is structurally unstable. The following two examples illustrates this point for the planar case.

The first example deals with a pair of purely imaginary exponents, $\lambda_{1,2} = \pm i\omega$, $\omega > 0$. Let us consider a nonlinear system

$$\begin{aligned} \dot{x} &= -\omega y + g_1(x,y)\,, \\ \dot{y} &= \omega x + g_2(x,y)\,, \end{aligned} \tag{2.5.3}$$

where we suppose that the functions g_1 and g_2 vanish at the origin along with their first derivatives. The general solution of the associated linearized system is given by

$$\begin{aligned} x &= x_0 \cos(\omega t) - y_0 \sin(\omega t)\,, \\ y &= y_0 \cos(\omega t) + x_0 \sin(\omega t)\,. \end{aligned}$$

Here, phase trajectories are closed curves (concentric circumferences) surrounding the origin (Fig. 2.5.1). Such an equilibrium state is called *a center.*

The phase portrait of the nonlinear system is rather different in the general case. For instance, if we assume $g_1 = -x(x^2 + y^2)$ and $g_2 = -y(x^2 + y^2)$,

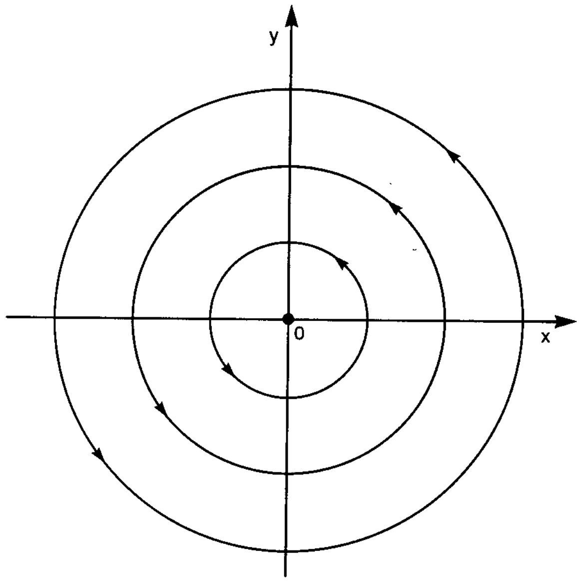


Fig. 2.5.1. A center. Every trajectory here is a "counter-clockwise" concentric cirle.

then the following general solution of Eq. (2.5.3) can be easily found in polar coordinates:

$$r^2 = \frac{1}{2t + r_0^{-2}}, \quad \varphi = \omega t + \varphi_0 .$$

In this case, all trajectories have the shape of spirals winding around the origin as shown in Fig. 2.5.2. Evidently, in any small neighborhoods of both equilibrium states there is no homeomorphism that maps trajectories of such a system onto those of the linearized system (since a homeomorphism maps closed curves onto closed curves). Thus, our system is not topologically equivalent to its linearization.

For our second example, let one exponent λ_1 be equal to zero and let the other exponent be equal to $\lambda_2 = -\lambda < 0$. This system can be written in the form

$$\begin{aligned} \dot{x} &= g_1(x, y) , \\ \dot{y} &= -\lambda y + g_2(x, y) , \end{aligned} \tag{2.5.4}$$

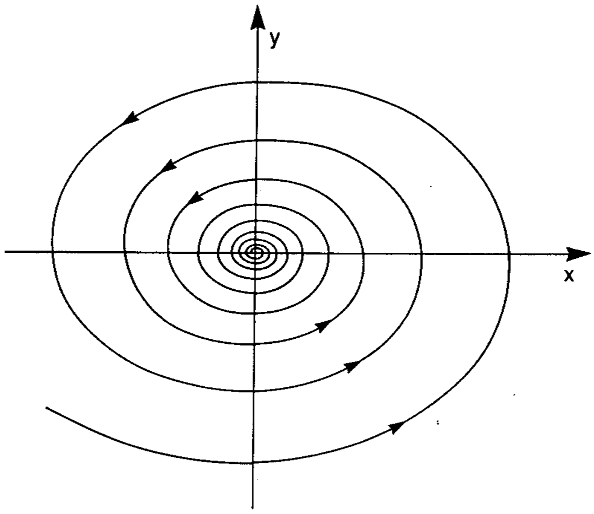


Fig. 2.5.2. Accounting for nonlinearities causes a change in the behavior of the trajectories near the center. They behave like spirals winding around O.

where the functions g_1 and g_2 vanish at the origin along with their first derivatives. The solution of the linearized system is

$$x = x_0 \,, \quad y = e^{-\lambda t} y_0 \,.$$

The phase portrait is shown in Fig. 2.5.3. The entire x-axis consists of equilibrium states of the linearized system, and, each equilibrium state attracts only one pair of trajectories. It is obvious that the nonlinear system may preserve a continuum of equilibrium states only for a very special choice of functions g_1 and g_2 and therefore topological equivalence between the original and linearized system is scarcely expected here.

Figure 2.5.4. demonstrates the phase portrait when $g_1 = x^2$, $g_2 = 0$. One can see that the two local phase portraits have nothing in common. The equilibrium state in Fig. 2.5.4 is called *a saddle-node*.

The problem of topological classification of *structurally stable* equilibrium states finds its solution in the following theorem:

Theorem 2.1. (Grobman–Hartman) *Let O be a structurally stable equilibrium state. Then, there are neighborhoods U_1 and U_2 of O where the original and the linearized systems are topologically equivalent.*

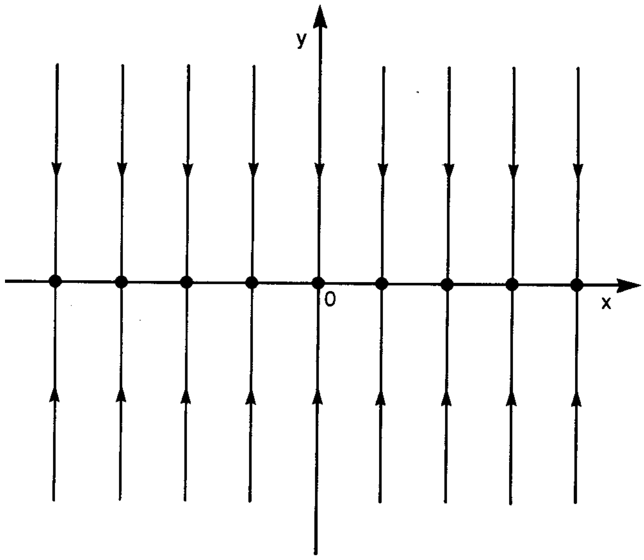


Fig. 2.5.3. Each point on the x-axis is a stable equilibrium state which attracts a pair of trajectories.

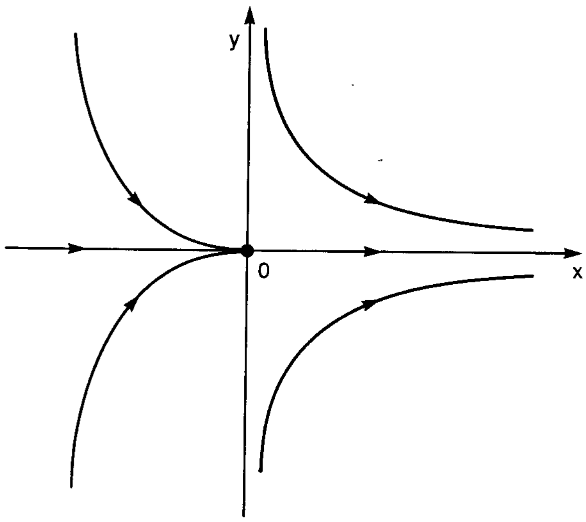


Fig. 2.5.4. A structurally unstable point of saddle-node type. The point O is stable in the "node" region ($x < 0$) but unstable in the "saddle" region ($x > 0$).

We note that the equilibrium state of the nonlinear system (2.5.2) is then said to be *locally topologically equivalent* to that of its linear part (2.5.1).

Let us go further and set up the question of topological equivalence among linear systems. Assign to a structurally stable equilibrium state the topological type $(k, n-k)$ when k characteristic exponents lie to the left of the imaginary axis, and $(n-k)$ to the right of it.

Theorem 2.2. *Linear systems with equilibrium states of the same type are topologically equivalent.*

The proof of this theorem is constructive in the sense that the homeomorphism η: $\mathbb{R}^n \mapsto \mathbb{R}^n$ may be found explicitly. For example, let us consider two linear systems, the first has a focus at the origin

$$\begin{aligned} \dot{x} &= -x + y\,, \\ \dot{y} &= -x - y \end{aligned} \tag{2.5.5}$$

and the second one has a node

$$\begin{aligned} \dot{x} &= -x\,, \\ \dot{y} &= -\frac{1}{3}y\,. \end{aligned} \tag{2.5.6}$$

Both systems are topologically equivalent, because the homeomorphism

$$(x, y) \mapsto (x\cos(\tau) + y^3\sin(\tau)\,,\ y^3\cos(\tau) - x\sin(\tau))\,,$$

where $\tau(x, y) = -\ln(x^2 + y^6)/2$, maps trajectories of (2.5.6) onto trajectories of (2.5.5).

A very important conclusion follows directly from Theorems 2.1 and 2.2, namely that *an n-dimensional system can have only* $(n+1)$ *different topological types of structurally stable equilibrium states.*

In particular, any system with a structurally stable equilibrium state of type $(k, n-k)$ is locally topologically equivalent to the system

$$\dot{x} = A_k x \tag{2.5.7}$$

with the matrix

$$A_k = \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix},$$

where by I_i we denote the i-dimensional identity matrix. If we assume $x = \binom{u}{v}$ where $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$, then the system (2.5.7) may be represented in the form

$$\begin{aligned} \dot{u} &= -u\,, \\ \dot{v} &= v\,. \end{aligned} \tag{2.5.8}$$

The solution of (2.5.8) is given by

$$u(t) = e^{-I_k t} u_0\,, \quad v(t) = e^{I_{n-k} t} v_0\,. \tag{2.5.9}$$

In the case $k = n$ all trajectories of the system (2.5.9) tend to the equilibrium state at the origin as $t \to +\infty$. Hence, by virtue of Theorems 2.1 and 2.2, any trajectory from a sufficiently small neighborhood of an equilibrium state of type $(n, 0)$ of the nonlinear system also tends to the equilibrium state as $t \to +\infty$. Such an equilibrium state is called *a stable topological node or sink*. We remark that the n-dimensional stable foci and nodes considered in the previous section are topologically equivalent by virtue of Theorem 2.2 and therefore both are stable sinks.

For an equilibrium state O of type $(0, n)$, any trajectory from a small neighborhood of O tends to O as $t \to -\infty$. As $t \to +\infty$ any trajectory, excluding O, leaves the neighborhood. Such an equilibrium state is called *an unstable topological node or source*.

We call the remaining structurally stable equilibrium states *topological saddles*. It follows from the Grobman–Hartman theorem that a topological saddle of the original nonlinear system has *local stable and local unstable manifolds* W^s_{loc} and W^u_{loc} of dimension k and $(n-k)$, respectively. Namely, if h is a local homeomorphism which maps the trajectories of the linearized system onto trajectories of the nonlinear system (such a homeomorphism exists here by virtue of Theorem 2.1), then the images $h\mathcal{E}^s$ and $h\mathcal{E}^u$ of the stable and unstable invariant subspaces of the linearized system are, exactly, the stable and unstable manifolds. As in the linear case, a positive semi-trajectory starting with any point in W^s_{loc} lies entirely in W^s_{loc} and tends to O as $t \to +\infty$. Similarly, a negative semi-trajectory starting with any point of W^u_{loc} lies entirely in W^u_{loc} and tends to O as $t \to -\infty$. The trajectories of points outside of $W^s_{loc} \cup W^u_{loc}$ escape from any neighborhood of the saddle as $t \to \pm\infty$ The manifolds W^s_{loc} and W^u_{loc} are *invariant manifolds*, *i.e.* they consist of whole trajectories (until they leave some neighborhood of the topological saddle).

It is obvious that if two systems X_1 and X_2 are topologically equivalent, the homeomorphism establishing this topological equivalence, maps the equilibrium states of system X_1 onto equilibrium states of system X_2. If O_1 is an equilibrium state of system X_1 and O_2 is the image of O_1 under the homeomorphism, then a trajectory asymptotic to O_1 as $t \to +\infty$ (resp. $t \to -\infty$) is mapped into a trajectory asymptotic to O_2 as $t \to +\infty$ ($t \to -\infty$). Consequently, the dimensions of the stable (unstable) manifolds of locally topologically equivalent saddles are identical. Thus, we arrive at the following theorem

Theorem 2.3. *Two structurally stable equilibrium states are locally topologically equivalent if and only if they are of the same topological type.*

The topological approach has excellently resolved the classification problem of structurally stable equilibrium states. However, it does not provide answers to a number of important questions such as the question concerning the exponential velocity of the convergence to an equilibrium state, of the character (monotonic or oscillating) of this convergence, the smoothness of the invariant manifolds, etc. These subtle (*i.e.* indistinguishable by local homeomorphisms) details of the behavior of the trajectories near equilibrium states are of the great importance in the study of various homoclinic bifurcations which play a principal role for dynamical systems with complex dynamics.

2.6. Stable equilibrium states. Leading and non-leading manifolds

Let an n-dimensional system of the $\mathbb{C}^r$ ($r \geq 1$) class of smoothness have a structurally stable equilibrium state O at the origin. Near O the system is written in the form

$$\dot{y} = Ay + h(y)\,, \tag{2.6.1}$$

where A is a constant $(n \times n)$-matrix whose spectrum, defined as the set of all eigenvalues of A, lies outside of the imaginary axis in the complex plane, h: $\mathbb{R}^n \to \mathbb{R}^n$ is a $\mathbb{C}^r$ function such that

$$h(0) = 0\,, \quad h'_y(0) = 0\,. \tag{2.6.2}$$

Let O be a stable equilibrium, *i.e.* all n characteristic exponents $(\lambda_1, \ldots, \lambda_n)$ have negative real parts. For a solution of the linearized system

$$\dot{y} = Ay \tag{2.6.3}$$

estimate (2.3.22) holds (according to Lemma 2.1) which implies that any trajectory tends to the origin exponentially. Does this property of exponential convergence of trajectories to the equilibrium state persist for the original nonlinear system? The following theorem answers this question in the affirmative.

Theorem 2.4. *For a sufficiently small $\delta > 0$ and for any y_0 such that $\|y_0\| < \delta$, the trajectory $y(t)$ of the nonlinear system* (2.6.1) *starting with y_0 satisfies the following inequality for all $t \geq 0$:*

$$\|y(t)\| \leq Ce^{(\max\, \mathrm{Re}\, \lambda_i + \varepsilon)t}\, \|y_0\|\,, \tag{2.6.4}$$

where the positive constant $\varepsilon > 0$ may be chosen infinitesimally small by means of decreasing δ, and $C > 0$ is some factor depending upon the choice of the basis in $\mathbb{R}^n$.

Proof. From (2.6.1), given the square of the norm

$$\|y(t)\|^2 = \langle y(t), y(t) \rangle\,,$$

where $\langle \cdot, \cdot \rangle$ denotes a scalar product in $\mathbb{R}^n$, we have

$$\frac{d}{dt}\|y(t)\|^2 = 2\langle y(t), \dot{y}(t) \rangle = 2\langle y, Ay \rangle + 2\langle y, h(y) \rangle\,. \tag{2.6.5}$$

By expanding the function $h(y)$ in a Taylor series near O, we have

$$h(y) = h(0) + h'_y(0)y + o(\|y\|)\,. \tag{2.6.6}$$

It follows from (2.6.2) that

$$h(y) = o(\|y\|)\,,$$

i.e. given $\varepsilon > 0$ we may choose $\delta > 0$ such that

$$\|h(y)\| \leq \frac{1}{2}\varepsilon\|y\| \tag{2.6.7}$$

for all y such that $\|y\| \leq \delta$. From (2.6.7) we obtain the estimate

$$|2\langle y, h(y) \rangle| \leq \varepsilon\|y\|^2 \tag{2.6.8}$$

for the second term in (2.6.5). In order to obtain an estimate for the first term $2\langle y, Ay \rangle$ in (2.6.5) we choose the Jordan basis similar to the case when we estimated solutions of linear systems (Lemma 2.1). In the given basis the matrix A is assigned by formulae (2.3.5)–(2.3.7), (2.3.10) and (2.3.11). Furthermore, we will choose the basis so that non-zero values δ_j in (2.3.11) are equal to $\varepsilon/2$. For components of the vector $z = Ay$ we have

$$z_i = \lambda_i y_i \tag{2.6.9}$$

when λ_i is real and simple; or

$$\begin{aligned} z_i &= y_i \mathrm{Re}\,\lambda_i - y_{i+1} \mathrm{Im}\,\lambda_i\,, \\ z_{i+1} &= y_{i+1} \mathrm{Re}\,\lambda_i + y_i \mathrm{Im}\,\lambda_i \end{aligned} \tag{2.6.10}$$

when λ_i, λ_{i+1} is a pair of complex-conjugate characteristic exponents; or

$$z_{i+j} = \lambda_{i+j} y_{i+j} + \delta_j y_{i+j+1}\,, \quad j = 1, \dots, k \tag{2.6.11}$$

when $\lambda_{i+1} = \cdots = \lambda_{i+k}$ is an exponent of multiplicity k (here $\delta_k \equiv 0$); or

$$\begin{aligned} z_{i+2j-1} &= y_{i+2j-1} \mathrm{Re}\,\lambda_{i+1} - y_{i+2j} \mathrm{Im}\,\lambda_{i+1} + \delta_j y_{i+2j+1} \\ z_{i+2j} &= y_{i+2j} \mathrm{Re}\,\lambda_{i+1} - y_{i+2j-1} \mathrm{Im}\,\lambda_{i+1} + \delta_j y_{j+2j+2} \\ &(j = 1, \dots, k; \delta_k \equiv 0) \end{aligned} \tag{2.6.12}$$

when $\lambda_{i+1} = \lambda_{i+3} = \cdots = \lambda_{i+2k-1}$ and $\lambda_{i+2} = \lambda_{i+4} = \cdots = \lambda_{i+2k}$ are the complex-conjugate characteristic exponents of multiplicity k. Recall that the quantities δ_j in (2.6.11) and (2.6.12) may be either 0 or $\varepsilon/2$.

It follows from formulae (2.6.9) and (2.6.10) that if the exponents are simple, then

$$\langle y, z \rangle = \sum_{i=1}^{n} y_i^2 \mathrm{Re}\,\lambda_i\,. \tag{2.6.13}$$

In the case of multiple characteristic exponents from (2.6.9)–(2.6.12) we obtain the estimate

$$|\langle y, z \rangle - \sum_{i=1}^{n} y_i^2 \,\mathrm{Re}\,\lambda_i| \le \frac{\varepsilon}{2} \left(\sum_{\mathrm{real}\,\lambda_i = \lambda_{i+1}} |y_i y_{i+1}| + \sum_{\mathrm{complex}\,\lambda_i = \lambda_{i+2}} |y_i y_{i+2}| \right)$$

and since $|y_i\, y_j| \le \frac{1}{2}(y_i^2 + y_j^2)$, we obtain

$$|\langle y, z \rangle - \sum_{i=1}^{n} y_i^2 \,\mathrm{Re}\,\lambda_i| \le \frac{\varepsilon}{2} \|y\|^2\,. \tag{2.6.14}$$

It follows from (2.6.5), (2.6.8) and (2.6.13)–(2.6.14) that we may estimate any trajectory $y(t)$ as

$$\frac{d}{dt}\|y(t)\|^2 \leq 2(\max \operatorname{Re}\lambda_i + \varepsilon)\|y(t)\|^2$$

or

$$\frac{d}{dt}\|y(t)\| \leq (\max \operatorname{Re}\lambda_i + \varepsilon)\|y(t)\| \tag{2.6.15}$$

unless $y(t)$ leaves the δ-neighborhood of the point O. By virtue of (2.6.15), the norm of $y(t)$ decays monotonically as t increases and therefore, if $\|y_0\| \leq \delta$, then $\|y(t)\| \leq \delta$ for $t \geq 0$. Thus, the inequality (2.6.15) is valid for any positive semi-trajectory starting inside the δ-neighborhood of the point O.

In order to integrate the inequality, we notice that (2.6.15) is equivalent to

$$\frac{d}{dt}\left(\|y(t)\|e^{-(\max \operatorname{Re}\lambda_i+\varepsilon)t}\right) \leq 0\,.$$

This implies that $\|y(t)\|e^{-(\max \operatorname{Re}\lambda_i+\varepsilon)t}$ decreases monotonically as t increases, whence we get

$$\|y(t)\|e^{-(\max \operatorname{Re}\lambda_i+\varepsilon)t} \leq \|y_0\|$$

which coincides with (2.6.4) when $C = 1$. The transition to an arbitrary basis leads to the appearance of the coefficient $C \neq 1$ in (2.6.4) (see (2.3.41)).

Remark. If the class of smoothness of the system is $\mathbb{C}^2$ or higher and if the characteristic exponent nearest to the imaginary axis is simple, one may set $\varepsilon = 0$ in the inequality (2.6.4).

Indeed, in this case the first term in the right-hand side of (2.6.5) is estimated as

$$2\langle y, Ay\rangle \leq \max(\operatorname{Re}\lambda_i)\sum_{i=1}^{n} y_i^2\,.$$

Therefore, the value ε in (2.6.15) is determined only by the inequality (2.6.7). If $h \in \mathbb{C}^r$ $(r \geq 2)$, then the reminder term of Taylor series (2.6.6) is estimated as

$$\|y\|^2(\max h'')\,,$$

where the maximum is taken in the δ-neighborhood of O. It follows that the constant ε in (2.6.7) may be chosen such that

$$\varepsilon \leq K\|y\|\,.$$

Since y decreases exponentially, the constant ε in (2.6.15) may be replaced by a time-dependent function $\varepsilon(t)$, which decays at least exponentially as $t \to +\infty$, and in particular, the integral $\int_0^\infty \varepsilon(s)ds$ is finite.

It follows from (2.6.15) that

$$\frac{d}{dt} \ln \|y(t)\| \leq \max \operatorname{Re} \lambda_i + \varepsilon(t)$$

whence

$$\ln \|y(t)\| \leq \ln \|y_0\| + t \max \operatorname{Re} \lambda_i + \int_0^t \varepsilon(s)\, ds$$

or, due to the convergence of the integral,

$$\ln \|y(t)\| \leq \ln \|y_0\| + t \max \operatorname{Re} \lambda_i + \ln C\,,$$

from which we obtain inequality (2.6.4) with $\varepsilon = 0$.

We remark that for $h \in \mathbb{C}^1$ the integral $\int_0^\infty \varepsilon(s)ds$ may in general diverge. For example, in the case where $y \in \mathbb{R}^1$ and

$$h(y) = \int_0^y \frac{ds}{\ln |s|}\,,$$

if y decays exponentially to zero, then the value $\varepsilon(t)$ tends to zero asymptotically as $\sim |h(y(t))|/|y(t)| \sim \frac{1}{t}$.

The above theorem asserts that a topologically stable node is an *exponentially* stable equilibrium state. As we have seen in Sec. 2.3, in the linear case the velocity and the character of convergence of the most of trajectories to the equilibrium are determined by the leading coordinates. This feature persists in the nonlinear case as well. Here, the role of the non-leading subspace is played by an invariant *non-leading manifold*, whose existence in high-dimensional nonlinear systems was discovered by Petrovsky.

Let us reorder the characteristic exponents so that

$$0 \geq \operatorname{Re} \lambda_1 \geq \operatorname{Re} \lambda_2 \geq \cdots \geq \operatorname{Re} \lambda_n\,.$$

Let $m > 0$ be such that the first m exponents have the same real part

$$\operatorname{Re} \lambda_i = \operatorname{Re} \lambda_1\,, \quad (i = 1, \ldots, m)$$

and

$$\operatorname{Re} \lambda_i < \operatorname{Re} \lambda_1\,, \quad (i = m+1, \ldots, n)\,.$$

Assume $m < n$. Each vector y may be uniquely decomposed as

$$y = u + v\,,$$

where $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_{n-m})$ are the projections onto the leading and the non-leading eigen-subspaces $\mathcal{E}^L$ and $\mathcal{E}^{ss}$ of the matrix A, respectively. With the new variables the system takes the form

$$\begin{aligned} \dot{u} &= A_1 u + f(u, v)\,, \\ \dot{v} &= A_2 v + g(u, v)\,, \end{aligned} \tag{2.6.16}$$

where spectr $A_1 = \{\lambda_1, \ldots, \lambda_m\}$ and spectr $A_2 = \{\lambda_{m+1}, \ldots, \lambda_n\}$, the functions f, $g \in \mathbb{C}^r$ and

$$f(0) = 0\,, \quad g(0) = 0\,, \quad f'(0) = 0\,, \quad g'(0) = 0 \tag{2.6.17}$$

Definition 2.3. *Let U be a neighborhood of the point* 0. *A set $W \subseteq U$ is said to be locally invariant if a trajectory starting with any point $M \in W$ lies entirely in W until it leaves U.*

Theorem 2.5. (On non-leading manifold) *In a neighborhood U of a stable equilibrium state O there exists an $(n - m)$-dimensional $\mathbb{C}^r$-smooth invariant manifold W^{ss}_{loc} (non-leading or strongly stable) which passes through O and is tangential at O to the non-leading subspace $\mathcal{E}^{ss}$: $u = 0$. A trajectory $y(t)$ starting with any point y_0 outside W^{ss}_{loc} tends to* 0 *tangentially to the leading subspace $v = 0$. Moreover, for $t \geq 0$*

$$\|y(t)\| \geq C e^{(\mathrm{Re}\,\lambda_1 - \varepsilon)t}\, \rho(y_0, W^{ss}_{loc})\,, \tag{2.6.18}$$

where $\rho(y_0, W^{ss}_{loc})$ denotes the distance between y_0 and W^{ss}_{loc}.
In contrast, all trajectories from W^{ss}_{loc} tend to O faster, namely

$$\|y(t)\| \leq C e^{(\mathrm{Re}\,\lambda_{m+1} + \varepsilon)t}\, \|y_0\|\,. \tag{2.6.19}$$

The proof of the existence and smoothness of the non-leading manifold will be given in Chap. 5. For now, let us prove the second part of the theorem, namely, inequalities (2.6.18) and (2.6.19).

Since W^{ss}_{loc} is tangential to the subspace $u = 0$, it is defined by an equation of the form

$$u = \varphi(v)\,, \tag{2.6.20}$$

where $\varphi \in \mathbb{C}^r$ and

$$\varphi(0) = 0\,, \quad \varphi'(0) = 0\,. \tag{2.6.21}$$

Since W^{ss}_{loc} is an invariant set it follows that if $u_0 = \varphi(v_0)$, then $u(t) = \varphi(v(t))$ for $t \geq 0$, and therefore

$$\dot{u} = \varphi'(v)\dot{v} \quad \text{for} \quad u = \varphi(v)\,,$$

or, by virtue of (2.6.16),

$$A_1\varphi(v) + f(\varphi, v) = \varphi'(v)(A_2 v + g(\varphi, v))\,. \tag{2.6.22}$$

Let us introduce a new variable

$$w = u - \varphi(v)$$

so that in the new coordinates the equation of W^{ss}_{loc} is $w = 0$. Such a change of variables is called *a straightening of a manifold.* The system (2.6.16) is now recast as

$$\begin{aligned}
\dot{v} &= A_2 v + g(w + \varphi(v),\, v)\,, \\
\dot{w} &= A_1 w + A_1\varphi(v) + f(w + \varphi(v),\, v) - \varphi'(v)\dot{v} \\
&= A_1 w + A_1\varphi(v) + f(w + \varphi(v),\, v) - \varphi'(v)(A_2 v + g(w + \varphi(v), v))\,.
\end{aligned} \tag{2.6.23}$$

Using (2.6.22), the last equation may be rewritten as

$$\dot{w} = A_1 w + [f(w + \varphi,\, v) - f(\varphi,\, v)] - \varphi'(v)\,[g(w + \varphi,\, v) - g(\varphi,\, v)]$$

or, since the terms in the square brackets vanish when $w = 0$, as

$$\dot{w} = (A_1 + \tilde{g}(w, v))\, w\,, \tag{2.6.24}$$

where

$$\tilde{g} \equiv \left[\int_0^1 f'_u(\varphi(v) + sw,\, v)\; ds - \varphi'(v)\int_0^1 g'_u(\varphi(v) + sw,\, v)\; ds\right] \in \mathbb{C}^{r-1}\,.$$

Moreover, by virtue of (2.6.17) and (2.6.21),

$$\tilde{g}(0, 0) = 0\,.$$

From (2.6.24) for the norm of the vector $w(t)$ we have

$$\|w(t)\|\frac{d}{dt}\|w(t)\| = \langle w(t), (A_1 + \tilde{g}(w, v))\, w(t)\rangle\,.$$

Because $\tilde{g}(0,0) = 0$, it follows that

$$|\langle w(t), \tilde{g}(w,v)w(t)\rangle| \leq \frac{\varepsilon}{2}\|w(t)\|^2$$

provided that $\|y_0\|$ is sufficiently small. Hence

$$\frac{d}{dt}\|w\| \geq \frac{\langle w, A_1 w\rangle}{\|w\|} - \frac{\varepsilon}{2}\|w\| \,.$$

Now, following the same steps as in the proof of Theorem 2.4, we get that

$$\frac{d}{dt}\|w\| \geq (\operatorname{Re}\lambda_1 - \varepsilon)\|w\| \,. \tag{2.6.25}$$

Finally we obtain

$$\|w(t)\| \geq e^{(\operatorname{Re}\lambda_1 - \varepsilon)t}\, \|w_0\| \,,$$

i.e. (2.6.18) is justified.

Let us now show that if an initial point lies outside of W^{ss}_{loc}, the associated trajectory tends to O tangentially to the leading subspace $v = 0$. To do this, we consider a value $z(t) = \|v(t)\|/\|w(t)\|$, $w \neq 0$, and show that $z(t) \to 0$ as $t \to +\infty$. For $d\|w(t)\|/dt$ we have estimate (2.6.25). Analogously, from (2.6.23) we can obtain

$$\frac{d}{dt}\|v\| \leq (\operatorname{Re}\lambda_{m+1} + \varepsilon)\|v\| + \|w\| \,\max\, \|g'_u\| \,, \tag{2.6.26}$$

where the maximum is taken in a neighborhood of diameter $\|y(t)\|$ of the point O. From here and (2.6.25) it follows that

$$\frac{d}{dt}z = \frac{1}{\|w\|}\frac{d}{dt}\|v\| - z\frac{1}{\|w\|}\frac{d}{dt}\|w\| \leq \kappa(t) - \mu z \,, \tag{2.6.27}$$

where $\mu = \operatorname{Re}\lambda_{m+1} - \operatorname{Re}\lambda_1 + 2\varepsilon > 0$ and

$$\kappa(t) \to 0 \quad \text{as} \quad t \to +\infty \,. \tag{2.6.28}$$

By (2.6.27) we obtain $\frac{d}{dt}e^{\mu t}z(t) \leq e^{\mu t}\kappa(t)$ or

$$z(t) \leq z_0 e^{-\mu t} + \int_0^t e^{-\mu(t-s)}\kappa(s)\, ds \,.$$

In order to prove that $z(t) \to 0$, we must show that

$$I(t) = \int_0^t e^{-\mu(t-s)}\kappa(s)\, ds \to 0 \quad \text{as } t \to \infty \,.$$

For arbitrary $T > 0$ we can write

$$I(t) = \int_0^T e^{-\mu(t-s)}\kappa(s)\,ds + \int_T^t e^{-\mu(t-s)}\kappa(s)\,ds$$

whence

$$\begin{aligned} I(t) &\leq e^{-\mu t}\left(\int_0^T e^{\mu s}\,ds\right)\max_{s\geq 0}\,\kappa(s) + \left(\int_0^t e^{-\mu(t-s)}\,ds\right)\max_{s\geq T}\,\kappa(s) \\ &\leq e^{-\mu t}\,\frac{1}{\mu}e^{\mu T}\,\max_{s\geq 0}\kappa(s) + \frac{1}{\mu}\max_{s\geq T}\,\kappa(s)\,. \end{aligned} \tag{2.6.29}$$

By virtue of (2.6.28), the second summand in (2.6.29) can be made to be infinitesimally small if we choose a sufficiently large T. By choosing a sufficiently large t, the first term in (2.6.29) can be made infinitesimally small too. Therefore, $I(t) \to 0$ as $t \to +\infty$.

Thus, if $w_0 \neq 0$, then $\|v(t)\|/\|w(t)\| \to 0$, *i.e.* any trajectory that does not lie in W^{ss}_{loc}, touches tangentially the leading subspace as $t \to +\infty$. In particular, this implies that W^{ss}_{loc} is a *unique* m-dimensional smooth invariant manifold which is tangential to the non-leading subspace at the point O.

In order to prove estimate (2.6.19) for trajectories from W^{ss}_{loc}, we notice that in the restriction of the system (2.6.23)–(2.6.24) to the non-leading manifold

$$\dot{v} = A_2 v + g(\varphi(v), v) \tag{2.6.30}$$

the point O is a stable equilibrium state with characteristic exponents $(\lambda_{m+1}, \ldots, \lambda_n)$. Therefore, the exponential estimate (2.6.19) holds for system (2.6.30) by virtue of Theorem 2.4. End of the proof.

It should also be noticed that the theorem on the non-leading manifold is valid for system (2.6.30). This implies that most trajectories from W^{ss}_{loc} tend to O at the rate $e^{\mathrm{Re}\,\lambda_{m+1}t}$. The exclusive trajectories that tend to O faster compose a $\mathbb{C}^r$-smooth manifold W^{sss}_{loc} which is tangential at O to the eigen-subspace corresponding to characteristic exponents λ_i of A such that $\mathrm{Re}\,\lambda_i < \mathrm{Re}\,\lambda_{m+1}$. The non-leading manifold theorem is also applied to the system on W^{sss}_{loc}, etc. As a result we obtain a hierarchy of non-leading manifolds $W^{ss}, W^{sss}, W^{ssss}, \ldots$, composed of trajectories which tend to the equilibrium point at ever increasing rates.

As in the linear case, there are two basic kinds of stable equilibrium states according to the behavior of the system in the leading coordinates: *a stable* *node* and *a stable focus*. The point O is called *a node* if $m = 1$, *i.e.* the leading characteristic exponent λ_1 is simple and real:

$$0 > -\lambda = \lambda_1 > \operatorname{Re}\lambda_i \quad (i = 2, \ldots, n)\,. \tag{2.6.31}$$

The point O is called *a focus* if $m = 2$ and the leading characteristic exponents comprise a pair of complex-conjugate numbers:

$$0 > \operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 > \operatorname{Re}\lambda_i \quad (i = 3, \ldots, n)\,. \tag{2.6.32}$$

It is clear that if the point O is either a node or a focus, then for any matrix close to A, it remains a node or a focus, respectively. Conversely, in the case where neither (2.6.31) nor (2.6.32) holds, a small perturbation of the matrix A always ensures the validity of at least one of these relations.

In the case where the equilibrium state is *a node*, the non-leading manifold is $(n-1)$-dimensional and it partitions a neighborhood of the point O into two components. Trajectories outside of W^{ss}_{loc} approach O tangentially to the w-axis along two opposite directions, from side $w > 0$ for the first component, and from side $w < 0$ for the second. Equation (2.6.24) for the leading coordinate w has the form

$$\dot{w} = -\lambda w + o(w)\,. \tag{2.6.33}$$

It is evident that trajectories from each component tend to O monotonically (see Figs. 2.6.1 and 2.6.2).

When the equilibrium state is *a focus*, the non-leading manifold is $(n-2)$-dimensional and it does not divide a neighborhood of the point O. If $\lambda_{1,2} = -\rho \pm i\omega$, then Eq. (2.6.24) of the leading coordinates may be written as

$$\begin{aligned} \dot{w}_1 &= (-\rho + \cdots)w_1 - (\omega + \cdots)w_2\,, \\ \dot{w}_2 &= (-\rho + \cdots)w_2 + (\omega + \cdots)w_1\,, \end{aligned} \tag{2.6.34}$$

or in polar coordinates as

$$\begin{aligned} \dot{r} &= (-\rho + \cdots)r\,, \\ \dot{\varphi} &= \omega + \cdots\,, \end{aligned} \tag{2.6.35}$$

hereafter the ellipsis denotes terms of a higher order.

We can see from (2.6.35) that the motion of trajectories tending to O has an oscillatory character. The trajectories that do not lie in W^{ss}_{loc} have the shape

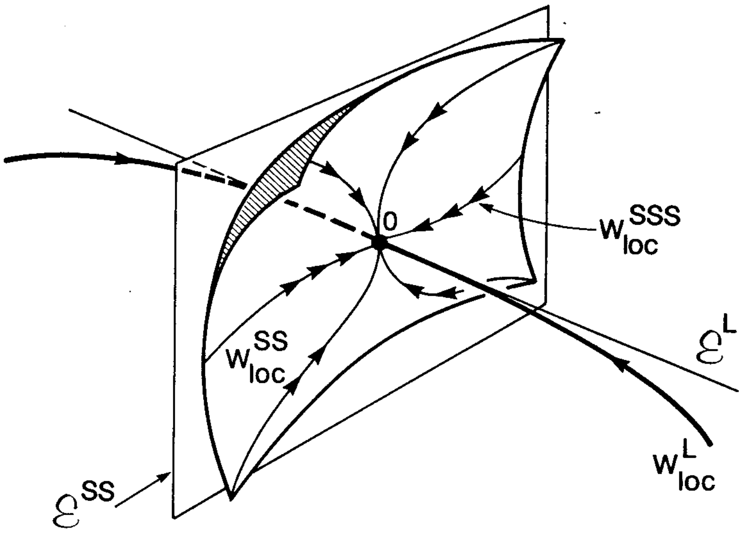


Fig. 2.6.1. A stable node. There is a certain hierarchy of strongly stable local manifolds. Any trajectory converges monotonically to the node O.

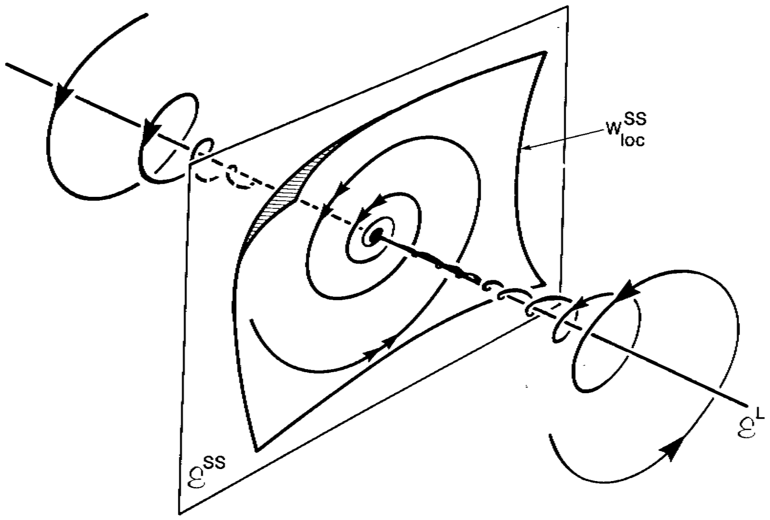


Fig. 2.6.2. In contrast to Fig. 2.6.1, the loci of convergence of trajectories to the node include an oscillating character, namely, a focus.

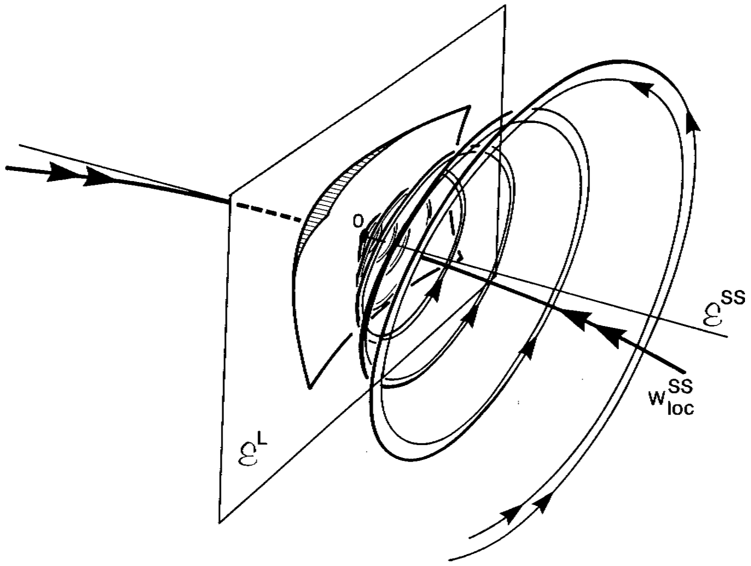


Fig. 2.6.3. A stable focus.

of spirals tending toward O without any definite direction but tangentially to the leading plane $v = 0$ as shown in Fig. 2.6.3.

For stable equilibrium states of nonlinear systems the non-leading manifold plays a role similar to that of the non-leading invariant subspace in the linear case. Recall that in the linear case an equilibrium point has also a leading invariant subspace which, however, has no adequate analogue in the non-linear case. The difference is that in general the leading manifold of a non-linear system may have a finite smoothness only.

Example. A two-dimensional system:

$$\begin{aligned} \dot{w} &= -w\,, \\ \dot{v} &= -2v + w^2\,, \end{aligned} \tag{2.6.36}$$

has a stable node at the origin O. The general solution is given by

$$w = w_0 e^{-t}\,, \quad v = v_0 e^{-2t} + w_0^2 t e^{-2t}\,. \tag{2.6.37}$$

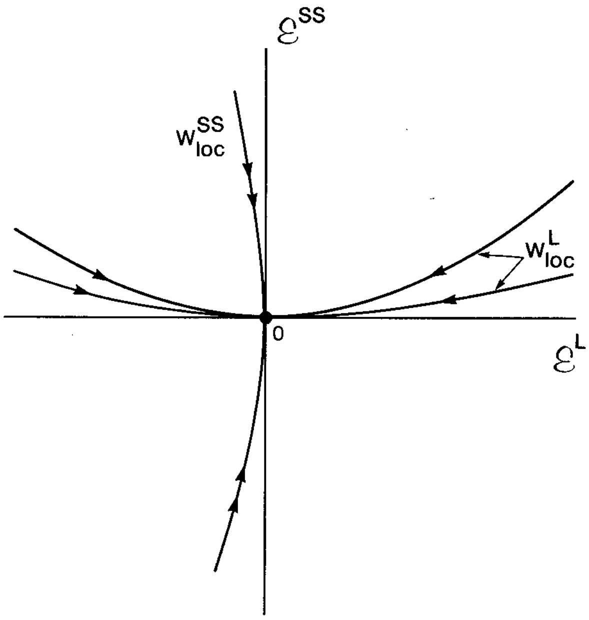


Fig. 2.6.4. The only non-leading local manifold W^{ss}_{loc}. The leading local manifold W^{L}_{loc} of a node cannot be uniquely defined.

If we take two trajectories, one from the region $w > 0$ and another from $w < 0$, their union with O forms an invariant manifold tangential to the leading subspace $\mathcal{E}^L$ at O, see Fig. 2.6.4. Any such manifold can be considered as a leading one but each has only $\mathbb{C}^1$-smoothness at the point O because, by virtue of (2.6.37),

$$\frac{dv}{dw} = e^{-t}\left(\frac{2v_0}{w_0} - w_0 + 2w_0 t\right),$$

whence

$$\frac{d^2v}{dw^2} = \frac{2v_0}{w_0^2} - 2 + 2t$$

hence, $d^2v/dw^2 \to +\infty$ as $t \to +\infty$. In the general case the following theorem holds (see Chap. 5 for the proof).

Theorem 2.6. *The system* (2.6.16) *has an m-dimensional invariant leading manifold W^L_{loc} which is tangential to the subspace $v = 0$ at the equilibrium state*

O and its smoothness is equal to $\min(r, r_L) \geq 1$, *where*

$$r_L = \left[\frac{\operatorname{Re}\lambda_{m+1}}{\operatorname{Re}\lambda_1}\right]. \tag{2.6.38}$$

Here, $[x]$ *denotes the largest integer which is strictly less than* x.

Remark 1. The leading manifold is, generally speaking, non-unique (see example above).

Remark 2. It follows from (2.6.38) that when $\operatorname{Re}\lambda_1$ tends to zero, the smoothness of the leading manifold increases to ∞.

The case where $\operatorname{Re}\lambda_i > 0$, $(i = 1, \ldots, n)$, is reduced to that considered above by reversion of time $t \to -t$. The following estimate therefore holds

$$\|y(t)\| \leq Ce^{(\min\ \operatorname{Re}\lambda_i - \varepsilon)|t|}\ y_0 \quad \text{for} \quad t \leq 0\,. \tag{2.6.39}$$

This means that such an equilibrium state is exponentially completely unstable. In complete analogy with stable equilibrium states (but for $t \to -\infty$), the non-leading manifold W^{uu}_{loc} and the leading manifold W^L_{loc} may be defined. In correspondence with the behavior of trajectories in the leading coordinates, two basic kinds of equilibrium states are selected:

— when the characteristic exponent nearest to the imaginary axis is real and simple (i.e. of multiplicity $m = 1$), the trajectories not lying in W^{uu}_{loc} leave O as $t \to +\infty$ along one of two opposite directions tangential to the leading axis. Such an equilibrium state is called *an unstable node*;

— in the case where a pair of simple complex-conjugate characteristic exponents is nearest to the imaginary axis, all trajectories not lying in W^{uu}_{loc} spiral away from O without any definite direction but tangentially to the leading plane as $t \to +\infty$. Such an equilibrium state is called *an unstable focus*.

2.7. Saddle equilibrium states. Invariant manifolds

Let k characteristic exponents of the structurally stable equilibrium state O lie to the left of the imaginary axis and $(n - k)$ to the right of it, *i.e.* $\operatorname{Re}\lambda_i < 0$, $i = 1, \ldots, k$ and $\operatorname{Re}\lambda_j > 0$, $j = k + 1, \ldots, n$, where $k \neq 0, n$.

We have already seen in Sec. 2.5 that such an equilibrium state has the stable and the unstable invariant sets W^s_{loc} and W^u_{loc} which are locally homeomorphic to a k-dimensional and an $(n-k)$-dimensional disk, respectively. The invariant sets W^s_{loc} and W^u_{loc} intersect at only one point, namely, the equilibrium state O. If we puncture O off, both sets consist of semi-trajectories: W^s_{loc} is composed of positive semi-trajectories and W^u_{loc} is composed of negative semi-trajectories.

Continuation of W^s_{loc} and W^u_{loc} along trajectories outside of a neighborhood of the saddle yields us *the global stable invariant manifold* W^s and *the global unstable invariant manifold* W^u of the equilibrium state O. In the case of a linear system they are just a k-dimensional and an $(n-k)$-dimensional invariant subspaces of the matrix A. In the nonlinear case the manifolds W^s and W^u may be embedded in $\mathbb{R}^n$ in a very complicated way. We will see below how the relative location of both W^s and W^u in the phase space greatly affects the global dynamics of the system. This is one reason why calculation (analytical, when possible, and numerical) of these manifolds is a key element in the qualitative study of specific systems.

We must emphasize that the results which may be obtained from the Grobman–Hartman theorem do not allow one to determine W^s or W^u, or to estimate their smoothness. At the same time, the local manifolds W^s_{loc} and W^u_{loc} are well-defined smooth objects. The existence of analytical invariant manifolds of a saddle in analytic systems was proven by Poincaré as well as by Lyapunov who used a different method (in terms of the so-called conditional stability). For smooth systems Perron and Hadamard obtained similar results.

A linear non-singular change of variables transforms a nonlinear system near a saddle equilibrium state into the following form

$$\begin{aligned} \dot{u} &= A^- u + f(u,v)\,, \\ \dot{v} &= A^+ v + g(u,v)\,, \end{aligned} \tag{2.7.1}$$

where $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$, $\operatorname{spectr} A^- = \{\lambda_1, \ldots, \lambda_k\}$, $\operatorname{spectr} A^+ = \{\lambda_{k+1}, \ldots, \lambda_n\}$ and f and g are some $\mathbb{C}^r$-smooth ($r \geq 1$) functions which vanish at the origin along with their first derivatives.

Theorem 2.7. *A structurally stable saddle O has $\mathbb{C}^r$-smooth invariant manifolds W^s_{loc} and W^u_{loc} (see Figs.* 2.7.1 *and* 2.7.2*) whose equations are*

$$W^s_{loc}: \quad v = \psi(u) \tag{2.7.2}$$

$$W^u_{loc}: \quad u = \varphi(v) \tag{2.7.3}$$

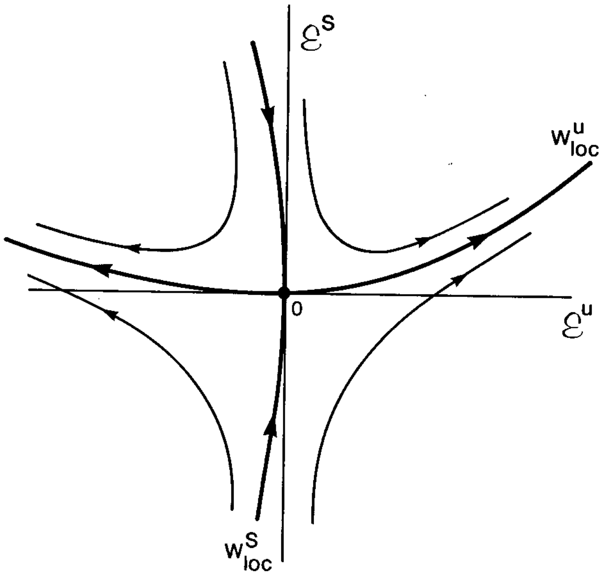


Fig. 2.7.1. The local stable W^s_{loc} and the unstable manifold W^u_{loc} of a saddle on the plane.

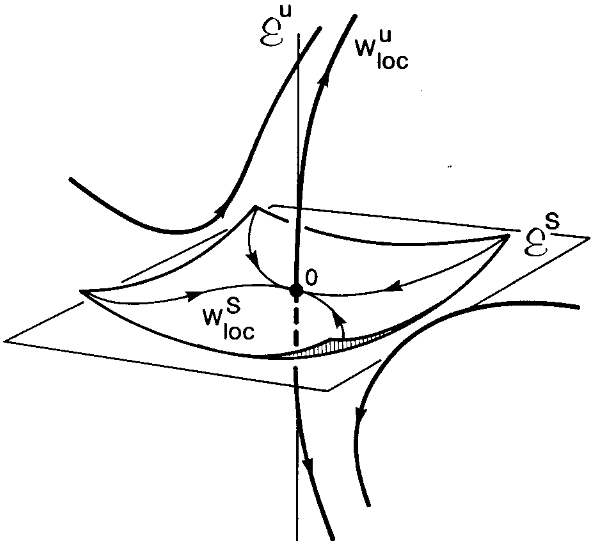


Fig. 2.7.2. Same as Fig. 2.7.1 but in $\mathbb{R}^3$.

where

$$\psi(0) = 0\,, \quad \psi'(0) = 0 \tag{2.7.4}$$

$$\varphi(0) = 0\,, \quad \varphi'(0) = 0\,. \tag{2.7.5}$$

We prove this theorem in the next section.

The condition of invariance of the manifolds W^s_{loc} and W^u_{loc} may be expressed as

$$\begin{aligned} \dot{v} &= \psi'(u)\dot{u} \quad \text{when } v = \psi(u)\,, \\ \dot{u} &= \varphi'(v)\dot{v} \quad \text{when } u = \varphi(v)\,, \end{aligned}$$

or

$$A^+\psi(u) + g(u,\psi) = \psi'(u)(A^-u + f(u,\psi)) \tag{2.7.6}$$

$$A^-\varphi(v) + f(\varphi,v) = \varphi'(v)(A^+v + g(\varphi,v))\,. \tag{2.7.7}$$

Relations (2.7.6) and (2.7.7) yield an algorithm for computing the invariant manifolds near the saddle. First of all we expand φ and ψ in the Taylor series with symbolic coefficients and then substitute them into (2.7.6) and (2.7.7) and collect similar terms. As a result the formulae obtained allow one to sequentially determine any number of terms in the expansions of the functions φ and ψ through the Taylor coefficients of the functions f and g.

For example, for the two-dimensional analytic system

$$\begin{aligned} \dot{u} &= -\lambda u + \sum_{i+j\geq 2} \alpha_{ij} u^i v^j\,, \\ \dot{v} &= \gamma v + \sum_{i+j\geq 2} \beta_{ij} u^i v^j \end{aligned}$$

Eq. (2.7.6) has the form

$$\gamma\psi + \sum_{i+j\geq 2} \beta_{ij} u^i \psi^j = \psi'(u)(-\lambda u + \sum_{i+j\geq 2} \alpha_{ij} u^i \psi^j)\,.$$

If we substitute the expression

$$\psi(u) = \psi_2 u^2 + \psi_3 u^3 + \cdots$$

with indefinite coefficients ψ_i into this equation and equate the coefficients of u^2, we obtain

$$\gamma\psi_2 + \beta_{20} = -2\lambda\psi_2\,,$$

hence we find

$$\psi_2 = -\beta_{20}/(2\lambda + \gamma)\,.$$

Having equated coefficients of u^3, we obtain

$$\gamma\psi_3 + \beta_{30} + \beta_{11}\psi_2 = -3\lambda\psi_3 + 2\psi_2\alpha_{20}\,,$$

yielding

$$\psi_3 = -(\beta_{30} + \beta_{11}\psi_2 - 2\psi_2\alpha_{20})/(3\lambda + \gamma)\,.$$

Repeating this procedure, we find step by step all of the coefficients of the Taylor expansion of the function ψ. The formula for calculating the m-th coefficient is the following:

$$(m\lambda + \gamma)\psi_m = \mathcal{F}_m(\beta_{ij}, \alpha_{ij}, \psi_2, \ldots, \psi_{m-1})\,,$$

where $\mathcal{F}_m$ is some expression which depends only upon a finite number of coefficients α and β, and on the first $(m-1)$ coefficients of the expansion of ψ. One may show that the coefficients of ψ_m decrease rapidly as m increases so that the series converges.

Following the scheme previously employed for the non-leading manifold we can locally straighten W^s and W^u near O, using the change of variables

$$\begin{aligned}\xi &= u - \psi(v)\,,\\ \eta &= v - \varphi(u)\,.\end{aligned}$$

In the new coordinates the equations of the invariant manifolds become

$$W^s_{loc}\colon\ \eta = 0\,, \quad W^u_{loc}\colon\ \xi = 0\,,$$

the invariance implying that $\dot\eta = 0$ when $\eta = 0$ and $\dot\xi = 0$ when $\xi = 0$.

The system may be written as

$$\begin{aligned}\dot\xi &= (A^- + h_1(\xi,\eta))\,\xi\,,\\ \dot\eta &= (A^+ + h_2(\xi,\eta))\,\eta\,,\end{aligned} \tag{2.7.8}$$

where $h_i \in \mathbb{C}^{r-1}$ and

$$h_i(0,0) = 0\,, \quad i = 1,2\,. \tag{2.7.9}$$

For convenience, let us denote the characteristic exponents with positive real parts as $\gamma_1, \ldots, \gamma_{n-k}$. We assume also that the characteristic exponents are ordered so that

$$\operatorname{Re}\lambda_k \leq \cdots \leq \operatorname{Re}\lambda_2 \leq \operatorname{Re}\lambda_1 < 0 < \operatorname{Re}\gamma_1 \leq \operatorname{Re}\gamma_2 \leq \cdots \leq \operatorname{Re}\gamma_{n-k}\,.$$

The functions $h_{1,2}$ are small near the saddle and therefore, as long as the trajectory remains in a neighborhood of the saddle, the inequalities

$$\begin{aligned}
\frac{d}{dt}\|\xi(t)\| &\leq (\operatorname{Re}\lambda_1 + \varepsilon)\|\xi(t)\| \quad \text{for} \quad t \geq 0\,,\\
\frac{d}{dt}\|\eta(t)\| &\leq (\operatorname{Re}\gamma_1 - \varepsilon)\|\eta(t)\| \quad \text{for} \quad t \leq 0
\end{aligned}$$

hold in the Jordan basis (see the proof of Theorem 2.4). After integrating we obtain

$$\|\xi(t)\| \leq e^{(\operatorname{Re}\lambda_1+\varepsilon)t}\,\|\xi_0\| \quad \text{for} \quad t \geq 0\,, \tag{2.7.10}$$

$$\|\eta(t)\| \leq e^{(\operatorname{Re}\gamma_1-\varepsilon)t}\,\|\eta_0\| \quad \text{for} \quad t \leq 0\,. \tag{2.7.11}$$

It follows from these estimates that a trajectory which lies neither in W^s_{loc} nor in W^u_{loc} must escape from any small neighborhood of the saddle as $t \to \pm\infty$. Moreover, the time that a positive semi-trajectory spends near the saddle is proportional to $\ln\|\eta_0\|$ and for the negative semi-trajectories it is proportional to $\ln\|\xi_0\|$.

The system (2.7.1) in its restriction to the stable manifold W^s_{loc}: $v = \psi(u)$ is defined by the equation

$$\dot{u} = A^- u + h_1(u, \psi(u))\,, \tag{2.7.12}$$

hence, the point O is a stable equilibrium state on W^s_{loc}. In the generic case it is either a node (provided there is only one leading coordinate), or a focus (provided there are two leading coordinates corresponding to a pair of complex-conjugate characteristic exponents).

In its restriction to W^u_{loc} the system becomes

$$\dot{v} = A^+ v + h_2(\varphi(v), v)\,. \tag{2.7.13}$$

The point O is here a completely unstable equilibrium state, in general, it is either a node or a focus.

Now, in complete analogy to the linear case, we can select four basic kinds of saddle equilibria according to their behavior in the leading coordinates:

- *saddle* : a node on W^s_{loc} and on W^u_{loc};
- *saddle-focus* (2,1) : a focus on W^s_{loc} and a node on W^u_{loc};
- *saddle-focus* (1,2) : a node on W^s_{loc} and a focus on W^u_{loc};
- *saddle-focus* (2,2) : a focus on W^s_{loc} and a focus on W^u_{loc}.

Theorems 2.5 and 2.6 hold for the systems (2.7.12) and (2.7.13). This implies that there exist *a non-leading and a leading stable invariant sub-manifolds* W^{ss}_{loc}, W^{sL}_{loc} in W^s_{loc} and *unstable invariant sub-manifolds* W^{uu}_{loc}, W^{uL}_{loc} in W^u_{loc}. In addition, we will close this section by describing the existence of three more smooth invariant manifolds of the saddle equilibrium state. Let us introduce the notation

$$r_{sL} = \left[\frac{\hat{\lambda}}{\operatorname{Re}\lambda_1}\right], \tag{2.7.14}$$

$$r_{uL} = \left[\frac{\hat{\gamma}}{\operatorname{Re}\gamma_1}\right], \tag{2.7.15}$$

where $\hat{\lambda}$ and $\hat{\gamma}$ are the real parts of the non-leading stable and unstable exponents nearest to the imaginary axis, respectively, and where $[x]$, as before, denotes the largest integer which is strictly less than x.

Theorem 2.8. *In a small neighborhood of a structurally stable equilibrium state of saddle type there exist the following smooth invariant manifolds*:

- *a* $\mathbb{C}^{\min(r, r_{uL})}$*-smooth extended stable manifold* W^{sE}_{loc} *which contains* W^s_{loc} *and which is tangential at the point* 0 *to the direct sum of the stable and the leading unstable eigen-space of the linearization matrix* (*thus,* W^{sE}_{loc} *is transverse to* W^{uu}_{loc});
- *a* $\mathbb{C}^{\min(r, r_{sL})}$*-smooth extended unstable manifold* W^{eE}_{loc} *which contains* W^u_{loc} *and which is tangential at the point* 0 *to the direct sum of the unstable and the leading stable eigen-space of the linearization matrix* (*it is transverse to* W^{ss}_{loc});
- *a* $\mathbb{C}^{\min(r, r_{sL}, r_{uL})}$*-smooth leading saddle manifold* $W^L_{loc} = W^{uE}_{loc} \cap W^{sE}_{loc}$.

The proof of this theorem is given in Chap. 5. We notice merely that the manifold W^{sE}_{loc}, generally speaking, is not unique, however *any two such manifolds have the same tangent at each point of* W^s_{loc}. Similarly, *any two manifolds* W^{uE}_{loc} *are tangential everywhere on* W^u_{loc}.

2.8. Solution near a saddle. The boundary-value problem

In this section we discuss a method for constructing a solution of a nonlinear system near a saddle equilibrium state. We will use this method throughout the book and, in particular, to prove the existence and the smoothness of the stable and the unstable manifolds of the saddle equilibrium point.

Let us consider an n-dimensional system

$$\begin{aligned} \dot{u} &= A^- u + f(u,v)\,, \\ \dot{v} &= A^+ v + g(u,v)\,, \end{aligned} \tag{2.8.1}$$

where $u \in \mathbb{R}^k$, $v \in \mathbb{R}^m$ $(k+m=n)$, and the functions f and g are $\mathbb{C}^r$-functions $(r \geq 1)$ which vanish at the origin along with their first derivatives. We assume that $\operatorname{spectr} A^- = \{\lambda_1, \ldots, \lambda_k\}$ lies strictly to the left of the imaginary axis and $\operatorname{spectr} A^+ = \{\gamma_1, \ldots, \gamma_m\}$ lies strictly to the right of it.

Taking into account the nonlinearities while constructing a solution of the system (2.8.1) near the saddle O, we may identify a number of difficulties. Firstly, a trajectory of the system can stay near the point O for a very long time; the closer an initial point is to the stable manifold W^s_{loc}, the larger this dwelling time will be. Moreover, this time is equal to infinity if the initial point lies on W^s_{loc}. Thus, we need formulae which can work on an infinitely large time interval. Obviously the major obstacle here is the instability of the initial-value problem near the saddle. Let us consider, for instance, a solution

$$u(t) = e^{A^- t}\, u_0\,, \quad v(t) = e^{A^+ t}\, v_0$$

of the initial-value problem of the linearized system. When we add a small perturbation Δv to v_0, we can estimate the corresponding increment of $v(t)$ which is given by

$$\|\Delta v(t)\| = \|e^{A^+ t}\, \Delta v\| \gg \|\Delta v\|\,.$$

This inequality implies that arbitrarily small perturbations of the initial data may cause finite changes in the solutions provided that the integration time is sufficiently long. This kind of instability occurs not only in computer simulations but also in recursive construction of analytical expressions. So, if we are looking for a solution of the system (2.8.1) using the method of successive approximations, starting with the solution of the linear system we will generate terms of the type $(e^{A^+ t}\, v_0)^m$ at the m-th step; namely, the terms with arbitrarily large exponents.

Of course, the same problem also arises near a completely unstable equilibrium point. However, in this case, a solution of the initial-value problem becomes stable upon reversing the time variable. In the case of a saddle, however, after the change $t \to -t$ a solution becomes stable with respect to the variables v, but unstable with respect to the variables u.

The principal idea for overcoming these obstacles is to integrate the system with respect to the variables u in forward time, and with respect to v in backward time. More specifically, instead of solving an initial-value problem, we must solve the following *boundary value problem*:

given any u_0 *and* v_1, *where* $\|u_0\| \leq \varepsilon$ *and* $\|v_1\| \leq \varepsilon$, *and any* $\tau > 0$, *find a solution of the system* (2.8.1) *on the interval* $t \in [0, \tau]$ *such that*

$$u(0) = u_0 \,, \quad v(\tau) = v_1 \,. \tag{2.8.2}$$

For linear systems the solution of the boundary value problem has the form

$$u(t) = e^{A^- t}\, u_0 \,, \quad v(t) = e^{-A^+(\tau - t)} v_1 \,. \tag{2.8.3}$$

As both $\|e^{A^- t}\|$ and $\|e^{-A^+(\tau-t)}\|$ are bounded for all $t \in [0, \tau]$ (see inequality (2.3.35)), this solution is stable with respect to perturbations of the initial values (u_0, v_1, τ). This technique can be applied in the nonlinear case as well. As we will see, a solution of the boundary value problem of a nonlinear system can be obtained by the method of successive approximations, starting with solution (2.8.3) of the linear problem.

Theorem 2.9. *For sufficiently small* $\varepsilon > 0$ *and any* $\tau \geq 0$, *and* u_0 *and* v_1 *such that* $\|u_0\| \leq \varepsilon$, $\|v_1\| \leq \varepsilon$ *a solution of the boundary value problem* (2.8.2) *exists, is unique and depends continuously on* (u_0, v_1, τ).

For a two-dimensional system, the existence of such a solution is geometrically obvious, see Fig. 2.8.1. In a small neighborhood of the point O there are infinitely many trajectories which begin on a straight line $u = u_0$ and end on the straight line $v = v_1$. The flight-time may vary from zero to infinity.

In the general case, the proof of Theorem 2.9 is analytical. Let us consider a system of integral equations

$$\begin{aligned} u(t) &= e^{A^- t}\, u_0 + \int_0^t e^{A^-(t-s)}\, f(u(s), v(s))\, ds \\ v(t) &= e^{-A^+(\tau - t)}\, v_1 - \int_t^\tau e^{-A^+(s-t)}\, g(u(s), v(s))\, ds \end{aligned} \tag{2.8.4}$$

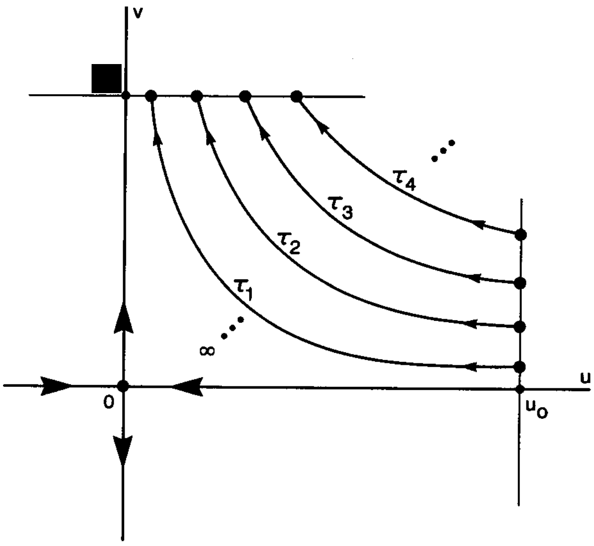


Fig. 2.8.1. The boundary value problem near a saddle. The flight time $\tau \to +\infty$ as a nearby initial point on $u = u_0$ tends to the stable manifold $v = 0$.

with respect to the functions $u(t)$ and $v(t)$ on the interval $t \in [0, \tau]$. By differentiating the right-hand side of formula (2.8.4) with respect to t one can easily verify that any continuous solution $\{u(t), v(t)\}$ of this system is a solution of the system (2.8.1). Moreover, as $u(0) = u_0$ and $v(\tau) = v_1$, the solution of the system (2.8.4) is the desired solution of the boundary value problem (2.8.2).

The converse is also true. Let $\{u(t), v(t)\}_{t \in [0,\tau]}$ be a solution of the boundary value problem. It follows from (2.8.1) that

$$\begin{aligned} \frac{d}{dt}(e^{-A^- t} u(t)) &= e^{-A^- t} f(u(t), v(t)), \\ \frac{d}{dt}(e^{-A^+ t} v(t)) &= e^{-A^+ t} g(u(t), v(t)), \end{aligned}$$

whence

$$\begin{aligned} e^{-A^- t} u(t) &= u_0 + \int_0^t e^{-A^- s} f(u(s, v(s)) \, ds, \\ e^{-A^+ t} v(t) + \int_t^\tau e^{-A^+ s} g(u(s), v(s)) \, ds &= e^{-A^+ \tau} v_1, \end{aligned}$$

and, consequently, (2.8.4) is valid. Thus, the solution of the boundary value problem is identical to that of the system of the integral equations (2.8.4).

We will construct a solution of system (2.8.4) by the method of successive approximations. As the first approximation we choose

$$u^{(1)}(t) = e^{A^- t} u_0 \,, \quad v^{(1)}(t) = e^{-A^+(\tau - t)} v_1 \,,$$

and for each successive approximation we will use the formula

$$\begin{aligned} u^{(n+1)}(t) &= e^{A^- t}\, u_0 + \int_0^t e^{A^-(t-s)}\, f(u^{(n)}(s), v^{(n)}(s))\, ds \,, \\ v^{(n+1)}(t) &= e^{-A^+(\tau-t)} v_1 - \int_t^\tau e^{-A^+(s-t)}\, g(u^{(n)}(s), v^{(n)}(s))\, ds \,. \end{aligned} \tag{2.8.5}$$

We will show that this sequence converges uniformly to some limit function $\{u^*(t), v^*(t)\}$. First, let us prove that

$$\|u^{(n)}(t)\| \leq 2\varepsilon \,, \quad \|v^{(n)}(t)\| \leq 2\varepsilon \tag{2.8.6}$$

for all n and $t \in [0, \tau]$. When $n = 1$ it follows directly from $\|u_0\| \leq \varepsilon$ and $\|v_1\| \leq \varepsilon$ as well as from the inequalities

$$\begin{aligned} \|e^{A^- t}\| &\leq e^{-\lambda t} \,, \\ \|e^{-A^+(\tau - t)}\| &\leq e^{-\gamma(\tau - t)} \,, \end{aligned} \tag{2.8.7}$$

where $\lambda > 0$ and $\gamma > 0$ are such that the spectrum of the matrix A^- lies strictly to the left of the straight-line $\operatorname{Re} z = -\lambda$ in the complex plane, and the spectrum of the matrix A^+ lies strictly to the right of the straight-line $\operatorname{Re} z = \gamma$.

We will use mathematical induction to prove inequality (2.8.6) for all n. Firstly, observe that since both f and g vanish at the point O along with their first derivatives, it follows that

$$\left\| \frac{\partial(f, g)}{\partial(u, v)} \right\| \leq \delta \,, \tag{2.8.8}$$

$$\|f, g\| \leq \delta \|u, v\| \,, \tag{2.8.9}$$

where $\|u, v\|$ denotes $\max\{\|u\|, \|v\|\}$. In this equation, the constant δ can be made arbitrarily small by decreasing the size of the neighborhood of the point

O. Choose ε small such that for all u and v from the 2ε-neighborhood of the saddle, the inequality

$$2\delta \max(\lambda^{-1}, \gamma^{-1}) \leq 1 \tag{2.8.10}$$

is satisfied. From (2.8.5) and (2.8.9) we obtain

$$\|u^{(n+1)}(t)\| \leq \|u_0\| + \delta \int_0^t e^{-\lambda(t-s)} \|u^{(n)}(s), v^{(n)}(s)\| \, ds \,,$$
$$\|v^{(n+1)}(t)\| \leq \|v_1\| + \delta \int_t^\tau e^{-\gamma(s-t)} \|u^{(n)}(s), v^{(n)}(s)\| \, ds \,,$$

whence

$$\|u^{(n+1)}(t), v^{(n+1)}(t)\| \leq \varepsilon + \delta \max(\lambda^{-1}, \gamma^{-1}) \max_{0 \leq s \leq \tau} \|u^{(n)}(s), v^{(n)}(s)\| \,.$$

Using (2.8.10) we have that if

$$\|u^{(n)}(t), v^{(n)}(t)\| \leq 2\varepsilon \,,$$

then

$$\|u^{(n+1)}(t), v^{(n+1)}(t)\| \leq 2\varepsilon \,,$$

i.e. (2.8.6) holds for all n. Let us now prove that

$$\max_{0 \leq t \leq \tau} \|u^{(n+1)}(t) - u^{(n)}(t), v^{(n+1)}(t) - v^{(n)}(t)\|$$
$$\leq \frac{1}{2} \max_{0 \leq s \leq \tau} \|u^{(n)}(s) - u^{(n-1)}(s), v^{(n)}(s) - v^{(n-1)}(s)\| \,. \tag{2.8.11}$$

Indeed, by (2.8.5)

$$\|u^{(n+1)}(t) - u^{(n)}(t)\|$$
$$\leq \int_0^t \|e^{A^-(t-s)}\| \, \|f(u^{(n)}(s), v^{(n)}(s)) - f(u^{(n-1)}(s), v^{(n-1)}(s))\| \, ds \,.$$

Hence, taking into account the inequality

$$\|f(u^{(n)}(s), v^{(n)}(s)) - f(u^{(n-1)}(s), v^{(n-1)}(s))\|$$
$$\leq \max_{(u,v)} \left\| \frac{\partial(f,g)}{\partial(u,v)} \right\| \|u^{(n)}(s) - u^{(n-1)}(s), \; v^{(n)}(s) - v^{(n-1)}(s)\|$$

we obtain from (2.8.7) and (2.8.8) that

$$\begin{aligned}&\|u^{(n+1)}(t) - u^{(n)}(t)\| \\ &\qquad \le \delta\lambda^{-1} \max_{0\le s\le\tau} \|u^{(n)}(s) - u^{(n-1)}(s),\, v^{(n)}(s) - v^{(n-1)}(s)\|\,.\end{aligned}$$

Analogously,

$$\begin{aligned}&\|v^{(n+1)}(t) - v^{(n)}(t)\| \\ &\qquad \le \delta\gamma^{-1} \max_{0\le s\le\tau} \|u^{(n)}(s) - u^{(n-1)}(s), v^{(n)}(s) - v^{(n-1)}(s)\|\,.\end{aligned}$$

Since the values $u^{(n)}$ and $v^{(n)}$ lie inside the 2ε-neighborhood of the saddle for all n, the value δ in the last two inequalities satisfies (2.8.10), and hence inequality (2.8.11) follows.

By virtue of (2.8.11) the series

$$\sum_{n=1}^{\infty} (u^{(n+1)}(t) - u^{(n)}(t),\, v^{(n+1)}(t) - v^{(n)}(t))$$

is majorized by a geometric progression with the coefficient 1/2. Therefore, this series converges uniformly to some continuous function $\{u^*(t), v^*(t)\}$. By construction, $\{u^*(t), v^*(t)\}$ is the limit of successive approximations (2.8.5). Taking the limit $n \to \infty$ in (2.8.5), we determine that $\{u^*(t), v^*(t)\}$ satisfies relation (2.8.4), *i.e.* we have the solution of the boundary-value problem. Because the convergence is uniform, $\{u^*(t), v^*(t)\}$ depends continuously upon the initial value (u_0, v_1, τ).

To prove uniqueness, let us suppose that Eqs. (2.8.4) have a second solution $\{u^{**}(t), v^{**}(t)\}_{t\in[0,\tau]}$. Then, by using the same algorithm as that in the proof of inequality (2.8.11) we may show that

$$\begin{aligned}&\|u^{**}(t) - u^*(t),\, v^{**}(t) - v^*(t)\| \\ &\qquad \le \frac{1}{2} \max_{0\le s\le\tau} \|u^{**}(s) - u^*(s),\, v^{**}(s) - v^*(s)\|\end{aligned}$$

for all $t \in [0, \tau]$, *i.e.* $u^{**} \equiv u^*$ and $v^{**} \equiv v^*$. End of the proof.

Remark. It is clear from the proof that the result on the existence and continuity of solutions of the boundary value problem holds when the functions f, g in the right-hand side of system (2.8.1) depend explicitly on time. The requirements here are that for all t the functions f and g must vanish at

$u = 0$, $v = 0$ and the norm of their derivatives with respect to u and v must be bounded by a small constant δ (see inequalities (2.8.8)–(2.8.10)), uniformly with respect to t. We emphasize that the smoothness of the functions f and g with respect to t is not required.

Theorem 2.10. *The solution of the boundary-value problem depends $\mathbb{C}^r$-smoothly on (u_0, v_1, t, τ).*

Proof. Let $\{u^*(t), v^*(t)\}_{t\in[0,\tau]}$ be a solution of the boundary value problem corresponding to (u_0, v_1, τ). Denote $v_0 = v^*(0)$. The trajectory $\{u^*, v^*\}$ depends $\mathbb{C}^r$-smoothly on (u_0, v_0, t, τ) as it is a solution of the initial-value problem. Therefore to prove the theorem we must show that v_0 depends smoothly on (u_0, v_1, t, τ). Since $v_1 = v^*(t = \tau)$ is a smooth function with respect to (u_0, v_0, t, τ), it is sufficient (according to the implicit function theorem) to verify that the derivative $\partial v_1/\partial v_0 = \partial v^*/\partial v_0|_{t=\tau}$ is non-singular.

The derivatives $\partial u^*/\partial v_0$ and $\partial v^*/\partial v_0$ can be found as solutions of the system of *variational equations*

$$\begin{aligned} \dot{U} &= A^- U + f'_u(u^*(t), v^*(t))\, U + f'_v(u^*(t), v^*(t))\, V\,, \\ \dot{V} &= A^+ V + g'_u(u^*(t), v^*(t))\, U + g'_v(u^*(t), v^*(t))\, V\,, \end{aligned} \tag{2.8.12}$$

with the initial conditions

$$U(0) = 0\,, \quad V(0) = I_m\,, \tag{2.8.13}$$

where $U \equiv \partial u^*/\partial v_0$, $V \equiv \partial v^*/\partial v_0$ and I_m is the $(m \times m)$ identity matrix. The fact that the matrix of derivatives $\partial v_1/\partial v_0 \equiv V(\tau)$ is non-singular means that there exists a matrix Q such that

$$V(\tau)\, Q = I_m\,. \tag{2.8.14}$$

In addition we notice that if such matrix Q exists, then

$$Q = \left(\frac{\partial v_1}{\partial v_0}\right)^{-1}\,. \tag{2.8.15}$$

Equations (2.8.12) are linear with respect to the variables U and V, therefore, if we postmultiply both the right and the left-hand sides of (2.8.12) by Q, it is easily seen that $\tilde{U} \equiv UQ$ and $\tilde{V} \equiv VQ$ also satisfy Eqs. (2.8.12). Moreover, in order that both satisfy (2.8.13) and (2.8.14) the following conditions must hold

$$\tilde{U}(0) \equiv 0\,, \quad \tilde{V}(\tau) \equiv I_m\,. \tag{2.8.16}$$

Thus, the derivative matrix $\partial v_1/\partial v_0$ is non-singular if and only if the boundary-value problem (2.8.16) for the system of variational equations has a solution.

To complete the proof of the theorem we note that the existence and uniqueness of a solution of boundary value-problem (2.8.16) for the system of variational equations (2.8.12) follows from the remark to the previous theorem. Moreover, since the derivatives of the right-hand side of system (2.8.12) with respect to U and V are (without the constant matrices A^+ and A^-) $f'(u^*(t), v^*(t))$ and $g'(u^*(t), v^*(t))$, respectively, it follows that they can be estimated by the same constant δ as the derivatives of the nonlinear part of the right-hand side of system (2.8.1) with respect to u and v. The boundary value problem for the system of variational equations is therefore solvable for $\|u_0, v_1\| \leq \varepsilon$ where ε may be taken the same as in the boundary value problem for the original system.

As it is seen from the proof, the derivative of the solution $(u^*(t), v^*(t))$ of the boundary-value problem for system (2.8.1) with respect to v_1 is given by

$$\begin{aligned}\frac{\partial(u^*(t), v^*(t))}{\partial v_1} &= \frac{\partial(u^*(t), v^*(t))}{\partial v_0}\left(\frac{\partial v_1}{\partial v_0}\right)^{-1}\\ &= (U(t), V(t))V(\tau)^{-1} = (\tilde{U}(t), \tilde{V}(t))\,,\end{aligned}$$

where (U, V) is the solution of initial value problem (2.8.12),(2.8.13) and $(\tilde{U}, \tilde{V})$ is the solution of boundary value problem (2.8.12),(2.8.16).

Analogously, one can prove that the derivative with respect to u_0 is found as a solution of the boundary value problem

$$U(0) = I_n\,, \quad V(\tau) = 0\,. \tag{2.8.17}$$

Summarizing: *The derivatives of the solution of the boundary value problem for system* (2.8.1) *with respect to* u_0 *and* v_1 *are found as the solutions of the corresponding boundary value problems obtained by formal differentiation of the equations and boundary conditions.*

In the same way, one can show that if system (2.8.1) depends smoothly on some set of parameters μ, then the derivative of $\{u^*, v^*\}$ with respect to μ is found as a solution of the boundary value problem

$$U(0) = 0\,, \quad V(\tau) = 0 \tag{2.8.18}$$

for a non-homogeneous system of variational equations derived from a formal differentiation of system (2.8.1) with respect to μ, namely

$$\begin{aligned}\dot{U} &= A^- U + f'_u(u^*(t), v^*(t), \mu)\, U \\ &\quad + f'_v(u^*(t), v^*(t), \mu)\, V + f'_\mu(u^*(t), v^*(t), \mu) \\ \dot{V} &= A^+ V + g'_u(u^*(t), v^*(t), \mu)\, U \\ &\quad + g'_v(u^*(t), v^*(t), \mu)\, V + g'_\mu(u^*(t), v^*(t), \mu)\,.\end{aligned} \tag{2.8.19}$$

Just like the solution of the boundary-value problem for the homogeneous equations, the solution of the problem (2.8.18),(2.8.19) is found as a limit of successive approximations for the system of integral equations

$$\begin{aligned}U(t) &= \int_0^t e^{A^-(t-s)}\, [f'_u(s)U(s) + f'_v(s)V(s) + f'_\mu(s)]\, ds \\ V(t) &= -\int_t^\tau e^{A^-(s-t)}\, [g'_u(s)U(s) + g'_v(s)V(s) + g'_\mu(s)]\, ds\,.\end{aligned} \tag{2.8.20}$$

The proof of uniform convergence of the successive approximations is similar to the proof of Theorem 2.9. The key inequality of (2.8.11) for convergence holds under the same condition

$$\left\| \frac{\partial(f, g)}{\partial(u, v)} \right\| \leq \delta$$

as in Theorem 2.9, *i.e.* the unique solution of the boundary value problem (2.8.18), (2.8.19) exists for the same ε as the solution of the boundary value problem for the original system.

Derivatives of a higher-order with respect to the variables (u_0, v_1) can also be found as solutions of a boundary-value problem by considering the variational equations for the first order variational equations (2.8.12). Because the variables (u_0, v_1) are no longer the boundary conditions for $\partial(u^*, v^*)/\partial(u_0, v_1)$ (see (2.8.16) and (2.8.17)) but they occur in Eqs. (2.8.12) as parameters only (this is because the right-hand side of the system (2.8.12) depends on $\{u^*(t), v^*(t)\}$, while $\{u^*(t), v^*(t)\}$ depends on (u_0, v_1)), it follows that the second derivatives and higher ones are found in a similar manner as the solutions of the non-homogeneous boundary value problems analogous to (2.8.18) and (2.8.19).

For example, the boundary value problem for $\partial^2(u^*, v^*)/\partial(v_1)^2$ is obtained by the formal differentiation, with respect to v_1, of variational equations

(2.8.12) with boundary conditions (2.8.16), *i.e.*

$$\begin{aligned}\dot{Z} &= A^- Z + f'_u Z + f'_v W \\ &\quad + f''_{uu} U^* U^* + f''_{uv} V^* U^* + f''_{vu} U^* V^* + f''_{vv} V^* V^* , \\ \dot{W} &= A^+ W + g'_u Z + g'_v W \\ &\quad + g''_{uu} U^* U^* + g''_{uv} V^* U^* + g''_{vu} U^* V^* + g''_{vv} V^* V^* , \\ &\qquad Z(0) = 0 , \quad W(\tau) = 0 ,\end{aligned}$$

where we have introduced the notation

$$U^*(t) \equiv \partial u^*(t)/\partial v_1 , \quad V^*(t) \equiv \partial v^*(t)/\partial v_1$$

and

$$Z \equiv \partial^2 (u^*)/\partial (v_1)^2 , \quad W \equiv \partial^2 (v^*)/\partial (v_1)^2 .$$

Let us show how this theory can be applied to prove the existence and smoothness of the stable and the unstable manifolds (Theorem 2.7 of the previous section). We will consider only the case of the stable manifold W^s; as for the unstable manifold W^u one repeats the method for the system obtained from (2.8.1) by reversing time.

Note that our conclusions concerning the uniform convergence of successive approximations (2.8.5) and, consequently, successive approximations for boundary value problems for variational equations (2.8.12) as well as for nonhomogeneous variational equations (2.8.19) and for the variational equations for the variational equations etc., remain valid for all $\tau \geq 0$ including $\tau = +\infty$. Thus, the system

$$\begin{aligned} u(t) &= e^{A^- t}\, u_0 + \int_0^t e^{A^-(t-s)}\, f(u(s), v(s))\, ds , \\ v(t) &= -\int_t^\infty e^{-A^+(s-t)}\, g(u(s), v(s))\, ds \end{aligned} \tag{2.8.21}$$

obtained from (2.8.4) with $\tau = +\infty$, for all $\|u_0\| \leq \varepsilon$ has a unique solution which lies in the 2ε-neighborhood of the point O for $t \geq 0$. Moreover, this solution depends $\mathbb{C}^r$-smoothly on u_0.

It is easy to verify by direct differentiation that a solution of the system (2.8.21) is also a solution of the system (2.8.1).

Conversely, it follows from the proof of Theorem 2.9 that for any solution $\{u(t), v(t)\}$ of system (2.8.1) which stays in a small neighborhood of O for all $t \geq 0$, relation (2.8.4) holds with $u_0 = u(0)$ and $v_1 = v(\tau)$ satisfying for any $\tau \geq 0$. Therefore, taking the limit in relation (2.8.4) we see that any bounded solution of system (2.8.1) satisfies the system of integral equations (2.8.21).

Thus, for any u_0 satisfying $\|u_0\| \leq \varepsilon$, there exists a unique v_0 such that the trajectory beginning with (u_0, v_0) does not leave a neighborhood of the saddle as $t \to +\infty$. Let us denote $v_0 = \psi(u_0)$. By definition, the union of all points $(u_0, \psi(u_0))$ is an invariant set of the system (2.8.1): It consists of all trajectories which remain in a neighborhood of the equilibrium state O for all $t \geq 0$. In particular, this set contains the point O itself; *i.e.* $\psi(0) = 0$. Because $\psi \in \mathbb{C}^r$, this set is a smooth invariant manifold. The system on this manifold is written as

$$\dot{u} = A^- u + f(u, \psi(u)) .$$

Since the derivatives f'_u and f'_v vanish at the origin, the linearized equation is

$$\dot{u} = A^- u .$$

The spectrum of A^- lies to the left of the imaginary axis, therefore O is an exponentially asymptotically stable on the invariant manifold under consideration. This means that all trajectories in this manifold tend to O as $t \to +\infty$; *i.e.* this smooth invariant manifold is the stable invariand manifold W^s of O. We have proved the existence and smoothness of W^s which completes the proof of Theorem 2.7.

In conclusion, we remark that this method of reduction of the problem of existence of the invariant manifold to solving an integral equation was proposed by Lyapunov and proved to be rather useful for studying non-autonomous systems especially.

2.9. Problem of smooth linearization. Resonances

We discussed earlier (see the Grobman–Hartman theorem in Sec. 2.5) that in a neighborhood of a structurally stable equilibrium state the system

$$\dot{x} = Ax + f(x) , \tag{2.9.1}$$

where

$$f(0) = 0, \qquad f'(0) = 0 ,$$

is topologically equivalent to the linearized system

$$\dot{y} = Ay\,. \tag{2.9.2}$$

Let us now ask a very natural question: Can system (2.9.1) be reduced to system (2.9.2) by some smooth change of variables

$$y = x + \varphi(x)\,, \tag{2.9.3}$$

where $\varphi(0) = 0$ and $\varphi'(0) = 0$? Poincaré was the first who posed this question and considered it in the analytic case. We remark that a smooth change of variables preserves the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of the matrix A and, moreover, when it is locally close to identity like (2.9.3) it preserves the matrix A itself. Observe that such a change of variables is local, *i.e.* the smooth equivalence is assumed to be valid only in a small neighborhood of the equilibrium state O.

While reducing the original nonlinear system to linear form we run into a number of difficulties, and the main one is caused by the presence of resonances.

The set $\{\lambda_1, \dots, \lambda_n\}$ of the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of the matrix A is called *a resonant set* if there exists a linear relationship

$$\lambda_k = (m, \lambda) = \sum_{j=1}^{n} m_j \lambda_j\,, \tag{2.9.4}$$

where $m = (m_1, \dots, m_n)$ is the row of non-negative integers such that $|m| = \sum\limits_{j=1}^{n} m_j \geq 2$. The relation itself is called *a resonance* and $|m|$ is called *the order of the resonance.*

Let the function $f(x)$ be $\mathbb{C}^N$-smooth, then its expansion

$$f(x) = f_2(x) + \cdots + f_N(x) + o_N(x) \tag{2.9.5}$$

is valid, where $f_l(x)\,(l = 2, \dots, N)$ is a homogeneous polynomial of the power l; hereafter $o_N(\cdot)$ stands for the terms which vanish at the origin along with the first N derivatives.

The following lemmas are well known.

Lemma 2.2. *Let $f(x) \in \mathbb{C}^N$ and assume there are no resonances of order $|m| \leq N$. Then there is a polynomial change of variables*

$$y = x + \varphi_2(x) + \cdots + \varphi_N(x) \tag{2.9.6}$$

(*where* $\varphi_l(x)\,(l = 2, \dots, N)$ *is a homogeneous polynomial of power* l) *which transforms system* (2.9.1) *into*

$$\dot{y} = Ay + o_N(y)\,. \tag{2.9.7}$$

Proof. Substituting (2.9.6) into (2.9.1) we obtain

$$\begin{aligned} \dot{y} &= \dot{x} + \sum_{l=2}^{N} \frac{\partial \varphi_l(x)}{\partial x}\dot{x} \\ &= Ax + f_2(x) + \dots + f_N(x) + o_N(x) \\ &\quad + \sum_{l=2}^{N} \frac{\partial \varphi_l(x)}{\partial x}[Ax + f_2(x) + \dots + f_N(x) + o_N(x)] \\ &= Ay - A\varphi_2(x) - \dots - A\varphi_N(x) + f_2(x) + \dots + f_N(x) \\ &\quad + \sum_{l=2}^{N} \frac{\partial \varphi_l(x)}{\partial x}[Ax + f_2(x) + \dots + f_N(x)] + \dots\,, \end{aligned} \tag{2.9.8}$$

where the last summand denoted by the ellipsis contains terms of degree $N+1$ and higher. The remaining terms (except for Ay) must cancel each other, therefore one must assume that $\varphi_2(x)$ satisfies the equation

$$-A\varphi_2(x) + f_2(x) + \frac{\partial \varphi_2(x)}{\partial x}Ax = 0\,; \tag{2.9.9}$$

$\varphi_3(x)$ satisfies

$$-A\varphi_3(x) + f_3(x) + \frac{\partial \varphi_3(x)}{\partial x}Ax + \frac{\partial \varphi_2(x)}{\partial x}f_2(x) = 0\,; \tag{2.9.10}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$\varphi_N(x)$ satisfies

$$-A\varphi_N(x) + f_N(x) + \frac{\partial \varphi_N(x)}{\partial x}Ax + \sum_{p+q=N+1} \frac{\partial \varphi_p(x)}{\partial x}f_q(x) = 0\,. \tag{2.9.11}$$

Let us now prove the lemma for the case where the matrix A is diagonal

$$A = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{pmatrix}.$$

Recall that $\varphi_l(x)$ and $f_l(x)$ are homogeneous vector-polynomials, *i.e.*

$$\varphi_l(x) = (\varphi_{l1}(x), \dots, \varphi_{lk}(x), \dots, \varphi_{ln}(x))\,,$$
$$f_l(x) = (f_{l1}(x), \dots, f_{lk}(x), \dots, f_{ln}(x))\,,$$
$$k = 1, \dots, n\,.$$

Represent the polynomials $\varphi_{lk}(x)$ and $f_{lk}(x)$ in the form

$$\varphi_{lk}(x) = \sum_{m_1+\cdots+m_n=l} c_{mk}x^m\,,$$

$$f_{lk}(x) = \sum_{m_1+\cdots+m_n=l} d_{mk}x^m\,.$$

Equations (2.9.9)–(2.9.11) may now be rewritten in the component-by-component form

$$-\lambda_k\varphi_{2k}(x) + \sum_{j=1}^{n}\lambda_j\frac{\partial\varphi_{2k}(x)}{\partial x_j}x_j + f_{2k}(x) = 0\,, \tag{2.9.12}$$

$$-\lambda_k\varphi_{3k}(x) + \sum_{j=1}^{n}\lambda_j\frac{\partial\varphi_{3k}(x)}{\partial x_j}x_j + f_{3k}(x) + \sum_{j=1}^{n}\frac{\partial\varphi_{2k}(x)}{\partial x_j}f_{2j}(x) = 0\,, \tag{2.9.13}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$-\lambda_k\varphi_{Nk}(x) + \sum_{j=1}^{n}\lambda_j\frac{\partial\varphi_{Nk}(x)}{\partial x_j}x_j + f_{Nk}(x) + \sum_{j=1}^{n}\sum_{p+q=N+1}\frac{\partial\varphi_{pk}(x)}{\partial x_j}f_{qj}(x) = 0\,, \tag{2.9.14}$$

where $k = 1, 2, \dots, n$.

First solve (2.9.12). Equating the coefficients of the similar terms we obtain the equation

$$[(m,\lambda) - \lambda_k]c_{mk} + d_{mk} = 0\,. \tag{2.9.15}$$

It is clear that this equation may be resolved since there are no resonances, *i.e.* we can find

$$c_{mk} = \frac{d_{mk}}{\lambda_k - (m,\lambda)}\,, \tag{2.9.16}$$

and, consequently, the expression for $\varphi_2(x)$.

Substituting $\varphi_2(x)$ into Eq. (2.9.13) we obtain the equation for unknown coefficients of $\varphi_3(x)$

$$[(m,\lambda) - \lambda_k]c_{mk} + \tilde{d}_{mk} = 0 \tag{2.9.17}$$

with $\tilde{d}_{mk} = d_{mk} + d'_{mk}$, where d'_{mk} is the coefficient of x^m in the second sum in (2.9.13). Proceeding in the analogous manner we obtain all φ_l $(l = 2, \ldots, N)$ satisfying Eqs. (2.9.9)–(2.9.11).

This proves the lemma when all eigenvalues of A are real and different (because A is brought to the diagonal form by a linear change of variables in this case). If there are simple complex eigenvalues, then A is brought to a block-diagonal form:

$$\begin{aligned} &A_{kk} = \lambda_k \qquad \text{if } \lambda_k \text{ is real} \\ &A_{kk} = A_{k+1,k+1} = \operatorname{Re}\lambda_k, \quad A_{k,k+1} = -A_{k+1,k} = \operatorname{Im}\lambda_k \quad \text{if } \lambda_k = \lambda^*_{k+1} \end{aligned}$$

where $*$ denotes complex conjugation. This matrix is made diagonal by a *complex* coordinate transformation

$$\begin{aligned} &x'_k = x_k \qquad \text{if } \lambda_k \text{ is real} \\ &x'_k = x_k + i x_{k+1}, \quad x'_{k+1} = x_k - i x_{k+1} \quad \text{if } \lambda_k = \lambda^*_{k+1}\,. \end{aligned}$$

Since $x'_{k+1} = x'^*_k$, it follows that the new function f satisfies

$$f_{l,k+1}(x') = f_{lk}(x')^*$$

for such k that $\lambda_k = \lambda^*_{k+1}$ is complex. Now, the coordinate transformation φ defined by (2.9.12)–(2.9.14) satisfies

$$\varphi_{l,k+1}(x') = \varphi_{lk}(x')^*$$

for those k. Obviously then, the *real* coordinate transformation

$$y_k = x_k + \textstyle\sum_{l=2}^N \operatorname{Re}\varphi_{lk}(x')$$

$$y_{k+1} = x_{k+1} + \textstyle\sum_{l=2}^N \operatorname{Im}\varphi_{lk}(x')$$

brings the system to the desired form (2.9.7).

In case multiple eigenvalues are present, the matrix A can be written in the Jordan form (real on complex): The eigenvalues λ_k fill the main diagonal, plus some of upper diagonal entries may be non-zero:

$$A_{k,k+1} = \delta_k\,.$$

Thus, additional terms

$$\delta_k \varphi_{l,k+1}(x) \quad \text{and} \quad \delta_j \frac{\partial \varphi_{lk}}{\partial x_j} x_{j+1}$$

may appear in (2.9.12)–(2.9.14). Obviously, this does not change the conclusion of the lemma: relation (2.9.17), by which the sought coefficients c_{mk} are defined inductively, remains the same with the only difference that d'_{mk} is expressed now in terms of a wider range of coefficients $c_{m'k'}$. Namely, d'_{mk} is a function of such $c_{m'k'}$ that:

$$\begin{aligned} &(1)\ |m'| < |m|, \quad \text{or} \ \ (2)\ m' = m \ \text{ and } \ k' > k, \\ &\text{or} \ \ (3)\ |m'| = |m| \ \text{ and } \ \sum_{j=1}^{n} j \cdot m'_j < \sum_{j=1}^{n} j \cdot m_j \end{aligned} \tag{2.9.18}$$

(only case 1) is possible when the eigenvalues are simple). Thus, one may introduce *the partial order* among the vector-monomials $x^m e_k$ (where $e_k = (\underbrace{0, 0, \ldots, 1}_{k}, 0, \ldots, 0))$ in the following way:

> $x^m e_k$ *is of a higher order than* $x^{m'} e_{k'}$ *if one of the three options of* (2.9.18) *holds.*

Formula (2.9.17) then allows for determining the coefficients c_{mk} sequentially, for monomials of higher and higher orders. End of the proof.

It is immediately seen from (2.9.15) that in the case of resonance $\lambda_k = (m, \lambda)$ it is impossible to eliminate monomials $d_{mk} x^m e_k$. Thus, *a sufficiently smooth linearization is impossible in the resonant case.* In the same way as Lemma 2.2, the following result may be proved.

Lemma 2.3. *Let* $f(x) \in \mathbb{C}^N$. *Then, there exists a polynomial change of variables which transforms system* (2.9.1) *into*

$$\dot{y} = Ay + R(y) + o_N(y) \tag{2.9.19}$$

with

$$R(y) = \sum_{(m,\lambda)=\lambda_k}^{2 \le |m| \le N} b_{mk} y^m e_k \,, \tag{2.9.20}$$

where e_k *is the* k*-th basis vector, and the coefficient* b_{mk} *of the resonant monomial* $y^m e_k$ *is found in terms of the coefficients of the polynomials* $f_l(x)$ *with* $l \le |m|$.

Let $f(x)$ now be a Taylor series. As N increases, the size of a neighborhood of the point O where such changes of variables are valid decreases. Furthermore, the neighborhood may shrink to the equilibrium state as $N \to \infty$.

Therefore, the two following theorems are only concerned with the formal series.

Theorem 2.11. (Poincaré) *If the eigenvalues of the matrix A are non-resonant, then a formal change*

$$y = x + \varphi_2(x) + \cdots + \varphi_l(x) + \dots \tag{2.9.21}$$

brings system (2.9.1) *to linear form* (2.9.2).

Theorem 2.12. (Dulac) *A formal change of variables brings the system* (2.9.1) *to*

$$\dot{y} = Ay + R(y)\,, \tag{2.9.22}$$

where $R(y)$ is a formal series

$$R(y) = \sum_{(m,\lambda)=\lambda_k}^{\infty} b_{mk} y^m e_k\,. \tag{2.9.23}$$

Let us discuss next the question of the convergence of these series. Following Arnold [10], we introduce a few useful preliminary notions.

Consider a complex n-dimensional space $\mathbb{C}^n$. The set

$$\lambda_k = (m, \lambda)\,, \qquad \sum_{j=1}^{n} m_j \geq 2\,, \qquad m_j \geq 0\,,$$

where m_j's are integers, is called *a resonant (hyper)plane*. By keeping $(\lambda_1, \dots, \lambda_n)$ fixed as k and m vary we may obtain a countable set of such planes.

Definition 2.4. *A collection $\lambda = \{\lambda_1, \dots, \lambda_n\}$ belongs to the Poincaré region if the convex hull of the n points $\lambda_1, \dots, \lambda_n$ in the complex plane does not contain zero. Otherwise, the collection $\lambda = \{\lambda_1, \dots, \lambda_n\}$ belongs to the Siegel region.*

Each point in the Poincaré region satisfies at most a finite number of resonances and lies in such a neighborhood which has no intersection with other resonant planes. In contrast, the resonant planes are dense in the Siegel region.

Theorem 2.13. (Poincaré) *If the eigenvalues of the matrix A are non-resonant and belong to the Poincaré region, then system* (2.9.1) *with an analytical right-hand side can be reduced to linear form by an analytical change of variables.*

Poincaré proved this theorem using majorant series. The principal requirement in his formulation is the condition of the existence of a straight-line passing through the origin in the complex plane such that all n eigenvalues $(\lambda_1, \ldots, \lambda_n)$ lie to one side of this line.

In the case where there are resonances in the Poincaré region, the formal series which defines the change of variables in Theorem 2.12 is convergent. Therefore, the following result holds.

Theorem 2.14. (Dulac) *If the eigenvalues of the matrix A belong to the Poincaré region, then an analytical change of coordinates transforms system* (2.9.1) *with the analytical right-hand side into the form*

$$\dot{y} = Ay + R(y)\,,$$

where $R(y)$ is a finite-order polynomial composed by resonant monomials.

The situation outside the Poincaré region is much more complicated. Specific conditions on eigenvalues under which the linearizing series converges and system (2.9.1) can still be reduced to linear form were found by Siegel. It is important to note that eigenvalues satisfying these conditions compose a dense set of positive measure.

Poincaré and Dulac considered complex analytical systems. We are interested in the real case. In the real case, the Poincaré region is determined by the conditions $\operatorname{Re}\lambda_i < 0$ or $\operatorname{Re}\lambda_i > 0$ $(i = 1, \ldots, n)$, *i.e.* where the equilibrium state is stable or completely unstable, respectively. If there are eigenvalues on the imaginary axis, or both in the left and right half-planes, then the system falls in the Siegel region. For example, suppose a two-dimensional system has a saddle equilibrium state with the eigenvalues $\lambda_1 < 0 < \lambda_2$. If the saddle index $\nu = -\lambda_1\lambda_2^{-1}$ is rational ($\nu = \frac{p}{q}$), there is an infinite set of resonances of the type

$$\begin{aligned} \lambda_1 &= (rq+1)\lambda_1 + pr\lambda_2\,, \\ \lambda_2 &= qr\lambda_1 + (pr+1)\lambda_2\,, \qquad r = 1, 2, \ldots\,. \end{aligned}$$

If the equilibrium state is of the saddle type, then even when the collection $\{\lambda_1, \ldots, \lambda_n\}$ is not resonant, zero is a limit point of the set

$$\{(m, \lambda) - \lambda_k\}_{|m|=2}^{\infty}\,, \qquad k = 1, \ldots, n\,.$$

In such situations, when determining the coefficients of the coordinate transformation by formula (2.9.17) we run into the problem of "small denominators".

This is a reason why the linearizing series may not converge even provided that there are no resonances (see Bruno [18]).

The situation becomes less difficult if we do not require that the system be analytic, but $\mathbb{C}^\infty$-smooth. In this case the following statement holds

Lemma 2.4. (Borel (see Hartman)) *For any formal power series $R(y)$ there exists a $\mathbb{C}^\infty$-smooth function whose formal Taylor series coincides with $R(y)$.*

Therefore, the following theorem is intuitively understandable.

Theorem 2.15. (Sternberg) *Let the system* (2.9.1) *be $\mathbb{C}^\infty$-smooth and let it have no resonances, then there exists a $\mathbb{C}^\infty$-smooth change of variables which brings* (2.9.1) *to linear form.*

There exists also an analogue to Dulac's Theorem 2.12

Theorem 2.16. *In the $\mathbb{C}^\infty$-smooth case there exists a $\mathbb{C}^\infty$-smooth change of variables which transforms the system* (2.9.1) *into*

$$\dot{y} = Ay + R(y)\,,$$

where $R(y) \in \mathbb{C}^\infty$ and its formal Taylor series coincides with the formal series (2.9.23)*.*

Thus, we can see that an individual $\mathbb{C}^\infty$-smooth system, in the neighborhood of an equilibrium state, may be reduced either to the Poincaré form

$$\dot{y} = Ay$$

or to the Dulac form

$$\dot{y} = Ay + R(y)\,.$$

Both forms are called *normal forms*. It follows from Theorems 2.15 and 2.16 that the dependence of the normal forms on the collection $\lambda = \{\lambda_1, \dots, \lambda_n\}$ has a discontinuous character in the Siegel region. The latter induces the question: Can a system of differential equations be reduced to a linear form by a *finitely*-smooth change of variables in a neighborhood of the structurally stable (saddle) equilibrium state? This question was posed by Sternberg [64,65] who proved the existence of a number $K(k)$, depending on the spectrum of the matrix A, such that any $\mathbb{C}^\infty$-system

$$\dot{x} = Ax + o_K(x)$$

may be reduced to linear form by a $\mathbb{C}^k$-smooth transformation. This is also true for the $\mathbb{C}^N$-smooth case provided that $N \geq K$. Later, Chen [20] showed that any $\mathbb{C}^N$-smooth ($N \geq K(k; A)$) system near a structurally stable equilibrium is $\mathbb{C}^k$-equivalent to

$$\dot{y} = Ay + \sum_{l=2}^{K(k;A)} f_l(y)\,, \tag{2.9.24}$$

i.e. to a polynomial vector field. Moreover, we can assume that $f(x)$ is a resonant polynomial here. As far as the question of the exact estimate for the number K is concerned it is still completely unsolved.

The above resonant-polynomial normal form (2.9.24) can be further simplified by $\mathbb{C}^k$-smooth changes of variables. We will not discuss these topics further and refer the reader to the book by Bronstein and Kopanskii [16], which contains the latest achievements and relevant references in this field. Observe that in this theory a special attention must be dedicated to the so-called "*weak*" resonances. Denote by $(\theta_1, \ldots, \theta_p)$ distinct values of $\operatorname{Re}\lambda_i$ ($i = 1, \ldots, n$). It is obvious that $p \leq n$. The values $(\theta_1, \ldots, \theta_p)$ are called *the Lyapunov exponents.* We call the relation

$$\theta_s = l_1\theta_1 + \cdots + l_p\theta_p = (l, \theta)\,, \quad \text{where} \quad \sum_{i=1}^{p} l_p \geq 2$$

a weak resonance. Observe that the notion of the weak resonance does not employ analysis in the complex plane and, consequently, the reduction of the linear part to the Jordan form. In contrast to the classical notion of resonances being an obstacle in the linearization by polynomial transformations, the notion of weak resonances arises in the problem of the reduction of systems of a finite smoothness to linear form when we use wider classes of changes of variables.

Of primary interest from the viewpoint of nonlinear dynamics are saddles. The reason is because a saddle may have bi-asymptotical trajectories which belong to both the stable and the unstable manifold. Such trajectories are called *homoclinic loops.* In the case where an equilibrium state is a saddle-focus, infinitely many periodic trajectories may arise from one homoclinic loop under certain conditions. The study of such phenomena begins with the reduction of the system near a saddle to a simpler form. It is obvious that the case where the system is reducible to linear form would be ideal. However, the study of global bifurcations requires the consideration of finite-parameter families of systems rather then an individual system. Reduction to linear form is difficult for the family of systems because resonances are dense in the Siegel regions. At the same time, most resonances, with a few exceptions, do not play an essential role in the study of homoclinic bifurcations.

Let us discuss in detail an example of a planar system which was studied by Andronov and Leontovich. Consider a one-parameter family of systems

$$\begin{aligned}\dot{x} &= \lambda_1(\mu)x + P(x,y,\mu)\,,\\ \dot{y} &= \lambda_2(\mu)y + Q(x,y,\mu)\,,\end{aligned}$$

where P and Q are $\mathbb{C}^N$ ($N \geq 1$)-smooth functions vanishing at the origin along with their first derivatives with respect to x and y, and $\lambda_1(0) < 0 < \lambda_2(0)$. Suppose that when $\mu = 0$ this system has a separatrix loop, see Fig. 2.9.1. Assume also that the saddle index $\nu = -\lambda_1(\mu)/\lambda_2(\mu)$ differs from 1 when $\mu = 0$, *i.e.* the so-called saddle value σ is non-zero:

$$\sigma(0) = \lambda_1(0) + \lambda_2(0) \neq 0\,.$$

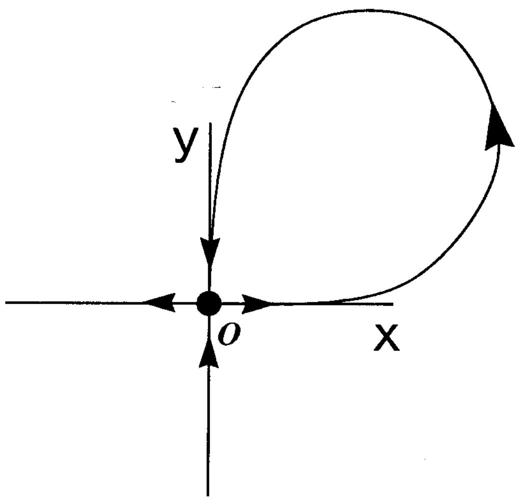


Fig. 2.9.1. A homoclinic loop to a saddle on the plane.

Andronov and Leontovich showed that under these conditions at most one periodic trajectory can arise from the separatrix loop. The condition

$$\sigma(0) = \lambda_1(0) + \lambda_2(0) = 0$$

leads to the appearance of an infinite set of resonances

$$\lambda_1(0) = (p+1)\lambda_1(0) + p\lambda_2(0), \lambda_2(0) = q\lambda_1(0) + (q+1)\lambda_2(0),$$

where p and q are positive integers.

The problem of a number of periodic trajectories arising from the homoclinic loop in the case $\sigma(0) = 0$ in a finite-parameter family was studied by Leontovich [40]. The principal difficulty here is in constructing the Dulac normal form for a finite-parameter family. At $\mu = 0$ this is given by

$$\dot{x} = \lambda_1(0)x + \sum_{p=1}^{\infty} a_p x^{p+1} y^p \,,$$

$$\dot{y} = \lambda_2(0)y + \sum_{p=1}^{\infty} b_p x^p y^{p+1} \,.$$

or, having excluded the time t, by

$$\frac{dy}{dx} = \frac{y}{x}\left(1 + \sum_{p=1}^{\infty} c_p (xy)^p\right) \,.$$

When $\mu \neq 0$ it does not make sense to apply the theory of normal forms because the dependence on the parameter is discontinuous on a dense set since the saddle index $\nu(\mu)$ may be rational or irrational. Nevertheless, Leontovich [40] was able to transform the family into

$$\frac{dy}{dx} = \frac{y}{x}\left[1 + \sum_{p=1}^{K-1} c_p(\mu)(xy)^p + (xy)^K \Phi(x, y, \mu)\right] \,. \tag{2.9.25}$$

The main method which she employed was that of sequentially eliminating the *non-resonant functions*, *i.e.* such whose all terms in the formal Taylor expansion are non-resonant. The procedure works for a $\mathbb{C}^N$-smooth family provided that $N \geq 4K + 1$. The function $\Phi(x, y, \mu)$ in Eq. (2.9.25) is then of smoothness $\mathbb{C}^K$.

The concept of eliminating non-resonant functions in the right-hand side has been effectively developed for multi-dimensional systems with homoclinic loops to saddle and saddle-foci in the papers by Ovsyannikov and Shilnikov [48,49].[3]

Consider a family $X(\mu)$ of dynamical systems which depends on parameters μ. We assume that $X(\mu)$ is of $\mathbb{C}^r$-class with respect to all variables and parameters. We may present $X(\mu)$ in the form

$$\begin{aligned} \dot{x} &= A_1(\mu)x + f_1(x, y, u, v, \mu)\,, \\ \dot{u} &= A_2(\mu)u + f_2(x, y, u, v, \mu)\,, \\ \dot{y} &= B_1(\mu)y + g_1(x, y, u, v, \mu)\,, \\ \dot{v} &= B_2(\mu)v + g_2(x, y, u, v, \mu)\,, \end{aligned} \tag{2.9.26}$$

where the eigenvalues of the matrix $A(0)$

$$A(0) \equiv \begin{pmatrix} A_1(0) & 0 \\ 0 & A_2(0) \end{pmatrix}$$

lie to the left of the imaginary axis in the complex plane and those of the matrix $B(0)$

$$B(0) \equiv \begin{pmatrix} B_1(0) & 0 \\ 0 & B_2(0) \end{pmatrix}$$

lie to the right of the imaginary axis.

We assume also that the real parts of the eigenvalues $(\lambda_1, \ldots, \lambda_{m_1})$ of the matrix $A_1(0)$ are equal, *i.e.*

$$\operatorname{Re}\lambda_1 = \cdots = \operatorname{Re}\lambda_{m_1} = \lambda < 0\,,$$

and that the real parts of the eigenvalues $(\gamma_1, \ldots, \gamma_{n_1})$ of the matrix $B_1(0)$ are also equal *i.e.*

$$\operatorname{Re}\gamma_1 = \cdots = \operatorname{Re}\gamma_{n_1} = \gamma > 0\,.$$

Regarding the eigenvalues of the matrices $A_2(0)$ and $B_2(0)$ we assume that the real parts of the eigenvalues of $A_2(0)$ are strictly smaller than λ, and those of $B_2(0)$ are strictly larger than γ. In this case the coordinates x and y are leading stable and unstable, respectively, and the coordinates u and v are non-leading.

[3] The same approach was applied near a saddle periodic orbit by Gonchenko and Shilnikov [27]; for other applications see the papers by Afraimovich and by Lerman and Umanskii in *Methods of the Qualitative Theory of Differential Equations*, edited by Leontovich, Gorky State University, Gorky, 1984.

Theorem 2.17. (Ovsyannikov-Shilnikov) *For all small μ system* (2.9.26) *is transformed locally to*

$$\begin{aligned} \dot{x} &= A_1(\mu)x + f_{11}(x,y,v,\mu)x + f_{12}(x,u,y,v,\mu)u\,, \\ \dot{u} &= A_2(\mu)u + f_{21}(x,y,v,\mu)x + f_{22}(x,u,y,v,\mu)u\,, \\ \dot{y} &= B_1(\mu)y + g_{11}(x,u,y,\mu)y + g_{12}(x,u,y,v,\mu)v\,, \\ \dot{v} &= B_2(\mu)v + g_{21}(x,u,y,\mu)y + f_{22}(x,u,y,v,\mu)v\,, \end{aligned} \tag{2.9.27}$$

where

$$\begin{aligned} f_{ij}|_{(x,u,y,v)=0} &= 0 \qquad g_{ij}|_{(x,u,y,v)=0} = 0 \\ f_{1j}|_{(y,v)=0} &= 0 \qquad g_{1j}|_{(x,u)=0} = 0 \\ f_{i1}|_{x=0} &= 0 \qquad g_{i1}|_{y=0} = 0 \qquad (i,j=1,2)\,. \end{aligned} \tag{2.9.28}$$

Since we will frequently use this theorem while studying homoclinic bifurcations, its complete proof is given in Appendix A. The smoothness of the coordinate transformation and the functions f_{ij}, g_{ij} is defined as follows: It is $\mathbb{C}^{r-1}$ with respect to (x,u,y,v) and the first derivatives with respect to (x,u,y,v) are $\mathbb{C}^{r-2}$ with respect to (x,u,y,v,μ). If $r=\infty$, then the transformation is $\mathbb{C}^\infty$ with respect to (x,y,u,v) (or even analytical in the real analytical case). Nevertheless, even when $r=\infty$ the smoothness of the transformation with respect to the parameters μ is, generically, only finite: It grows unboundedly as $\mu_0 \to 0$ (where $\|\mu\| \leq \mu_0$ is the range of parameter values under consideration).

Let us consider next the case where some eigenvalues of the matrix A in (2.9.1) lie on the imaginary axis. It is obvious that if there is one zero eigenvalue $\lambda_1 = 0$, then there exists an infinite set of resonances of the type:

$$\lambda_1 = m\lambda_1\,, \qquad m \geq 2\,. \tag{2.9.29}$$

In the case of a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega$, $\omega \neq 0$, there is also an infinite set of resonances of the type:

$$\begin{aligned} \lambda_1 &= (s_1+1)\lambda_1 + s_1\lambda_2\,, \\ \lambda_2 &= s_2\lambda_1 + (s_2+1)\lambda_2\,, \end{aligned} \tag{2.9.30}$$

where s_1 and s_2 are positive integers.

Here, the theory of normal forms is especially valuable since these resonances determine conditions of the stability and as a result the types of local bifurcations of equilibrium states in the critical cases.

Since the equilibria under consideration are now structurally unstable, it is very natural not to restrict the consideration by studying a concrete system but by including the latter in a finite-parameter q-dimensional family

$$\dot{x} = Ax + f(x) + h(x,\, \mu)\,, \tag{2.9.31}$$

where $\mu = (\mu_1, \ldots, \mu_q)$; the functions $f(x)$ and $h(x, \mu)$ are sufficiently smooth and

$$f(0) = f'(0) = 0\,, \quad h(x, 0) \equiv 0\,.$$

Assume that the eigenvalues $(\lambda_1, \ldots, \lambda_p)$ of the matrix A lie on the imaginary axis in the complex plane. The assumption is not burdensome since in this case the general family may be reduced to system (2.9.31) by virtue of the center manifold theorem (see Chap. 5). Let us now consider the following $(p+q)$-dimensional system in the triangular form

$$\begin{aligned} \dot{x} &= Ax + f(x) + h(x, \mu)\,, \\ \dot{\mu} &= 0\,. \end{aligned} \tag{2.9.32}$$

This system has a fixed point $O(0, 0)$ with Jacobian given by

$$\tilde{A} = \begin{pmatrix} A & h'_\mu(0,0) \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of the matrix $\tilde{A}$ are $\lambda_1, \ldots \lambda_p$ and $\gamma_1 = \cdots = \gamma_q = 0$. System (2.9.32) has resonances of the type (2.9.29), (2.9.30) as well as the following resonances:

$$\lambda_k = \lambda_k + (l, \gamma)\,, \tag{2.9.33}$$

$$\lambda_k = (m, \lambda) + (l, \gamma)\,, \tag{2.9.34}$$

$$\gamma_j = (l, \gamma)\,, \tag{2.9.35}$$

where $(l, \gamma) = \sum\limits_{j=1}^{q} l_j \gamma_j$ and $\sum\limits_{j=1}^{q} l_j \geq 2$. System (2.9.32) may be reduced to the normal form by the change of variables

$$\begin{aligned} y &= x + \varphi(x, \mu) \\ \mu &= \mu \end{aligned} \tag{2.9.36}$$

which leaves the second equation in (2.9.32) unchanged. Thus, we do not need to consider the resonances of the type (2.9.35). In an analogy with Lemma 2.3 system (2.9.32) may be transformed into

$$\dot{y} = Ay + R_0(\mu) + R_1(\mu)y + R_N(y, \mu) + o_N(y, \mu)\,, \tag{2.9.37}$$

where $R_1(\mu)$ is a polynomial of degree not higher than $(N-1)$, $R_1(0) = 0$ and

$$R_N(y,\mu) = \sum_{(m,\lambda)=\lambda_k}^{|m|\leq N} b_{mk}(\mu) y^m e_k\,, \tag{2.9.38}$$

where $b_{mk}(\mu)$ are certain polynomials of degree not exceeding $(N - |m|)$.

If the matrix A is non-degenerate, then $R_0(\mu) \equiv 0$. Otherwise, when among the eigenvalues $(\lambda_1, \ldots, \lambda_p)$ there is $\lambda_k = 0$, $R_0(\mu)$ is a polynomial of degree N such that $R_0(0) = 0$. The appearance of the term $R_0(\mu)$ in (2.9.37) is due to the existence of resonances of the kind

$$0 = \lambda_k = (l, \gamma)\,.$$

The family

$$\dot{y} = Ay + R_0(\mu) + R_1(\mu)y + \sum_{(m,\lambda)=\lambda_k}^{|m|\leq N} b_{mk}(\mu) y^m e_k \tag{2.9.39}$$

is called a *truncated* or *shortened* normal form. In many cases one may try to restrict the consideration of the behavior of trajectories in a small fixed neighborhood of an equilibrium state, as well as the study of the bifurcation unfolding for small values of the control parameters by the investigation of the truncated normal form for a suitable choice of N and q. Of course, the information obtained from the analysis of the truncated system must be justified before it is applied to the original family. Following this scheme we will carry out the study of principal local bifurcations of equilibrium states in the second part of this book.

Chapter 3

STRUCTURALLY STABLE PERIODIC TRAJECTORIES OF DYNAMICAL SYSTEMS

Let us consider an autonomous system of differential equations in $\mathbb{R}^{n+1}$, $(n \geq 1)$

$$\dot{x} = X(x)\,,$$

where $x = (x_1, \ldots, x_n, x_{n+1})$, $X \in \mathbb{C}^r$ $(r \geq 1)$. The subject of this chapter is *periodic trajectories*, *i.e.* non-stationary, periodic solutions of the form $x = \varphi(t)$, where

$$\varphi(t) \equiv \varphi(t + \tau)$$

for some $\tau \neq 0$. Observe that $\varphi(t) \in \mathbb{C}^{r+1}$. The periodic motion is associated in the phase space with a smooth closed curve, called *a limit cycle*, *a periodic trajectory*, or *a periodic orbit*. By definition, any point on a periodic trajectory returns to the initial position over the interval of time equal exactly to τ. The same occurs at 2τ, 3τ and so on. The smallest of such return times is called *the period*.

The periodic motion is one of the most important objects of nonlinear dynamics. There are at least two reasons for this. Firstly, a *stable* periodic trajectory is the mathematical image of such physical phenomena as self-oscillations. Secondly, *saddle* periodic trajectories are the key components of strange attractors which dynamical chaos is associated with.

In contrast to equilibrium states, searching for periodic motions in phase space is presently an art, particularly, in the phase space of high-dimensional systems. So, for example, the number of equilibrium states in a system with a polynomial right-hand side can be estimated exactly, but the estimation of the number of periodic trajectories of even just a planar system is the subject of the famous Hilbert's 16-th problem which is still unsolved. One exception is, perhaps, nearly integrable two-dimensional systems for which the problem of finding the periodic trajectories is reduced to that of finding the zeros of some special integrals which can be explicitly calculated in certain specific cases.

In this chapter we will examine the behavior of trajectories near a structurally stable periodic trajectory. The main idea of the study is based on constructing the Poincaré return map.

3.1. A Poincaré map. A fixed point. Multipliers

Assume that a system of differential equations in $\mathbb{R}^{n+1}$ $(n \geq 1)$

$$\dot{x} = X(x)\,, \tag{3.1.1}$$

where $x = (x_1, \ldots, x_n, x_{n+1})$, $X \in \mathbb{C}^r$ $(r \geq 1)$ possesses a periodic trajectory L.

Let us choose a point M^* on L and translate the origin to M^*; see Fig. 3.1.1. Without loss of generality we can assume that the last component of the velocity vector at the point M^* is non-zero, *i.e.*

$$X_{n+1}(0) \neq 0\,. \tag{3.1.2}$$

This can be always achieved by a reordering of the coordinates, because nowhere on the periodic trajectory is the velocity vector equal to zero. Condition (3.1.2) allows us to choose a small *cross-section* S on the plane $x_{n+1} = 0$ so that $M^* \in S$. By construction, all trajectories of the system (3.1.1) near the periodic trajectory L flow through the cross-section S transversely.

It follows from the theorem on continuous dependence on initial conditions that a trajectory starting from a point $M \in S$ sufficiently close to M^*, returns to S at some point $\bar{M}$ over a time interval $t(M)$ close to the period of the periodic trajectory L. Thus a map $T\colon\ M \mapsto \bar{M}$, called *the Poincaré map*, can be defined along such trajectories.

Let $x = \varphi(t, x_0)$ be the trajectory which passes through the point $M(x_0) \in S$ at $t = 0$. The return time $t(x_0)$ from M to $\bar{M}$ can be found from the equation

$$\varphi_{n+1}(t, x_0) = 0\,. \tag{3.1.3}$$

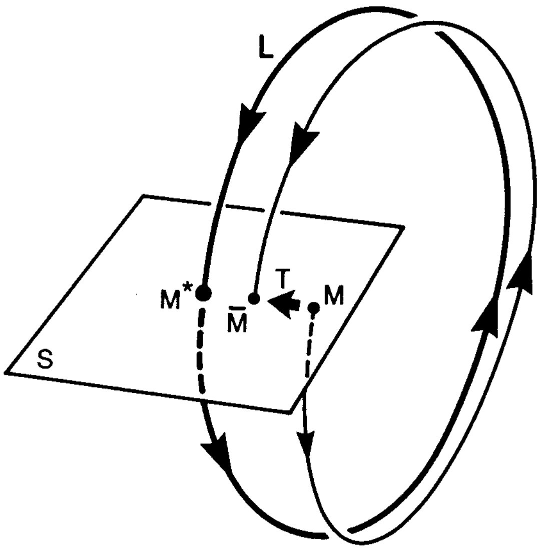


Fig. 3.1.1. A cross-section S is chosen to be transverse to a periodic trajectory L as well as to the trajectories close to L.

Since the periodic trajectory L passes through the origin, this equation has a solution $t = \tau$ for $x_0 = 0$, where τ is the period of the periodic trajectory. By virtue of (3.1.2) we may apply the implicit function theorem to Eq. (3.1.3), whence the return time $t(x_0)$ is uniquely defined. Moreover, the function $t(x_0)$ has the same smoothness as the original system.

The map T may be written in the following form

$$\bar{x}_k = \varphi_k(t(x),\, x)$$

or as

$$\bar{x} = f(x)\,, \tag{3.1.4}$$

where x is an n-dimensional vector of the coordinates on the cross-section S, $f(x) \in \mathbb{C}^r$.

If, at $t = 0$, we let a trajectory flow out from the point $\bar{M}$ on S in backward time, then it must return to S at the point M over the time interval $t(M)$. Thus, the map T^{-1}, the inverse of the Poincaré map, is also defined on the cross-section S. Because the property of smooth dependence of the return time on the initial point persists in backward time, we can assert that the inverse map T^{-1} also belongs to the class $\mathbb{C}^r$. This implies that the Poincaré map is *a $\mathbb{C}^r$-smooth diffeomorphism.*

If we write down

$$x = f^{-1}(f(x)),$$

then after differentiating we obtain

$$[f^{-1}(f(x))]' \cdot f'(x) = I,$$

where I is the identity matrix. Thus,

$$\det \left| \frac{df(x)}{dx} \right| \neq 0$$

for small x.

A trajectory of system (3.1.1) which passes through an arbitrary point M on S intersects S consequently at the points $(\ldots, M_{-1}, M_0 \equiv M, M_1, \ldots,)$. By construction, the points $\{M_j\}$ are the images of the point M under the action of the Poincaré map: $M_j \equiv T^j M$. The sequence $\{M_j\}$ is called *a trajectory of the point M with respect to the map T*. It is obvious that the behavior of the trajectories of the original system (3.1.1) close to the periodic trajectory L is completely determined by the behavior of the trajectories of the map T close to *a fixed point* $M^* = L \cap S$ (the point M^* is called the fixed point because $TM^* = M^*$). So, for example, a trajectory of system (3.1.1) tends towards the periodic trajectory L as $t \to +\infty$, if and only if the corresponding trajectory of the map T converges to the fixed point M^* as $j \to +\infty$.

In a neighborhood of the fixed point at the origin the map T may be written in the form

$$\bar{x} = Ax + g(x), \tag{3.1.5}$$

where

$$A \equiv \frac{df}{dx}\Big|_{x=0}$$

is a non-singular $(n \times n)$-matrix, $g(0) = g'(0) = 0$.

Before we examine the nonlinear map (3.1.5), it is useful to examine the behavior of the trajectories of *the linearized map*

$$\bar{x} = Ax. \tag{3.1.6}$$

We will see below that just like the characteristic exponents of an equilibrium state, the key role in determining the dynamics of a Poincaré map near a fixed point belongs to the eigenvalues of the matrix A. These eigenvalues are called *the multipliers of the fixed point or the multipliers of the associated periodic trajectory.*

It is not hard to see that the multipliers are not changed by a smooth transformation of variables. Indeed, if we make the following change of coordinates

$$x = By + \psi(y)\,,$$

where $\det B \neq 0$, $\psi(0) = 0$ and $\psi'(0) = 0$, so that it does not move the origin, then in the new coordinates the map (3.1.5) becomes

$$B\bar{y} + \psi(\bar{y}) = ABy + A\psi(y) + g(By + \psi(y))$$

or

$$\bar{y} = B^{-1}ABy + \cdots\,,$$

where the ellipsis denotes nonlinear terms. Because the matrix $B^{-1}AB$ is similar to the matrix A, they have the same eigenvalues.

It is also obvious that the multipliers of the Poincaré map depend neither on the choice of the point M^* on L nor on the particular choice of S with respect to the periodic trajectory. Since the flight time from one transverse cross-section to another depends smoothly on the initial point, the map of one cross-section onto another cross-section along trajectories of the system is a $\mathbb{C}^r$-diffeomorphism, and, therefore, the change of the cross-section may be simply considered as a change of coordinates.

In the following sections we will examine the behavior of the trajectories of dynamical systems near *structurally stable* (*rough*) *periodic trajectories*, *i.e.* those which have no multipliers equal to 1 in absolute value. We shall begin with the study of structurally stable fixed points of the Poincaré map. We remark here that the theory of fixed points amounts to a partial, though not absolute, repetition of the theory of equilibrium states. We therefore pursue our study by following the same scheme as in Chap. 2: the linear case followed by the nonlinear case and, by the correspondence between nonlinear and linear maps.

3.2. Non-degenerate linear one- and two-dimensional maps

In the present and the consequent sections we will study linear maps. We are interested in the linearization of the Poincaré map near a periodic trajectory, in other words in a linear map with non-singular Jacobian matrix

$$\bar{x} = Ax\,, \quad \det A \neq 0\,.$$

Let us start with a linear map of dimension one. It is written in the form

$$\bar{x} = \rho x\,, \tag{3.2.1}$$

where $\rho \neq 0$, $x \in \mathbb{R}^1$.

It is easy to see that the fixed point at the origin O is stable when $|\rho| < 1$. The iterations x_j of a point x_0 are given by the formula

$$x_j = \rho^j x_0\,,$$

where

$$\lim_{j \to +\infty} x_j = 0$$

if $|\rho| < 1$.

On the other hand, the fixed point is unstable when $|\rho| > 1$.

The behavior of the iterations of points in the one-dimensional case is conveniently interpreted by means of *a Lamerey diagram* which is constructed as follows. For the map

$$\bar{x} = f(x)$$

the graph of the function $f(x)$ and the bisectrix[1] $\bar{x} = x$ are drawn in the plane $(x,\, \bar{x})$. Trajectories are represented as polygonal lines: let $\{x_j\}$ be a trajectory; each point with coordinates $(x_j,\, x_{j+1})$ lies on the graph $f(x)$ while each point $(x_j,\, x_j)$ lies on the bisectrix $\bar{x} = x$. Each point $(x_j,\, x_j)$ is connected vertically with the subsequent point $(x_j,\, x_{j+1})$, which in turn is connected horizontally with the subsequent point $(x_{j+1},\, x_{j+1})$ and so on. This process is iterated repeatedly, as shown in the four typical Lamerey diagrams in Figs. 3.2.1 to 3.2.4.

When the function $f(x)$ increases monotonically, then the construction obtained is called *a Lamerey stair* (Figs. 3.2.1 and 3.2.2). When $f(x)$ decreases monotonically the construction is called *a Lamerey spiral*, see Figs. 3.2.3 and 3.2.4.

[1] The 45° line.

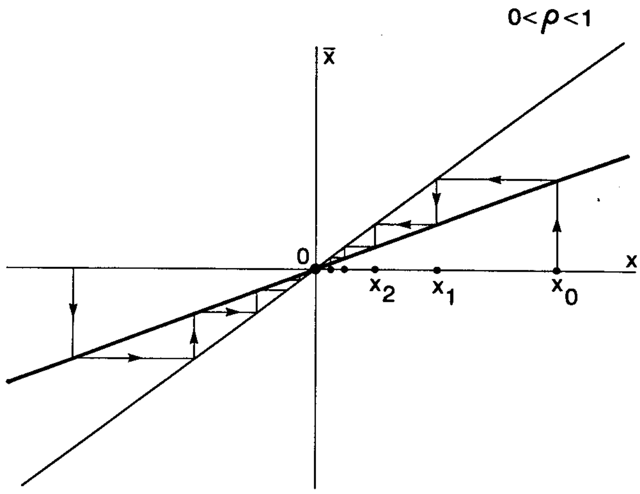


Fig. 3.2.1. A Lamerey stair. The origin is a stable fixed point: all points in its neighborhood converge to O.

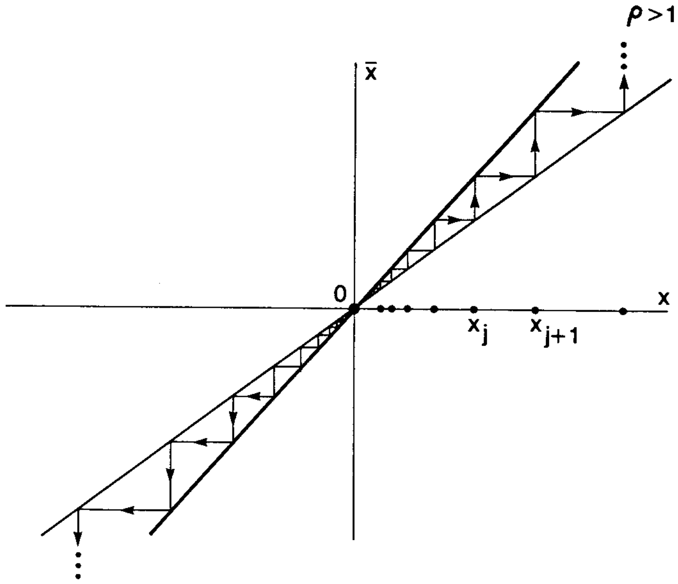


Fig. 3.2.2. A Lamerey stair where the origin is an unstable fixed point.

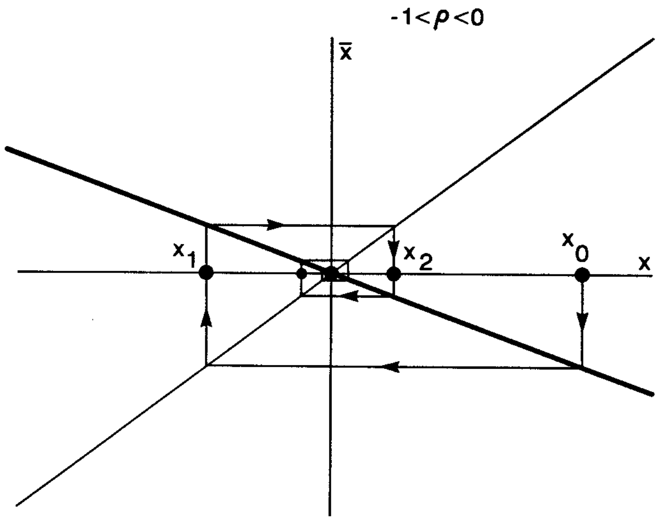


Fig. 3.2.3. An example of a Lamerey spiral; a trajectory starting from x_0 looks like a *clockwise-right-angled spiral.*

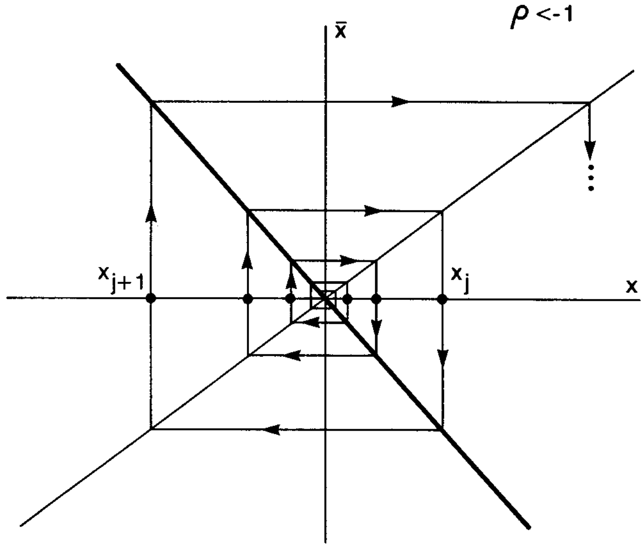


Fig. 3.2.4. An example of an "unstable" Lamerey spiral. A trajectory $\{x_i\}$ diverges from the fixed point at the origin.

In the degenerate case where $\rho = 1$, all points are fixed. When $\rho = -1$ all points, except for the origin O, are *periodic with period equal to* 2, *i.e.* the fixed points of the map T^2 defined by:

$$\bar{\bar{x}} = x \qquad (\bar{\bar{x}} = -(\bar{x}) \Leftarrow \bar{x} = -x)\,.$$

Let us now consider a two-dimensional Poincaré map

$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{3.2.2}$$

When both eigenvalues of the matrix A are real and different, a non-degenerate linear change of coordinates brings the Poincaré map to the form

$$\bar{x} = \rho_1 x\,, \quad \bar{y} = \rho_2 y\,. \tag{3.2.3}$$

The iterations of an initial point (x_0, y_0) are given by

$$x_j = \rho_1^j\, x_0\,, \quad y_j = \rho_2^j\, y_0\,. \tag{3.2.4}$$

There are four cases to be considered:

(1) $|\rho_i| < 1$ $(i = 1, 2)$. In this case the fixed point at the origin O is exponentially stable and is called *a stable node*. Let us assume $|\rho_1| > |\rho_2|$, then all trajectories, except for those which lie on *the non-leading* (*strongly stable*) axis y, tend to O tangentially to *the leading* axis x as $j \to +\infty$. The behavior of the trajectories in a neighborhood of a stable node is shown in Figs. 3.2.5 and 3.2.6.

(2) $|\rho_i| > 1$ $(i = 1, 2)$. This case is reduced to the previous one if we consider the inverse map T^{-1}. The fixed point is called *an unstable node*.

(3) $|\rho_1| < 1$ and $|\rho_2| > 1$. A fixed point with multipliers of this type is called *a saddle*. It is seen from (3.2.4) that both x and y axes are invariant with respect to the map (3.2.3). Points on the x-axis tend to O as $j \to +\infty$,

whereas points on the y-axis tend to O as $j \to -\infty$. For this reason, the x-axis and the y-axis are called *the stable subspace* and *the unstable subspace* respectively of the saddle O. All other trajectories pass close but away from the saddle. Their loci depend upon the signs of the multipliers; four possible variants are presented in Figs. 3.2.7a–3.2.7d.

(4) Complex-conjugate multipliers $\rho_{1,2} = \rho e^{\pm i\omega}$. In this case the Poincaré map may be written in the form

$$\begin{aligned} \bar{x} &= \rho(x \cos\omega \;-\; y \sin\omega)\,, \\ \bar{y} &= \rho(y \cos\omega \;+\; x \sin\omega)\,, \end{aligned}$$

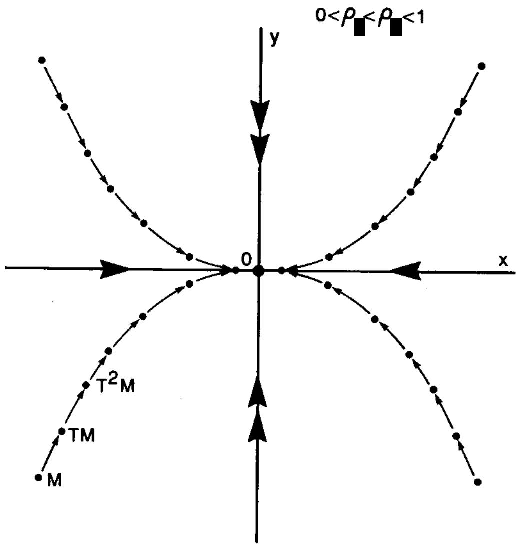


Fig. 3.2.5. A stable node (+) with positive multipliers $0 < \rho_2 < \rho_1 < 1$. A trajectory $\{T^i M\}$ of the point M enters the origin tangentially to the leading direction x.

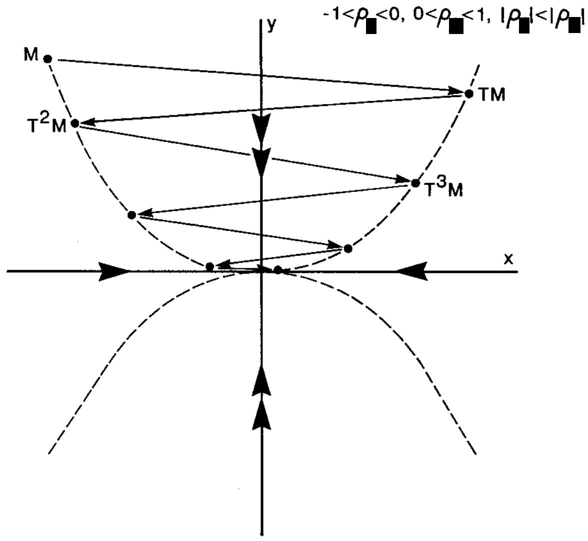
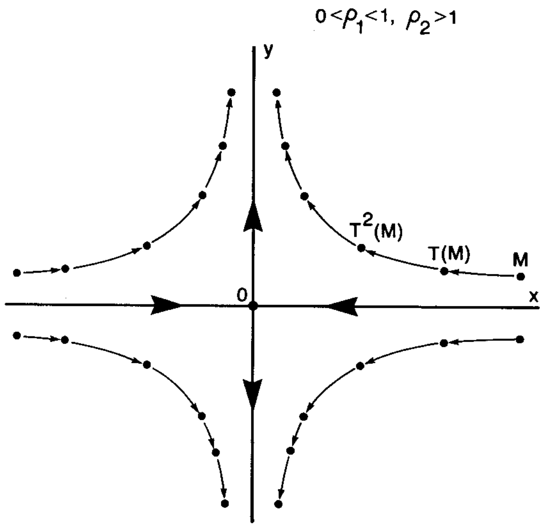
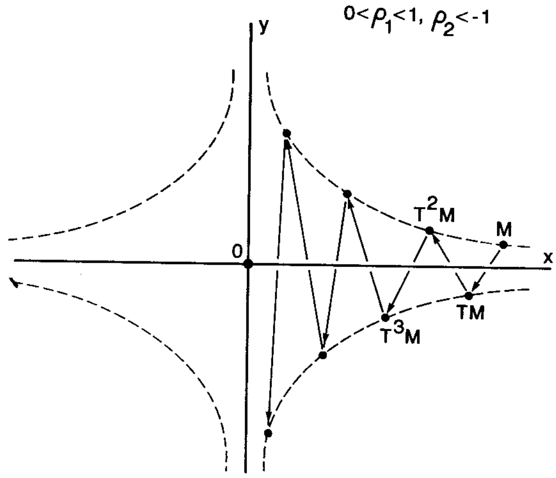


Fig. 3.2.6. A stable node "−" with the negative leading multiplier ρ_1; therefore, the x-coordinate changes its sign after each iteration.



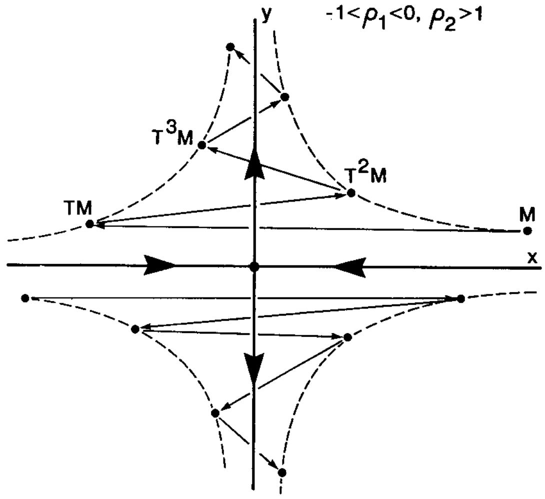
(a)

Fig. 3.2.7(a). A saddle (+,+). The y-axis coincides with the unstable direction, the x-axis with the stable one.



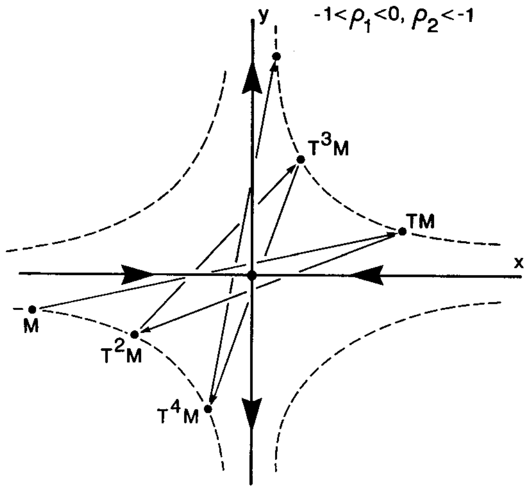
(b)

Fig. 3.2.7(b). A saddle $(+,-)$. The sign of the y-coordinate of the trajectory $\{T^i M\}$ changes after each iteration.



(c)

Fig. 3.2.7(c). A saddle $(-,+)$. The "jumping" direction is here the x-axis because the corresponding multiplier ρ_1 is negative.



(d)

Fig. 3.2.7(d). A saddle $(-,-)$. The trajectory of the initial point M runs away from the origin along the hyperbolas located in 1st and 3rd quadrants.

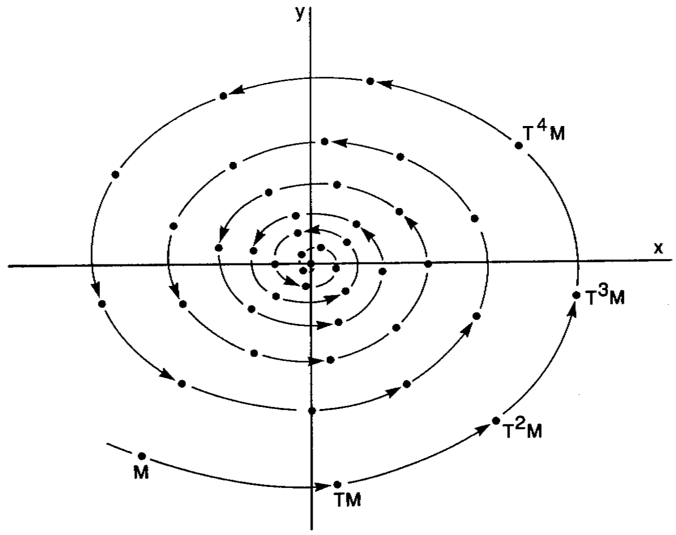


Fig. 3.2.8. A stable focus.

or in polar coordinates (r, φ), in the form

$$\begin{aligned} \bar{r} &= \rho\, r\,, \\ \bar{\varphi} &= \varphi + \omega\,. \end{aligned} \tag{3.2.5}$$

The j-th iteration of the point (r_0, φ_0) is given by

$$\begin{aligned} r_j &= \rho^j\, r_0\,, \\ \varphi_j &= \varphi_0 + \omega\, j\,. \end{aligned}$$

When $\rho < 1$, the point O is called *a stable focus*. In this case, all trajectories lie on logarithmic spirals winding into the origin as shown in Fig. 3.2.8.

When $\rho > 1$, the fixed point O is called *an unstable focus*. In this case, all trajectories diverge from any neighborhood of the point O as $j \to +\infty$.

In the degenerate case where $\rho = 1$, we note from (3.2.5) that $\bar{r} = r$, *i.e.* any circle with center at the origin O is invariant with respect to the map. In its restriction to an invariant circle, the map has the form

$$\bar{\varphi} = \varphi + \omega \qquad (\mathrm{mod}\ 2\pi)\,.$$

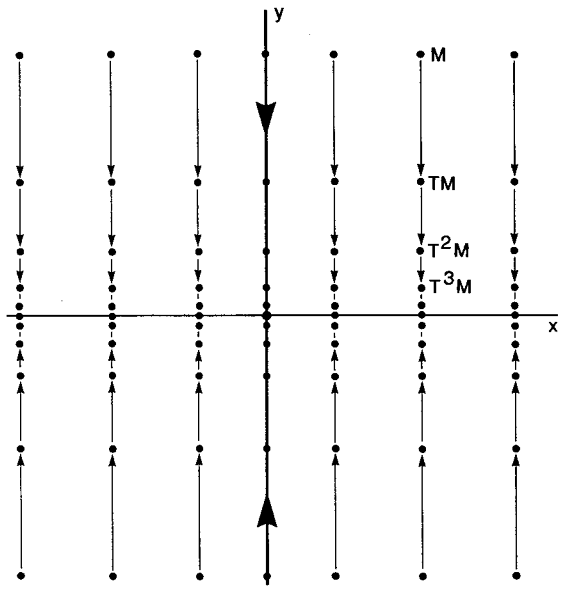


Fig. 3.2.9. An example of the behavior of the trajectories of a degenerate map. The entire x-axis consists of fixed points.

If ω is commensurable with 2π, *i.e.* $\omega = 2\pi M/N$ for some integers M and N, then all points are periodic with period N as

$$\varphi_N = \varphi + N\,\omega = \varphi + 2\pi M = \varphi \pmod{2\pi}\,.$$

Thus, all points are the fixed point of the N-th iterate of the map T. This implies that T^N is *the identity map.*

If ω is not commensurable with 2π, then the trajectory of any point φ_0 is non-periodic. Moreover, one can show that the set of the points

$$\{\varphi_j\}_{j=-\infty}^{j=+\infty}$$

is dense on any circle.

Figure 3.2.9 illustrates one more degenerate case: $\rho_1 = 1$, $|\rho_2| < 1$. Here, all points on the x-axis are fixed points. Any straight-line $x =$ *constant* is invariant with respect to the map. The trajectories on each straight-line tend to the corresponding fixed point.

3.3. Fixed points of high-dimensional linear maps

Let us consider an n-dimensional map

$$\bar{x} = Ax\,, \quad A \in \mathbb{R}^n\,. \tag{3.3.1}$$

As in our previous discussion of linear systems of differential equations, let us choose the coordinates such that the matrix A is represented in the real Jordan form:

$$A = A^0 + \Delta A\,, \tag{3.3.2}$$

where A^0 is the block-diagonal matrix

$$A^0 = \begin{pmatrix} A_1 & & & \mathbf{0} \\ & A_2 & & \\ & & \ddots & \\ \mathbf{0} & & & A_n \end{pmatrix}. \tag{3.3.3}$$

The block

$$A_i = (\rho)\,, \tag{3.3.4}$$

corresponds to each real eigenvalue (multiplier) ρ of the matrix A. The block

$$A_i = \rho \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \tag{3.3.5}$$

corresponds to each pair of complex-conjugate multipliers $\rho e^{\pm i\omega}$. The matrix ΔA is non-zero only when the matrix A has multiple eigenvalues. In this case a basis may be chosen in $\mathbb{R}^n$ such that

$$\|\Delta A\| \le \varepsilon \tag{3.3.6}$$

for an infinitesimally small constant $\varepsilon > 0$ (see Sec. 2.3).

It is evident that

$$\|A^0\| \le \rho'\,,$$

where ρ' is the maximum of the absolute values of all eigenvalues of the matrix A. Therefore

$$\|A\| \le \rho' + \varepsilon\,. \tag{3.3.7}$$

A trajectory of map (3.3.1) is given by the equation

$$x_j = A^j\, x_0\,. \tag{3.3.8}$$

When *all eigenvalues of the matrix A lie strictly inside the unit circle*, it follows from (3.3.7) that

$$\|x_j\| \leq \|A\|^j \|x_0\| \leq (\rho' + \varepsilon)^j \, \|x_0\| \quad \text{for} \quad j \geq 0\,, \tag{3.3.9}$$

i.e. all trajectories converge exponentially to the fixed point at the origin as $j \to +\infty$.

A transition to an arbitrary basis alters the estimate (3.3.9) so that some constant, generally speaking greater than 1 (see formulas (2.3.17) and (2.3.18) in Sec. 2.3), appears in the right-hand side.

As in the case of a stable equilibrium state, we may introduce the notions of leading and non-leading multipliers of a stable fixed point.

Let us arrange the multipliers in order of decreasing absolute value, and let the first m multipliers be of equal absolute value, *i.e.*

$$|\rho_1| = \cdots = |\rho_m| = \rho'\,, \quad |\rho_i| < \rho' < 1 \quad \text{for} \quad i \geq m+1\,.$$

Denote by $\mathcal{E}^L$ the m-dimensional eigen-subspace of the matrix A which corresponds to multipliers $(\rho_1, \ldots, \rho_m)$, and by $\mathcal{E}^{ss}$ the $(n-m)$-dimensional eigen-subspace which corresponds to multipliers $(\rho_{m+1}, \ldots, \rho_n)$. The subspace $\mathcal{E}^L$ is called *the leading* invariant subspace and $\mathcal{E}^{ss}$ is called *the non-leading*, or *strongly stable*, *invariant subspace*.

Each vector $x \in \mathbb{R}^n$ is uniquely represented in the form

$$x = u + v\,,$$

where $u \in \mathcal{E}^L$ and $v \in \mathcal{E}^{ss}$. In coordinate system (u, v) the map (3.3.1) may be written as follows

$$\begin{aligned} \bar{u} &= A_L u\,, \\ \bar{v} &= A_{ss} v\,, \end{aligned}$$

where spectr $A_L = \{\rho_1, \ldots, \rho_m\}$ and spectr $A_{ss} = \{\rho_{m+1}, \ldots, \rho_n\}$. A trajectory of the map is given by the formula

$$\begin{aligned} u_j &= A_L^j u_0\,, \\ v_j &= A_{ss}^j v_0\,. \end{aligned} \tag{3.3.10}$$

As in (3.3.7) we have

$$\begin{aligned} \|u_j\| &\geq (\rho' - \varepsilon)^j \, \|u_0\|\,, \\ \|v_j\| &\leq (|\rho_{m+1}| + \varepsilon)^j \, \|v_0\|\,, \end{aligned} \tag{3.3.11}$$

where ε can be made arbitrarily small by an appropriate choice of bases for $\mathcal{E}^L$ and $\mathcal{E}^{ss}$. Hence we obtain the following inequality

$$\|v_j\| \cdot \|u_0\|^\nu \;\le\; \|v_0\| \cdot \|u_j\|^\nu\,, \tag{3.3.12}$$

for some constant $\nu > 1$.

Thus, all trajectories for which $u_0 \neq 0$ (*i.e.* outside of $\mathcal{E}^{ss}$) tend to O tangentially to the subspace $\mathcal{E}^L$ as $j \to +\infty$. Moreover, any such trajectory converges to O not faster than $(\rho' - \varepsilon)^j$, whereas the trajectories in $\mathcal{E}^{ss}$ converge to O faster than $(|\rho_{m+1}| + \varepsilon)^j$, where the constant $\varepsilon > 0$ may be chosen arbitrarily small.

As regards the behavior of trajectories in the leading coordinates u we can point out three major classes of stable fixed points:

(1) In the case $m = 1$, *i.e.* when ρ_1 is real and $|\rho_i| < \rho_1$ $(i = 2, \dots, n)$, the leading subspace is a straight line. When $\rho_1 > 0$ all trajectories outside of $\mathcal{E}^{ss}$ converge to O along a certain direction, either from $u > 0$ or from $u < 0$, as shown in Fig. 3.2.5. Such a fixed point is called *a stable node* (+).

(2) When $m = 1$ and $\rho_1 < 0$ all trajectories except those in $\mathcal{E}^{ss}$ converge to O along the u-axis but after each successive iteration the sign of the u-coordinate changes as shown in Fig. 3.2.6. Such a fixed point is called *a stable node* (−).

(3) When $m = 2$ and $\rho_{1,2} = \rho' \, e^{\pm i\omega}$, $\omega \notin \{0, \pi\}$, the fixed point is called *a stable focus*. The leading subspace of the fixed point is two-dimensional and all trajectories outside of $\mathcal{E}^{ss}$ approach O along spirals tangential to the plane u.

The case of *a completely unstable fixed point* where the absolute values of all of its multipliers ρ_i are greater than 1, is reduced to the previous case via its inverse map (as the eigenvalues of the matrix A^{-1} are equal exactly to ρ_i^{-1}). Therefore, an estimate

$$\|x_j\| \le \|A^{-1}\|^j \| \, x_0\| \le (\rho' - \varepsilon)^j \, \|x_0\| \quad \text{for} \quad j \le 0\,, \tag{3.3.13}$$

analogous to (3.3.9), is valid, where ρ' is the smallest absolute value of the multipliers ρ_i, $(i = 1, \dots, n)$. All trajectories tend exponentially to O as $j \to -\infty$.

As in the case of stable fixed points, we may now define leading and non-leading invariant subspaces and select three principal classes of completely unstable fixed points: *an unstable node* (+), *an unstable node* (−) and *an unstable focus* according to the signs of the multipliers.

When some multipliers of the fixed point lie strictly inside the unit cycle $|\rho_i| < 1$ $(i = 1, \dots, k)$ and all others lie outside of it: $|\rho_j| > 1$ $(j = k+1, \dots, n)$, the fixed point is called *a saddle fixed point*. A linear non-degenerate change of coordinates transforms the map into the form

$$\begin{aligned} \bar{u} &= A^- u\,, \\ \bar{v} &= A^+ v\,, \end{aligned} \tag{3.3.14}$$

where spectr $A^- = \{\rho_1, \dots, \rho_k\}$, spectr $A^+ = \{\rho_{k+1}, \dots, \rho_n\}$, $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$. For u and v we have estimates analogous to (3.3.9) and (3.3.13), respectively. This means that the trajectories from *the stable invariant subspace* $\mathcal{E}^s\colon v = 0$ and *the unstable invariant subspace* $\mathcal{E}^u\colon u = 0$ tend exponentially to the saddle point O as $j \to +\infty$ and $j \to -\infty$, respectively. All other trajectories escape from a small neighborhood of O.

Thus, in the stable subspace $\mathcal{E}^s$ the saddle is a stable fixed point and in the unstable subspace $\mathcal{E}^u$ it is a completely unstable fixed point. Furthermore, in $\mathcal{E}^s$ and $\mathcal{E}^u$ we may select *stable and unstable leading and non-leading manifolds* $\mathcal{E}^{sL}$, $\mathcal{E}^{uL}$, $\mathcal{E}^{ss}$ and $\mathcal{E}^{uu}$. We will call the direct sum $\mathcal{E}^{sE} = \mathcal{E}^s \oplus \mathcal{E}^{uL}$ *the extended stable invariant subspace* and $\mathcal{E}^{uE} = \mathcal{E}^u \oplus \mathcal{E}^{sL}$ *the extended unstable invariant subspace*. The invariant subspace $\mathcal{E}^L = \mathcal{E}^{uE} \cap \mathcal{E}^{sE}$ is called *the leading saddle invariant subspace*.

When the point O is a node in both $\mathcal{E}^s$ and $\mathcal{E}^u$, O is called *a saddle*. When O is a focus in at least one of the subspaces $\mathcal{E}^s$ or $\mathcal{E}^u$, it is called *a saddle-focus*.

3.4. Topological classification of fixed points

We saw in Chap. 2 that near a structurally stable equilibrium state a system of differential equations is topologically equivalent to its linearization. A similar statement pertains to structurally stable fixed points. This allows us to present a complete classification of systems of differential equations near a structurally stable periodic trajectory.

In this context an appropriate analog of topological equivalence is the notion of *topological conjugacy*.

Definition 3.1. *Two diffeomorphisms T_1 and T_2 defined in regions D_1 and D_2, respectively, are said to be topologically conjugate in the regions $U_1 \subseteq D_1$ and $U_2 \subseteq D_2$ if there exists a homeomorphism $\eta\colon U_1 \to U_2$ which transforms trajectories (semi-trajectories, intervals of trajectories) of the first diffeomorphism onto trajectories (semi-trajectories, intervals of trajectories) of the second diffeomorphism.*

In other words, for any point $x \in D_1$ the following equality holds (see Fig. 3.4.1).

$$\eta(T_1(x)) = T_2(\eta(x))\,. \tag{3.4.1}$$

Theorem 3.1. (Grobman–Hartman) *Let O be a structurally stable fixed point of a diffeomorphism T. Then there exist neighborhoods U_1 and U_2 of the point O where the diffeomorphism T and its linear part are topologically conjugate.*

In the structurally unstable (non-rough) case an analogous statement does not hold. It is easy to show that when the matrix A of a linear map

$$\bar{x} = Ax \tag{3.4.2}$$

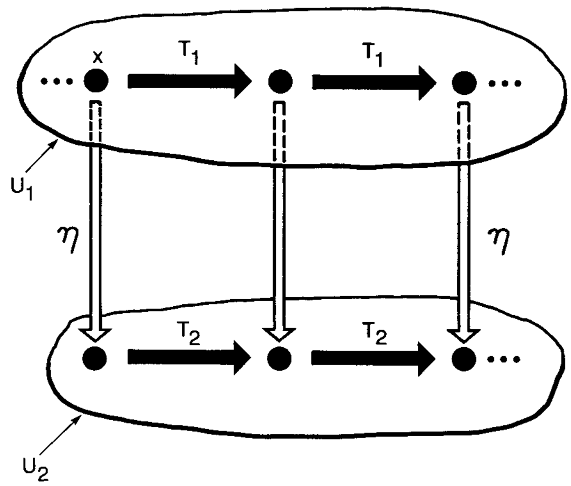


Fig. 3.4.1. Graphical representation of the homeomorphism $\eta(T_1(x)) = T_2(\eta(x))$ realizing the topological conjugacy between two maps T_1 and T_2 defined in subregions U_1 and U_2, respectively.

has some multipliers equal to 1 in absolute value, then one can add a nonlinear term $g(x)$ such that the map

$$\bar{x} = Ax + g(x) \tag{3.4.3}$$

is not topologically conjugate to its linear part (3.4.2). For example, the one-dimensional map

$$\bar{x} = x + x^2$$

has only one fixed point O (see Fig. 3.4.2), whereas all points on the bisectrix are fixed points of the associated linearized map

$$\bar{x} = x\,.$$

Consider another example: the map

$$\bar{x} = -x + x^3\,,$$

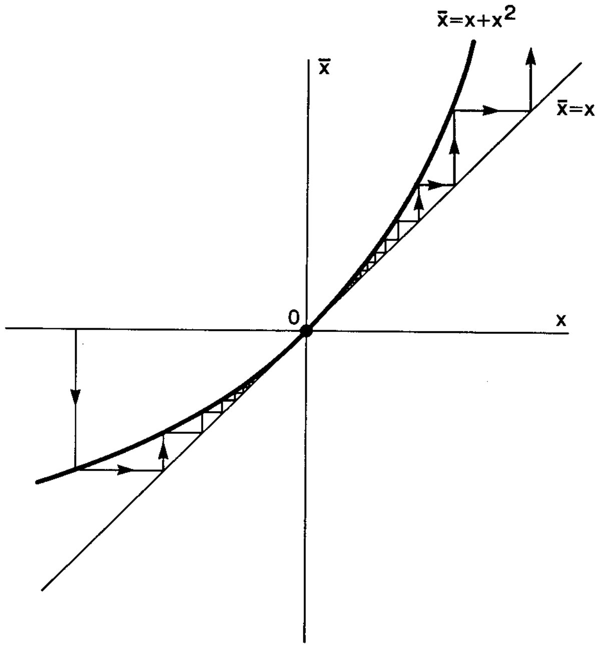


Fig. 3.4.2. A Lamerey stair. The graph of the function $f(x) = x + x^2$ is tangent to the bisectrix at the fixed point of the saddle-node type.

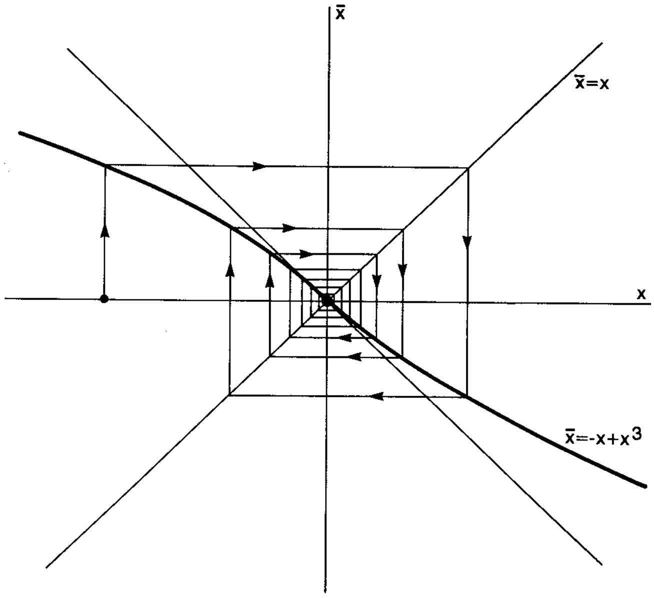


Fig. 3.4.3. A Lamerey spiral of the map $\bar{x} = -x + x^3$. Outside of the origin the derivative of the map is less than 1 in absolute value: the fixed point is stable.

possessing the stable fixed point O (see Fig. 3.4.3) is not topologically conjugate to its linear part

$$\bar{x} = -x\,,$$

for which all points (apart from O) are periodic trajectories of period 2.

For our next example let us examine a linear map possessing a fixed point with a pair of complex-conjugate multipliers $e^{\pm i\omega}$

$$\begin{aligned} \bar{x} &= x\cos\omega - y\sin\omega\,, \\ \bar{y} &= x\sin\omega + y\cos\omega\,. \end{aligned} \tag{3.4.4}$$

All trajectories lie on invariant circles centered at the point $O(0,0)$ (see Sec. 3.2). This map is not conjugate to the map

$$\begin{aligned} \bar{x} &= x\cos\omega - y\sin\omega - x(x^2+y^2)\cos\omega\,, \\ \bar{y} &= x\sin\omega + y\cos\omega - y(x^2+y^2)\cos\omega \end{aligned} \tag{3.4.5}$$

whose trajectories tend to O along spirals (the analogous example of an equilibrium state is given in Sec. 2.5).

It follows from Theorem 3.1 that when all multipliers of the fixed point O of the diffeomorphism T are less than 1 in absolute value, then all forward trajectories tend to O. When all multipliers lie outside of the unit circle, the backward trajectory of a point from a small neighborhood of the point O tends to the fixed point. Forward iterations of the map T force all trajectories (excluding the fixed point) to escape from a neighborhood of the fixed point.

In the saddle case where there are multipliers both inside and outside of the unit circle, the fixed point has (locally) *a stable invariant manifold* W^s_{loc} and *an unstable invariant manifold* W^u_{loc} which are the images of the invariant subspaces $\mathcal{E}^s$ and $\mathcal{E}^u$ of the associated linearized system by the homeomorphism η which establishes the topological conjugacy. Therefore, the forward semi-trajectory of any point in W^s_{loc} lies entirely in W^s_{loc} and tends to the saddle point O. On the other hand, given any point in W^u_{loc} its backward semi-trajectory lies entirely in W^u_{loc} and tends to O. The dimension of the stable manifold is equal to the number of the multipliers inside the unit circle and the dimension of the unstable manifold is equal to the number of multipliers outside of the unit circle. Trajectories of the points which do not lie in $W^s_{loc} \cup W^u_{loc}$ diverge from any neighborhood of the saddle.

It is obvious that if one diffeomorphism near a saddle fixed point is topologically conjugate to another diffeomorphism near another fixed point, then the dimensions of the stable (unstable) manifolds of both saddle points must be equal (for generality, in the case of a stable fixed point we can assume that $W^u = \{\emptyset\}$ and, therefore dim $W^u = 0$; for a completely unstable point assume that $W^s = \{\emptyset\}$ and dim $W^s = 0$). However, in contrast to the case of structurally stable equilibrium states, the dimensions of the stable and the unstable manifolds are not *the only* invariants of the topological conjugacy near the fixed points.

In order to find new invariants we notice that the Grobman–Hartman theorem may be generalized as follows

> *In a neighborhood of the origin a linear non-singular map which has no multipliers on the unit circle, is topologically conjugate to any, sufficiently close map.*

This implies in particular that any two close, linear maps are topologically conjugate. It follows that for two arbitrary matrices A_0 and A_1 the maps

$$\bar{x} = A_0 x \qquad \text{and} \qquad \bar{x} = A_1 x$$

are topologically conjugate if we can construct a family of the matrices $A(s)$ depending continuously on the parameter $s \in [0,1]$ such that $A(0) = A_0$ and $A(1) = A_1$, provided that all matrices $A(s)$ are non-singular and have no eigenvalues on the unit circle.

It is not hard to verify that such a family exists if and only if both matrices A_0 and A_1 have the same number of multipliers both inside and outside of the unit circle and the same values δ_s and δ_u, where $\delta_s = \text{sign} \prod_{i=1}^{k} \rho_i$ as well as $\delta_u = \text{sign} \prod_{i=k+1}^{n} \rho_i$, where $(\rho_1, \ldots, \rho_k)$ are the multipliers inside the unit circle, and $(\rho_{k+1}, \ldots, \rho_n)$ are the multipliers which lie outside of the unit circle. The values δ_s and δ_u remain unchanged while A_0 is changed continuously to A_1 because the product of the multipliers can change its sign only when at least one multiplier vanishes, which is only possible for degenerate maps.

Thus, when some fixed points have the same *topological type*, *i.e.* the same set of four numbers $(k,\ \delta_s,\ n-k,\ \delta_u)$, then they are *topologically conjugate*, *i.e.* there exists a homeomorphism establishing topological conjugacy among the diffeomorphisms defined near these fixed points. In particular, near a fixed point any diffeomorphism is topologically conjugate to a map

$$\bar{x} = A^s x\,, \quad \bar{y} = A^u y\,, \tag{3.4.6}$$

where A^s and A^u are, respectively, the $(k \times k)$-matrix and the $(n-k) \times (n-k)$-matrices:

$$A^s = \begin{pmatrix} 1/2 & & & \mathbf{0} \\ & 1/2 & & \\ & & \ddots & \\ \mathbf{0} & & & \delta_s/2 \end{pmatrix}, \qquad A^u = \begin{pmatrix} 2 & & & \mathbf{0} \\ & 2 & & \\ & & \ddots & \\ \mathbf{0} & & & 2\delta_u \end{pmatrix}.$$

We should emphasize that maps of the kind (3.4.6) with different values of δ_s cannot be topologically conjugate since the restriction $\bar{x} = A^s x$ of the map (3.4.6) to the stable invariant subspace $y = 0$ preserves the orientation in $\mathbb{R}^k$ provided that $\delta_s = 1$, but does not preserve it if $\delta_s = -1$. This assertion applies to δ_u as well. In conclusion we come to the following theorem.

Theorem 3.2. *Two structurally stable fixed points are topologically conjugate if and only if they are of the same topological type.*

Thus, the number of topological types of structurally stable fixed points of n-dimensional maps exceeds the number of topological types of structurally stable equilibrium states of differential equations in $\mathbb{R}^n$. So, a two-dimensional diffeomorphism may have two types of stable and unstable fixed points and four types of saddle fixed points. For example, the two linear maps

$$\begin{cases} \bar{x} = x/2 \\ \bar{y} = y/2 \end{cases} \quad \text{and} \quad \begin{cases} \bar{x} = x/2 \\ \bar{y} = -y/2 \end{cases}$$

have the fixed points of node type but the first point is orientable whereas the second is non-orientable.

(Note: a fixed point possessing a pair of complex-conjugate multipliers has also the topological type of orientable node.)

Observe next that the maps

$$\begin{cases} \bar{x} = 2x \\ \bar{y} = 2y \end{cases} \quad \text{and} \quad \begin{cases} \bar{x} = 2x \\ \bar{y} = -2y \end{cases}$$

have, respectively, an orientable and a non-orientable unstable topological node at the origin.

Examples of topological saddles $(\delta_s = (+), \delta_u = (+))$ and $(\delta_s = (-), \delta_u = (-))$ are given, respectively, by:

$$\begin{cases} \bar{x} = x/2 \\ \bar{y} = 2y \end{cases} \quad \text{and} \quad \begin{cases} \bar{x} = -x/2 \\ \bar{y} = -2y \end{cases}$$

Examples of topological saddles $(+,-)$ and $(-,+)$ are given, respectively, by

$$\begin{cases} \bar{x} = x/2 \\ \bar{y} = -2y \end{cases} \quad \text{and} \quad \begin{cases} \bar{x} = -x/2 \\ \bar{y} = 2y \end{cases}$$

It is clear that in the case $\delta_s > 0$ the stable invariant manifold (here, the x-axis) of the saddle point is subdivided by the point O into the two parts, namely, the rays $x > 0$ and $x < 0$, each of which is invariant with respect to the map. In the case $\delta_s < 0$ these rays are no longer separately invariant in the sense that the map takes one onto the other. Analogously, the value δ_u determines the structure of the unstable manifold of the saddle point O.

We must emphasize that the Poincaré map always preserves orientation in $\mathbb{R}^n$, *i.e.* the product of all multipliers of a periodic trajectory is positive.

This implies that the values δ_s and δ_u of the fixed point of the Poincaré map must have the same sign. Nevertheless, restricted to an invariant (stable or unstable) manifold, the Poincaré map *may not* continue to preserve orientation (for example, when both δ_s and δ_u are negative). Thus, the study of fixed points of non-orientable maps makes the sense.

We showed in Sec. 3.1 that the values of the multipliers of a fixed point of the Poincaré map (and, consequently, its topological type) are independent of the choice of the cross-section, *i.e.* we can always correctly define the topological type $(k,\ \delta_s,\ n-k,\ \delta_u)$ of a periodic trajectory. In order to complete our classification we refer to the following simple statement.

Lemma 3.1. *Let a system X_1 of differential equations have a periodic trajectory L_1 and a system X_2 have a periodic trajectory L_2. Then, the system X_1 in a neighborhood of the trajectory L_1 is topologically equivalent to the system X_2 in a neighborhood of the trajectory L_2 if and only if the respective Poincaré maps are topologically conjugate near the corresponding fixed points (regardless of the choice of cross-sections)*

From this lemma and from Theorem 3.2 we have the following theorem.

Theorem 3.3. *Two structurally stable periodic trajectories are locally topologically equivalent if and only if they have the same topological type.*

3.5. Properties of nonlinear maps near a stable fixed point

In the previous section we formally have presented a complete description of dynamical systems near a structurally stable periodic trajectory. However, such a topological classification of periodic trajectories, as well as equilibrium states, is too crude. For example, the assertion concerning the equivalence of a node and a focus seems to be rather strange from a practical point of view. We will consider below more subtle (and more significant) features of structurally stable fixed points.

Suppose the map

$$\bar{x} = A\,x + h(x)\,, \tag{3.5.1}$$

where

$$h(0) = 0\,, \quad h'(0) = 0\,, \tag{3.5.2}$$

has a stable fixed point at the origin. This means that the absolute values of all multipliers ρ_i $(i = 1, \dots, n)$ of the matrix A are strictly less than 1. It is not hard to verify that all trajectories beginning from a small neighborhood of the origin tend *exponentially* to O. Indeed, if we choose the Jordan basis, we can verify that the estimate (3.3.9) for the norm of the matrix A is fulfilled; namely,

$$\|A\| \leq \rho' + \varepsilon \,,$$

where $\rho' = \max\limits_{i=1,\dots,n} |\rho_i| < 1$. From the relation

$$h(x) = h(0) + x \int_0^1 h'(sx) ds$$

we have

$$\|h(x)\| \leq \|x\| \max_{\|y\| \leq \|x\|} \|h'(y)\| \,.$$

Hence

$$\|h(x)\| \leq \varepsilon \|x\| \,, \tag{3.5.3}$$

where $\varepsilon > 0$ can be chosen arbitrarily small since x is assumed to be small.

Consequently, for map (3.5.1) we obtain

$$\|\bar{x}\| \leq \|A\| \, \|x\| + \|h(x)\| \leq (\rho' + 2\varepsilon) \|x\| \,,$$

or, equivalently for the j-th iteration of an initial point x_0

$$\|x_j\| \leq (\rho' + 2\varepsilon)^j \|x_0\| \tag{3.5.4}$$

where $x_j \to 0$ as $j \to +\infty$ since $\rho' < 1$ and since ε can be made arbitrarily small.

Let us now reorder the eigenvalues of the matrix A so that

$$|\rho_1| = \cdots = |\rho_m| = \rho' \,, \quad |\rho_i| < \rho' \quad \text{for} \quad i = m+1, \dots, n \,.$$

The matrix A can be then represented in the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $\mathrm{spectr} A_1 = \{\rho_1, \dots, \rho_m\}$ and $\mathrm{spectr} A_2 = \{\rho_{m+1}, \dots, \rho_n\}$. The map (3.5.1) now takes the form

$$\begin{aligned} \bar{u} &= A_1 u + f(u, v) \,, \\ \bar{v} &= A_2 v + g(u, v) \,, \end{aligned} \tag{3.5.5}$$

where $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_{n-m})$ are the projections of x onto the leading and the non-leading subspaces of the matrix A, the functions f, $g \in \mathbb{C}^r$, and

$$f(0) = 0\,, \quad g(0) = 0\,, \quad f'(0) = 0\,, \quad g'(0) = 0\,. \tag{3.5.6}$$

In Chap. 5 we will prove the following theorem

Theorem 3.4. (On the non-leading manifold) *In a neighborhood U of the point O there exists a unique $(n-m)$-dimensional $\mathbb{C}^r$-smooth non-leading (strongly stable) invariant manifold W^{ss}_{loc} of the form*

$$u = \varphi(v)$$

where

$$\varphi(0) = \varphi'_v(0) = 0\,. \tag{3.5.7}$$

The following result follows from Theorem 3.4.

Theorem 3.5. *For any point x_0 that does not lie in W^{ss}_{loc} the associated trajectory $\{x_j\}_{j\geq 0}$ tends to O along the leading direction $v = 0$, and*

$$\|x_j\| \geq C\,(\rho' - \varepsilon)^j \operatorname{dist}\,(x_0,\ W^{ss}_{loc})\,, \tag{3.5.8}$$

where C is some positive constant and ε can be made arbitrarily small by choosing suitably small x_0.

Proof. Firstly, let us straighten the manifold W^{ss}_{loc} by the following change of variables

$$w = u - \varphi(v)\,. \tag{3.5.9}$$

In the new variables the equation of W^{ss}_{loc} is $w = 0$. The invariance of the manifold implies that $\bar{w} = 0$ when $w = 0$. Since the function $\varphi(v)$ contains no linear terms (see (3.5.7)), the transformation (3.5.9) does not change the linear part of the map. As a result, in the new variables map (3.5.5) becomes

$$\bar{w} = (A_1 + \tilde{f}(w,v))\,w \tag{3.5.10}$$

$$\bar{v} = A_2 v + g(w + \varphi(v),\ v)\,, \tag{3.5.11}$$

where

$$\tilde{f}(0,\,0) = 0\,. \tag{3.5.12}$$

In the Jordan basis (see Sec. 3.3), we have for the norm of the matrix A_1

$$\|A_1^{-1}\|^{-1} \geq \rho' - \varepsilon/2\,.$$

By (3.5.10) and (3.5.12) this implies that

$$\|\bar{w}\| \geq (\rho' - \varepsilon)\;\|w\| \tag{3.5.13}$$

provided that $\|w,\, v\|$ is sufficiently small. As we have shown already, when the norm of the initial point $\|w_0,\, v_0\|$ is sufficiently small, the norm $\|w_j,\, v_j\|$ is also small for all $j \geq 0$. Therefore, inequality (3.5.13) is valid for any pair $(w_j, w_{j+1} = \bar{w}_j)$. Hence, we obtain the estimate

$$\|w_j\| \geq (\rho' - \varepsilon)^j \|w_0\|$$

i.e. the inequality (3.5.8) is proven.

Let us now verify that when an initial point does not belong to W^{ss}_{loc}, its trajectory tends to O along the leading subspace $v = 0$. When $w \neq 0$ let us consider the value $z = \|v\|/\|w\|$. We seek to show that $z_j \to 0$ along the trajectories $(w_j, v_j)_{j\geq 0}$ of the map (3.5.10)–(3.5.11).

For $\|\bar{w}\|$ we have the estimate (3.5.13). Similarly, from (3.5.11) one obtains

$$\|\bar{v}\| \leq (|\rho_{m+1}| + \varepsilon)\;\|v\| + \|w\|\;\max\|f'_u\|\,,$$

where the maximum is taken over a neighborhood of O of diameter equal to $\|w, v\|$. By (3.5.13), it follows that

$$\begin{aligned} z_{j+1} \leq (|\rho_{m+1}| + \varepsilon)\|v_j\|/(\rho' - \varepsilon)\|w_j\| \\ + \max\|f'_u\|/(\rho' - \varepsilon) \equiv \mu z_j + \kappa_j\,, \end{aligned} \tag{3.5.14}$$

where $\mu = (|\rho_{m+1}| + \varepsilon)/(\rho' - \varepsilon) < 1$ and

$$\kappa_j \to 0 \qquad \text{as} \qquad j \to +\infty\,. \tag{3.5.15}$$

From (3.5.14) we obtain

$$\mu^{-(j+1)} z_{j+1} \leq \mu^{-j} z_j + \mu^{-(j+1)}\kappa_j\,,$$

$$\mu^{-(j+1)} z_{j+1} \leq z_0 + \sum_{i=0}^{j} \mu^{-(i+1)}\kappa_i\,,$$

$$z_j \leq z_0\mu^j + \sum_{i=0}^{j-1} \mu^{j-(i+1)}\kappa_i\,.$$

Since $\mu < 1$, the first summand decays to zero as $j \to \infty$. Hence, in order to prove that $z_j \to 0$ we need to show that the sum

$$I_j = \sum_{i=0}^{j-1} \mu^{j-(i+1)} \kappa_i \to 0\,.$$

This expression readily holds if we let $\kappa_j \to 0$. Let us choose a natural number J and break the sum into two parts:

$$I_j = \sum_{i=0}^{J-1} \mu^{j-(i+1)} \kappa_i + \sum_{i=J}^{j-1} \mu^{j-(i+1)} \kappa_i\,.$$

Observe that

$$\begin{aligned} I_j &\leq \mu^{j-J} \sum_{i=0}^{J-1} \kappa_i + \left(\sum_{i=J}^{j-1} \mu^{j-(i+1)} \right) \max_{i \leq J} \kappa_i \\ &\leq \mu^{j-J} \sum_{i=0}^{J-1} \kappa_i + (1-\mu)^{-1} \max_{i \geq J} \kappa_i\,. \end{aligned} \tag{3.5.16}$$

By virtue of (3.5.15) the second term in (3.5.16) can be made arbitrarily small by increasing J. By choosing J sufficiently large we may make the first term in (3.5.16) arbitrarily small too and, therefore, I_j arbitrary small if let $j \to +\infty$.

Thus, when $w_0 \neq 0$, $\|v_j\|/\|w_j\| \to 0$, *i.e.* any trajectory which does not lie in W^{ss}_{loc} converges to the leading manifold as $j \to +\infty$. Thus, we have proven Theorem 3.5

The map (3.5.10)–(3.5.11) on the non-leading manifold $w = 0$ is written in the form

$$\bar{v} = A_2\, v + g(\varphi(v),\, v)\,. \tag{3.5.17}$$

On this manifold the point O is a stable fixed point with the multipliers $(\rho_{m+1}, \ldots, \rho_n)$. The results obtained above can be applied to this mapping. In particular, the following exponential estimate analogous to (3.5.4) holds:

$$\|v_j\| \leq C\, (|\rho_{m+1}| + \varepsilon)^j\, \|v_0\|\,, \tag{3.5.18}$$

i.e. any trajectory from W^{ss}_{loc} tends to O exceedingly fast. Since Theorems 3.4 and 3.5 hold for the map (3.5.17), it follows that almost all trajectories in W^{ss}_{loc}

tends to O with the exponential rate equal asymptotically $|\rho_{m+1}|$. Those particular trajectories which tend to O faster, form a $\mathbb{C}^r$-smooth manifold W^{sss}_{loc} tangential at O to the eigen-subspace of the matrix A which corresponds to the multipliers whose absolute values are less than $|\rho_{m+1}|$. For the restriction of the map to W^{sss}_{loc}, the theorem on the non-leading manifold can also be applied, and so on. We again obtain a hierarchy of non-leading manifolds: W^{ss}_{loc}, W^{sss}_{loc}, $W^{ssss}_{loc}, \dots$, composed of trajectories with higher and higher velocities of convergence to the fixed point.

Just like the linear case, we can select three main types of stable fixed points which depend on the behavior of the map in the leading coordinates: *a node* (+), *a node* (−) and *a focus*.

The point O is called *a node* when $m = 1$, *i.e.* when the leading multiplier ρ_1 is unique and real:

$$1 > |\rho_1| > |\rho_i| \qquad (i = 2, \dots, n)\,. \tag{3.5.19}$$

Moreover, the point is called *a node* (+) when $0 < \rho_1 < 1$, and it is called *a node* (−) when $-1 < \rho_1 < 0$.

The point O is called *a focus* when $m = 2$ and the leading multipliers are complex:

$$1 > |\rho_1| = |\rho_2| > |\rho_i| \qquad (i = 3, \dots, n)\,. \tag{3.5.20}$$

In the case of the node, the $(n-1)$-dimensional non-leading manifold partitions a neighborhood of the fixed point O into two parts, namely, $w > 0$ and $w < 0$. Here, Eq. (3.5.10) for the leading coordinate w can be written in the form

$$\bar{w} = \rho_1 w + o(w)\,. \tag{3.5.21}$$

One can see that when $\rho_1 > 0$ each part is invariant with respect to the map. The trajectories which do not lie in W^{ss}_{loc} tend monotonically to O strictly along one of two directions, either from the region $w > 0$, or from the opposite side $w < 0$.

When $\rho_1 < 0$ the parts cycle under the action of the map, *i.e.* the sign of the leading coordinate changes with every iteration.

In the case of the focus, the non-leading manifold is $(n-2)$-dimensional and it no longer partitions a neighborhood of the point O. Having introduced $\rho_{1,2} = \rho e^{\pm i\omega}$, Eq. (3.5.10) for the leading coordinates becomes

$$\begin{aligned} \bar{w}_1 &= \rho(\cos\omega + \cdots)\, w_1 - \rho(\sin\omega + \cdots)\, w_2\,, \\ \bar{w}_2 &= \rho(\sin\omega + \cdots)\, w_1 + \rho(\cos\omega + \cdots)\, w_2\,, \end{aligned} \tag{3.5.22}$$

or in polar coordinates

$$\bar{r} = (\rho + \cdots)\, r \tag{3.5.23}$$

$$\bar{\varphi} = \varphi + \omega + \cdots\,, \tag{3.5.24}$$

where the ellipsis denotes the terms of a higher order. It follows from (3.5.23) and (3.5.24) that all trajectories which do not belong to W^{ss}_{loc} must spiral towards to O (tangentially to the leading plane $v = 0$).

The case where $|\rho_i| > 1$, $(i = 1, \ldots, n)$ is reduced to that discussed above by considering the inverse map. In this case a trajectory is estimated by

$$\|x_j\| \leq \left(\min_{i=1,\ldots,n} \rho_i - 2\varepsilon \right)^j \|x_0\| \qquad \text{for} \qquad j \leq 0\,. \tag{3.5.25}$$

The associated fixed point is exponentially completely unstable. The existence of the smooth non-leading manifold W^{uu}_{loc} can be established in the same way as for W^{ss}_{loc} of the stable fixed point but assuming on this occasion that $j \to -\infty$. Depending on the behavior of the trajectories in the leading coordinates there exist fixed points of the following types: *node* $(+)$, *node* $(-)$ and *a focus*.

We conclude this section with a theorem on *the leading invariant manifold* (see its proof in Chap. 5.)

Theorem 3.6. (On the leading invariant manifold) *A stable fixed point O has an m-dimensional $\mathbb{C}^{\min(r,r_L)}$-smooth invariant manifold W^L_{loc} (not unique in general) which is tangent at the point O to the subspace $v = 0$; here*

$$r_L = \left[\frac{\ln \rho_{m+1}}{\ln \rho_1} \right] \geq 1\,, \tag{3.5.26}$$

where $[x]$ denotes the largest integer strictly less than x, and m is the number of the leading multipliers.

3.6. Saddle fixed points. Invariant manifolds

Let us consider next a map T possessing a structurally stable fixed point O of saddle type whose first k multipliers lie inside the unit circle and whose remaining $(n - k)$ multipliers lie outside of the unit circle, *i.e.* $|\rho_i| < 1$ $(i = 1, \ldots, k)$, $|\rho_j| > 1$, $(j = k + 1, \ldots, n)$, where $k \neq 0$, n. For convenience, let us denote the multipliers inside the unit circle by $(\lambda_1, \dots, \lambda_k)$, and those outside by $(\gamma_1, \dots, \gamma_{n-k})$. We will also assume that the multipliers are ordered in the following manner

$$|\lambda_k| \leq \cdots \leq |\lambda_2| \leq |\lambda_1| < 1 < |\gamma_1| \leq |\gamma_2| \leq \cdots \leq |\gamma_{n-k}| .$$

A linear non-degenerate change of variables near the point O transforms the map T to the form

$$\begin{aligned} \bar{u} &= A^- u + f(u,v)\,, \\ \bar{v} &= A^+ v + g(u,v)\,, \end{aligned} \tag{3.6.1}$$

where $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^{n-k}$, *Spectrum* $A^- = \{\lambda_1, \dots, \lambda_k\}$ and *Spectrum* $A^+ = \{\gamma_1, \dots, \gamma_{n-k}\}$; f, g are $\mathbb{C}^r$-smooth ($r \geq 1$) functions which vanish at the origin along with their first derivatives.

The study of the map T near a saddle fixed point resembles that of a system of differential equations near a saddle equilibrium point. It can be reduced to the problem of the existence of a stable and an unstable invariant manifolds of the point O. We examine this problem in detail below. Poincaré proved the existence of analytical invariant manifolds for analytical maps. Later, the smooth case was considered by Hadamard [31] who proved the existence of invariant manifolds satisfying a Lipschitz condition.

Theorem 3.7. (Hadamard's theorem) *The saddle fixed point O has two invariant manifolds: a stable manifold W^s_{loc}: $v = \psi^*(u)$ and an unstable manifold W^u_{loc}: $u = \varphi^*(v)$, where $\psi^*(u)$ and $\varphi^*(v)$ satisfy the following Lipschitz conditions:*

$$\|\psi^*(u_2) - \psi^*(u_1)\| \leq N\|u_2 - u_1\|\,, \tag{3.6.2}$$

$$\|\varphi^*(v_2) - \varphi^*(v_1)\| \leq L\|v_2 - v_1\|\,, \tag{3.6.3}$$

for some constants $N > 0$ and $L > 0$.

Proof. We prove only the existence of W^u_{loc} as the inverse map T^{-1} can also be represented in the form (3.6.1) with the only difference that the variables u and v exchange roles. Therefore, having proven that the map T has an invariant manifold W^u_{loc} of the form $u = \varphi^*(v)$, by repeating the same arguments for the map T^{-1} we can prove that there exists an invariant manifold of the form $v = \psi^*(u)$, *i.e.* the desired manifold W^s_{loc}.

Let us select a small $\delta > 0$ and surround the point O by a neighborhood $D_1 \otimes D_2$ in $\mathbb{R}^n$, where D_1 and D_2 are spheres of diameter equal to δ in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively. Choose an arbitrary surface $\mathcal{W}$ of the form $u = \varphi(v)$ such that

$$\|\varphi\| \le \delta \tag{3.6.4}$$

$$\|\varphi'\| \le L \tag{3.6.5}$$

for some $L > 0$. We will show below that when δ is sufficiently small then the intersection $T(\mathcal{W}) \cap (D_1 \otimes D_2)$ is a surface of the same form $\bar{u} = \tilde{\varphi}(\bar{v})$ where $\tilde{\varphi}$ satisfies conditions (3.6.4)–(3.6.5) with the same constant L. This allows us to consider a sequence of surfaces $\{\mathcal{W}_j : u = \varphi_j(v)\}_{j=0}^{j=\infty}$ which are the sequential images of the initial surface $\mathcal{W}$ under the map T: $\varphi_j = T^j\varphi$. We will show further that this sequence converges uniformly to some surface $u = \varphi^*(v)$ satisfying a Lipschitz condition. Moreover φ^* does not depend on the initial function φ. By construction, $\varphi^* = \tilde{\varphi}^*$, *i.e.* its graph is invariant with respect to the map T. Thus, the surface $u = \varphi^*(v)$ is the desired invariant manifold W^u_{loc}: $T\,(W^u_{loc}) \cap (D_1 \otimes D_2) = W^u_{loc}$.

Step 1. Let us choose an arbitrary surface $\mathcal{W}$ of the form $u = \varphi(v)$ which satisfies conditions (3.6.4) and (3.6.5) for some L. By substituting $u = \varphi(v)$ into (3.6.1) we obtain a parametric representation

$$\bar{u} = A^- \varphi(v) + f(\varphi(v),\, v)\,, \tag{3.6.6}$$

$$\bar{v} = A^+ v + g(\varphi(v),\, v)\,, \tag{3.6.7}$$

for the image of the surface $\mathcal{W}$ under the map T, where v can take arbitrary values in D_2.

Let us now show that for any $\bar{v}$ whose norm does not exceed δ the value $\bar{u}$ is uniquely defined by (3.6.6) and (3.6.7). To do this we rewrite (3.6.7) in the form

$$v = (A^+)^{-1}(\bar{v} - g(\varphi(v), v))\,. \tag{3.6.8}$$

When δ is sufficiently small the norm $\|\partial(g, f)/\partial(u, v)\|$ is also small. It follows that[2]

$$\left\| \frac{dg(\varphi(v), v)}{dv} \right\| \le \|g'_u\|_\circ \cdot \|\varphi'\|_\circ + \|g'_v\|_\circ$$

---

[2] Here $\|\cdot\|_\circ = \sup \|\cdot\|$.

is small and, therefore, by virtue of the implicit function theorem, v is uniquely expressed in terms of $\bar{v}$ from (3.6.8). It should also be noted that $\|v\|$ also does not exceed δ. Indeed, it follows from (3.6.8) that

$$\|v\| \leq \|(A^+)^{-1}\| \big(\|\bar{v}\| + \|g'_u\|_\circ \cdot \|\varphi(v)\|_\circ + \|g'_v\|_\circ \|v\|\big)$$

whence

$$\|v\| \leq \|(A^+)^{-1}\| \frac{(\|\bar{v}\| + \|g'_u\|_\circ \cdot \|\varphi(v)\|_\circ)}{1 - \|(A^+)^{-1}\| \, \|g'_v\|_\circ} \, .$$

Thus, if $\|\bar{v}\| \leq \delta$, then $\|v\| \leq \delta$ because $\|(A^+)^{-1}\| < 1$, $\|\varphi(v)\|_\circ \leq \delta$ and $\|g'_{(u,v)}\|_\circ$ is small.

So, by expressing v in terms of $\bar{v}$ from (3.6.8) and by substituting the resulting expression into (3.6.6), we determine that for each $\bar{v}$ such that $\|\bar{v}\| \leq \delta$, there exists a uniquely defined $\bar{u}$ such that the point $(\bar{u}, \bar{v})$ is the image of some point $(u, v) \in \mathcal{W}$. Let us denote this $\bar{u}$ by $\bar{u} = \tilde{\varphi}(\bar{v})$.

Step 2. Let us show that the surface $T\mathcal{W}$: $\bar{u} = \tilde{\varphi}(\bar{v})$ satisfies conditions (3.6.4) and (3.6.5). In other words we will show that $T\mathcal{W}$ lies entirely in the δ-neighborhood $(D_1 \otimes D_2)$ of the point O, and that the norm of the derivative of the function $\tilde{\varphi}$ does not exceed L. It follows from (3.6.4) and (3.6.6) that

$$\begin{aligned}\|\bar{u}\| &\leq \|A^-\| \, \|\varphi(v)\| + \|f'_u\|_\circ \cdot \|\varphi(v)\| + \|f'_v\|_\circ \cdot \|u\| \\ &\leq (\|A^-\| + \|f'_u\|_\circ + \|f'_v\|_\circ) \, \delta \, .\end{aligned}$$

It follows that

$$\|\tilde{\varphi}(\bar{v})\| \equiv \|\bar{u}\| \leq \delta$$

as $\|A^-\| < 1$ and the norm $\|f'_{(u,v)}\|_\circ$ is small; *i.e.* condition (3.6.4) holds for $\tilde{\varphi}$ indeed.

Furthermore, from (3.6.6) and (3.6.7) we have

$$\begin{aligned}\frac{d\bar{u}}{dv} &= A^-\varphi' + f'_u(\varphi(v), v)\varphi' + f'_v(\varphi(v), v) \, , \\ \frac{d\bar{v}}{dv} &= A^+ + g'_u(\varphi(v), v)\varphi' + g'_v(\varphi(v), v)\end{aligned}$$

whence

$$\begin{aligned}\tilde{\varphi}'(\bar{v}) \equiv \frac{d\bar{u}}{d\bar{v}} &= (A^-\varphi' + f'_u(\varphi(v), v)\varphi' + f'_v(\varphi(v), v)) \cdot \\ &\quad \cdot [A^+ + g'_u(\varphi(v), v)\varphi' + g'_v(\varphi(v), v)]^{-1} \, .\end{aligned}$$

Finally

$$\tilde{\varphi}'(\bar{v}) = A^- \varphi'(v)(A^+)^{-1} + \cdots ,$$

where the ellipsis denotes the terms of order $\|(f,g)'_{(u,v)}\|_\circ$ which tend to zero as $\delta \to 0$. It is easy to see that since $\|A^-\| < 1$ and $\|(A^+)^{-1}\| < 1$, then $\|\tilde{\varphi}'\|_\circ \leq L$ provided $\|\varphi'\|_\circ \leq L$ and δ is sufficiently small.

Step 3. We have shown that the map T takes any surface $\mathcal{W}$ satisfying conditions (3.6.4) and (3.6.5), onto a surface satisfying the same conditions. Hence, all iterations of the surface $\mathcal{W}$ are defined under the action of the map T. Let us now show that the sequence of these iterations $\mathcal{W}_j\colon\ u = \varphi_j(v)$ converges uniformly to some surface $\mathcal{W}^*\colon u = \varphi^*(v)$. Since $\mathcal{W}_{j+1} = T(\mathcal{W}_j) \cap (D_1 \otimes D_2)$, it follows from continuity that $\mathcal{W}^* = T(\mathcal{W}^*) \cap (D_1 \otimes D_2)$, *i.e.* this surface is invariant with respect to the map T.

In order to prove this, we will show that there exists $K < 1$ such that

$$\sup_{\bar{v} \in D_2} \|\varphi_{j+2}(\bar{v}) - \varphi_{j+1}(\bar{v})\| \leq K \sup_{v \in D_2} \|\varphi_{j+1}(v) - \varphi_j(v)\| . \tag{3.6.9}$$

Let us choose any $\bar{v} \in D_2$ and consider a pair of points $\bar{M}_1(\varphi_{j+1}(\bar{v}), \bar{v})$ and $\bar{M}_2(\varphi_{j+2}(\bar{v}), \bar{v})$. By construction, each point $\bar{M}_i$ has a pre-image M_i on the surface $u = \varphi_{j+i-1}(v)$. Assume $M_1(u_1 = \varphi_j(v_1), v_1)$ and $M_2(u_2 = \varphi_{j+1}(v_2), v_2)$. Because $\bar{M}_1 = TM_1$ and $\bar{M}_2 = TM_2$, we have from (3.6.1)

$$\begin{aligned} \bar{u}_1 &= A^- u_1 + f(u_1, v_1) , & \bar{u}_2 &= A^- u_2 + f(u_2, v_2) , \\ \bar{v} &= A^+ v_1 + g(u_1, v_1) , & \bar{v} &= A^+ v_2 + g(u_2, v_2) . \end{aligned}$$

Hence it follows that

$$\|\bar{u}_2 - \bar{u}_1\| \leq (\|(A^-\| + \|f'_u\|_\circ) \|u_2 - u_1\| + \|f'_v\|_\circ \|v_2 - v_1\| \tag{3.6.10}$$

and

$$\|v_2 - v_1\| \leq \|(A^+)^{-1}\| (\|g'_u\|_\circ \|u_2 - u_1\| + \|g'_v\|_\circ \|v_2 - v_1\|)$$

whence

$$\|v_2 - v_1\| \leq \frac{\|(A^+)^{-1}\| \, \|g'_u\|_\circ \|u_2 - u_1\|}{1 - \|(A^+)^{-1}\| \, \|g'_v\|_\circ} . \tag{3.6.11}$$

For $\|u_2 - u_1\| = \|\varphi_{j+1}(v_2) - \varphi_j(v_1)\|$ we have the following estimate

$$\|\varphi_{j+1}(v_2) - \varphi_j(v_1)\| \leq \|\varphi_{j+1}(v_2) - \varphi_{j+1}(v_1)\| + \|\varphi_{j+1}(v_1) - \varphi_j(v_1)\| .$$

The first term in this sum is estimated by

$$\|\varphi_{j+1}(v_2) - \varphi_{j+1}(v_1)\| \le \|\varphi'_{j+1}\|_\circ \|v_2 - v_1\| \le L\|v_2 - v_1\|$$

and the second by

$$\|\varphi_{j+1}(v_1) - \varphi_j(v_1)\| \le \rho\,,$$

where $\rho = \sup\limits_{v\in D_2} \|\varphi_{j+1}(v) - \varphi_j(v)\|$. Thus, we obtain

$$\|u_2 - u_1\| \le \rho + L\|v_2 - v_1\|\,. \tag{3.6.12}$$

It follows from (3.6.11) that

$$\|v_2 - v_1\| \le \frac{\rho\|(A^+)^{-1}\|\,\|g'_u\|_\circ}{1 - \|(A^+)^{-1}\|(\|g'_v\|_\circ + L\|g'_u\|_\circ))}$$

which, after substitution into (3.6.10) with (3.6.12), gives

$$\|\bar{u}_2 - \bar{u}_1\| \le \rho(\|A^-\| + \cdots)$$

where the ellipsis denotes terms of the order $\|(f,g)'_{(u,v)}\|_\circ$. Since $\|A^-\| < 1$, we have

$$\|\bar{u}_2 - \bar{u}_1\| = \|\varphi_{j+2}(\bar{v}) - \varphi_{j+1}(\bar{v})\| \le K \sup \|\varphi_{j+1}(v) - \varphi_j(v)\|$$

for some $K < 1$ which does not depend on $\bar{v}$ (provided that $\|\bar{v}\| \le \delta$). If we take the supremum in the left-hand side of this inequality with respect to all $\bar{v}$, then we obtain the desired inequality (3.6.9).

From (3.6.9) we obtain

$$\|\varphi_{j+2}(v) - \varphi_{j+1}(v)\| \le K^j \sup \|\varphi_2(v) - \varphi_1(v)\|\,,$$

i.e. the series

$$\sum_{j=1}^{\infty}(\varphi_{j+1}(v) - \varphi_j(v))$$

is majorized by a geometrical progression with the coefficient $K < 1$, and, therefore, converges uniformly. Since the partial sums of this series are $(\varphi_{j+1}(v) - \varphi_1(v))$, its uniform convergence implies the uniform convergence of the sequence $\{\varphi_j\}$ to some limit function φ^*.

Step 4. Let us denote the graph $u = \varphi^*(v)$ of the function φ^* by W^u_{loc}. By construction, this graph is invariant with respect to the map T. Note that as $\varphi^*(v)$ is the uniform limit of a sequence of continuous functions, it is also continuous. Generally speaking, the smoothness of the function φ^* does not follow from our arguments (a limit of a series of smooth functions may be non-smooth). Nevertheless, we remark that the derivatives of all functions φ_j are bounded by the same constant L:

$$\|\varphi_j'(v)\| \leq L\,.$$

It follows from the inequality

$$\|\varphi_j(v_1) - \varphi_j(v_2)\| \leq \|\varphi_j'\|_\circ \|v_1 - v_2\|$$

that all functions φ_j satisfy a Lipschitz condition

$$\|\varphi_j(v_1) - \varphi_j(v_2)\| \leq L\|v_1 - v_2\|$$

for any (v_1, v_2) in D_2. If we take a limit of this inequality as $j \to \infty$, we obtain

$$\|\varphi^*(v_1) - \varphi^*(v_2)\| \leq L\|v_1 - v_2\|\,.$$

Thus, we have established the existence of the invariant manifold W^u_{loc} satisfying a Lipschitz condition. It should be noted that one can choose the initial surface $\mathcal{W}$ such that it passes through the point O. It is obvious then that all iterations of the surface $\mathcal{W}$ also contain the point O, and hence, the limit surface W^u_{loc} also contains O.

Before we prove the smoothness of the invariant manifolds, let us examine the behavior of the map in its restriction to W^u_{loc} and W^s_{loc}. The restriction of the map to W^s_{loc} is given by the formula

$$\bar{u} = A^- u + f(u,\, \psi^*(u))\,. \tag{3.6.13}$$

Hence it follows from (3.6.2) that

$$\|\bar{u}\| \leq \|A^-\|\,\|u\| + \|f_u'\|\,\|u\| + N\,\|f_v'\|\,\|u\| \leq \big(\|A^-\| + \|f_u'\| + N\|f_v'\|\big)\|u\|\,.$$

Thus, since $\|A^-\| < 1$ and since the norm $\|f_u'\|$ is small, the iterations of any point on W^s_{loc} converge exponentially to the point O under the action of the map T.

By the symmetry, an analogous result may be obtained for the map (3.6.1) in its restriction to W^u_{loc}:

$$\bar{v} = A^+ v + g(\varphi^*(v), v)\,, \tag{3.6.14}$$

namely, for any point $\bar{v}$ on W^u_{loc} there exists a uniquely defined image $v = T^{-1}\bar{v}$ which iterations under the map T^{-1} tend uniformly and exponentially to the point O.

Theorem 3.8. *The invariant manifolds W^s_{loc} and W^u_{loc} belong to the $\mathbb{C}^r$-class of smoothness and at the point O they are tangent respectively to the stable eigen-subspace $v = 0$ and the unstable eigen-subspace $u = 0$, i.e.*

$$\varphi^{*}_v{}'(0) = 0\,, \quad \psi^{*}_u{}'(0) = 0\,.$$

Proof. As above we will only prove that part of this theorem which concerns W^u_{loc}. Smoothness of the manifold W^s_{loc} follows from the symmetry of the problem. The invariance of the manifold W^u_{loc} implies that if some point $M(u,v)$ belongs to W^u_{loc}, *i.e.* $u = \varphi^*(v)$ and if its image $\bar{M}(\bar{u},\bar{v})$ remains in a δ-neighborhood of the point O, then the point $\bar{M}$ also belongs to W^u_{loc}, *i.e.* its coordinates satisfy $\bar{u} = \varphi^*(\bar{v})$.

From (3.6.1) we have

$$A^-\varphi^*(v) + f(\varphi^*(v), v) = \varphi^*(A^+v + g(\varphi^*(v), v))\,. \tag{3.6.15}$$

By formal differentiation of this identity we determine that if the function φ^* is differentiable, its derivative $\eta^* \equiv d\varphi^*/dv$ satisfies the identity

$$\begin{aligned} &A^-\eta^*(v) + f'_u(\varphi^*(v), v)\eta^*(v) + f'_v(\varphi^*(v), v) \\ &\qquad = \eta^*(\bar{v})\left[I + (g'_u(\varphi^*(v), v)\eta^*(v) + g'_v(\varphi^*(v), v))(A^+)^{-1}\right]A^+\,, \end{aligned} \tag{3.6.16}$$

where I is the identity $(n-k)\times(n-k)$-matrix, and the value of $\bar{v}$ being given by formula (3.6.14).

We show below that there exists a continuous function $\eta^*(v)$ satisfying (3.6.16) such that $\eta^*(0) = 0$, and that this function is the derivative of $\varphi^*(v)$, thereby establishing C^1-smoothness of the manifold W^u_{loc}. Later, by induction, we prove that W^u_{loc} is C^r-smooth.

Step 1. Formula (3.6.16) implies that the graph $\eta = \eta^*(v)$ of the derivative of the function defining the invariant manifold is itself an invariant manifold of the map T^*: $(v,\eta) \mapsto (\bar{v},\bar{\eta})$, where $\bar{v}$ is given by Eq. (3.6.14) and $\bar{\eta}$ is given by the following equation

$$\begin{aligned} \bar{\eta} = &\left[A^-\eta(A^+)^{-1} + (f'_u(\varphi^*(v), v)\eta + f'_v(\varphi^*(v), v))(A^+)^{-1})\right] \\ &\times\left[I + (g'_u(\varphi^*(v), v)\eta + g'_v(\varphi^*(v), v))(A^+)^{-1}\right]^{-1}\,. \end{aligned} \tag{3.6.17}$$

The map T^* can be represented schematically in the form

$$\begin{aligned}\bar{\eta} &= A^-\eta(A^+)^{-1} + F(v,\eta)\,,\\ \bar{v} &= A^+v + G(v)\,,\end{aligned} \tag{3.6.18}$$

where F and G are certain continuous functions

$$F(0,0) = 0\,,\quad G(0) = 0\,. \tag{3.6.19}$$

Moreover, F depends smoothly on η and

$$F'_\eta(0,0) = 0\,. \tag{3.6.20}$$

The function G satisfies a Lipschitz condition with a constant ε which may be made infinitesimally small by decreasing the size of the δ-neighborhood of the point O:

$$\begin{aligned}\|G(v_2) - G(v_1)\| &\equiv \|g(\varphi^*(v_2), v_2) - g(\varphi^*(v_1), v_1)\|\\ &\le \|g'_u\|_\circ\, \|\varphi^*(v_2) - \varphi^*(v_1)\| + \|g'_v\|_\circ\, \|v_2 - v_1\|\\ &\le (\|g'_u\|_\circ\, L + \|g'_v\|_\circ)\, \|v_2 - v_1\| \le \varepsilon\|v_2 - v_1\|\end{aligned} \tag{3.6.21}$$

(see (3.6.14), (3.6.3)).

The point ($v = 0$, $\eta = 0$) is a fixed point of map (3.6.18). The existence of an invariant manifold $\eta = \eta^*(v)$ of the map T^* which passes through this point can easily be proven by repeating the arguments used in proving the existence of the invariant manifold of the map (3.6.1). Indeed, by using the fact that $\|A^-\| < 1$ and $\|(A^+)^{-1}\| < 1$, and also the relations (3.6.19)–(3.6.21), one can directly verify that by choosing an arbitrary continuous surface of the form $\eta = \eta_1(v)$ such that $\|\eta_1(v)\| \le L$, the image of this surface under the map T^* is a surface of the same type. One may therefore consider the sequence of the surfaces $\{\eta = \eta_j(v)\}$ obtained from the initial one by successive iterations of the map T^*. It may be verified that this sequence satisfies an inequality of the kind (3.6.9) from which it follows that it converges to a continuous surface $\eta = \eta^*(v)$ for which

$$\|\eta^*(v)\| \le L\,. \tag{3.6.22}$$

By construction, this surface is invariant with respect to the map T^*, *i.e.* (3.6.16) is satisfied.[3]

[3]Here, in contrast to the map (3.6.1), in order to prove inequality (3.6.9) which is essential for the convergence of successive approximations, the functions η_j are no longer required to be smooth and to have the bounded derivatives. The reason is that the map T^* is a triangular map, and the second equation in (3.6.18) does not depend on the variable η.

Step 2. We have established the existence of a continuous bounded solution η^* of the functional equation (3.6.18) which is thereby a formal derivative of the function φ^*. Let us now show that η^* is really the derivative of φ^*. Consider the value

$$z(v) = \overline{\lim_{\|\Delta v\| \to 0}} \frac{\|\varphi^*(v + \Delta v) - \varphi^*(v) - \eta^*(v)\|\Delta v\|\|}{\|\Delta v\|} . \tag{3.6.23}$$

By the definition of the derivative, $\eta^* \equiv d\varphi^*/dv$ if and only if $z(v) \equiv 0$. Let us prove this identity. To begin let us note that the value of z is bounded: by virtue of (3.6.3) and (3.6.22)

$$\|\varphi^*(v + \Delta v) - \varphi^*(v) - \eta^*(v)\Delta v\| \le 2L\|\Delta v\| . \tag{3.6.24}$$

Let us determine the relation between the values of $z(v)$ and $z(\bar{v})$, where $\bar{v}$ is given by (3.6.14). From (3.6.14) we have

$$\begin{aligned} \Delta\bar{v} = \big(A^+ + g'_v(\varphi^*(v), v) + g'_u(\varphi^*(v), v)\eta^*(v)\big)\, \Delta v \\ + g'_u(\varphi^*(v), v)(\Delta\varphi - \eta^*(v)\Delta v) + o(\Delta v) , \end{aligned} \tag{3.6.25}$$

where $\Delta\varphi \equiv \varphi^*(v + \Delta v) - \varphi^*(v)$.

From (3.6.25) and (3.6.16) we obtain

$$\begin{aligned} \eta^*(\bar{v})\Delta\bar{v} = \big(A^- + f'_u\,(\varphi^*(v), v)\big)\eta^*(v)\Delta v + f'_v(\varphi^*(v), v)\Delta v \\ + \eta^*(\bar{v})g'_u(\varphi^*(v), v)\,(\Delta\varphi - \eta^*(v)\Delta v) + o(\Delta v) . \end{aligned} \tag{3.6.26}$$

From (3.6.15) we find

$$\varphi^*(\bar{v} + \Delta\bar{v}) - \varphi^*(\bar{v}) = \big(A^- + f'_u(\varphi^*(v), v)\big)\Delta\varphi + f'_v(\varphi^*(v), v)\Delta v + o(\Delta v) .$$

Now it follows that

$$\begin{aligned} &\Delta\bar{\varphi} - \eta^*(\bar{v})\Delta\bar{v} \\ &\quad = (A^- + f'_u(\varphi^*(v), v) - \eta^*(\bar{v})f'_u(\varphi^*(v), v))(\Delta\varphi - \eta^*(v)\Delta v) + o(\Delta v) \end{aligned}$$

whence

$$\|\Delta\bar{\varphi} - \eta^*(\bar{v})\Delta\bar{v}\| \le (\|A^-\| + \cdots)\|\Delta\varphi - \eta^*(v)\Delta v\| + o(\Delta v) , \tag{3.6.27}$$

where the ellipsis denotes terms of order $\|(f, g)'_{(u,v)}\|$.

From (3.6.24) and (3.6.25) we have the following estimate for $\|\Delta v\|$:

$$\|\Delta v\| \leq (\|(A^+)^{-1}\| + \cdots)\|\Delta \bar{v}\|\,.$$

From there and (3.6.27), and by definition of the function z, we obtain

$$z(\bar{v}) \leq (\|A^-\|\,\|(A^+)^{-1}\| + \cdots)\, z(v)\,. \tag{3.6.28}$$

It was already noted that for any point $\bar{v}$ which does not exceed δ in the norm, there exists a pre-image v such that $\|v\| \leq \delta$. Therefore, for any point v_0 the infinite sequence $\{v_j\}$ is defined such that $v_j = \bar{v}_{j+1}$. By virtue of (3.6.28)

$$z(v_0) \leq \left(\|A^-\|\,\|(A^+)^{-1}\| + \cdots\right)^j z(v_j)\,,$$

and since $z(v_j)$ is bounded and $\|A^-\| < 1$, $\|(A^+)^{-1}\| < 1$, it follows that $z(v_0) = 0$. As v_0 was chosen arbitrarily, it follows that $z(v) \equiv 0$, *i.e.* the smoothness of the function φ^* is established.

It should be noted that the statement concerning the existence of the invariant manifold $\{\eta = \eta^*(v)\}$ of the map (3.6.18) is, generally speaking, satisfied only for sufficiently small $v\colon \|v\| \leq \delta_1$, $\delta_1 > 0$. We have disregarded the fact that the value δ_1 may be less than δ, which is the diameter of the neighborhood of the origin in which the function φ^* is defined. Nevertheless, one can show that the function φ^* is a smooth function for all v in a δ-neighborhood of the origin. To do this we first note that since the backward iterations of any point $\bar{v}$ in W^u_{loc} in a δ-neighborhood of the origin converge uniformly to the point O, the image of the δ_1-neighborhood of the origin on W^u_{loc} will cover the δ-neighborhood after a number of forward iterations of the map T. Thus it is implied that because the map T is smooth and because in the δ_1-neighborhood of the point O the manifold W^u_{loc} is also smooth, W^u_{loc} is smooth inside the original δ-neighborhood.

Step 3. We have established that the map T has a smooth invariant manifold of the form $u = \varphi^*(v)$. Moreover, the graph of the derivative $\eta^* = d\varphi^*/dv$ is itself an invariant manifold of the map T^* given by formulae (3.6.17) and (3.6.18). If the smoothness of the right-hand side of the map T is greater than one, then the right-hand side of the map T^* belongs to the $\mathbb{C}^1$-class (because it is expressed in terms of φ^* and g). As the fixed point $(v = 0,\ \eta = 0)$ of the map T^* is a saddle point, all the arguments used for the map T can be repeated, leading to the conclusion that the invariant manifold $\eta = \eta^*(v)$ of the map T^* is smooth and consequently that the function φ^* belongs to the $\mathbb{C}^2$-class.[4]

Thus, when the smoothness of the right-hand side of the map T is greater than two, the right-hand side of the map T^* is then already $\mathbb{C}^2$-smooth. Therefore, by virtue of the previous arguments the function η^* is of $\mathbb{C}^2$-class, and the function φ^* is of $\mathbb{C}^3$-class, respectively, and so on. By induction we arrive at the existence of a $\mathbb{C}^r$-smooth invariant manifold W^u_{loc}.
End of the proof.

As in the case of equilibrium states, the invariant manifolds W^s_{loc} and W^u_{loc} can be locally straightened by a change of variables:

$$\begin{aligned} \xi &= u - \varphi^*(v)\,, \\ \eta &= v - \psi^*(u)\,. \end{aligned}$$

In the new variables the invariant manifolds take the form

$$W^s_{loc}\colon \eta = 0\,, \quad \text{and} \quad W^u_{loc}\colon \xi = 0\,.$$

The invariance of the manifolds implies that $\bar{\eta} = 0$ when $\eta = 0$ and $\bar{\xi} = 0$ when $\xi = 0$. In terms of the variables ξ and η, the original system recasts in the form

$$\begin{aligned} \bar{\eta} &= (A^- + h_1(\xi, \eta))\xi\,, \\ \bar{\xi} &= (A^+ + h_2(\xi, \eta))\eta\,, \end{aligned} \tag{3.6.29}$$

where $h_i \in \mathbb{C}^{r-1}$ and

$$h_i(0, 0) = 0\,, \quad i = 1, 2\,. \tag{3.6.30}$$

In a small neighborhood of the saddle the functions $h_{1,2}$ are small in norm and as long as a trajectory remains in a neighborhood of the saddle, the inequalities

$$\|\bar{\xi}\| \leq (|\lambda_1| + \varepsilon)\|\xi\|$$

and

$$\|\bar{\eta}\| \geq (|\gamma_1| - \varepsilon)\|\eta\|$$

[4] With the only difference being that the linear part of the map T^* is not block-diagonal, whence $\frac{d\eta^*}{dv}(0) \neq 0$, in general.

hold in the Jordan basis. Hence, we obtain

$$\|\xi_j\| \leq (|\lambda_1| + \varepsilon)^j \|\xi_0\| \qquad \text{for } j \geq 0 \tag{3.6.31}$$

$$\|\eta_j\| \leq (|\gamma_1| - \varepsilon)^{|j|} \|\eta_0\| \qquad \text{for } j \leq 0 \tag{3.6.32}$$

(see the proof of the analogous formulae (3.5.4) and (3.5.25) in the previous section). Thus, a trajectory that lies in neither W^s_{loc} nor in W^u_{loc}, leaves a neighborhood of the saddle as $j \to \pm\infty$. Moreover, the number of iterations needed for a forward trajectory to escape from the neighborhood of the saddle is of the order $\ln \|\eta_0\|$, and that for a backward trajectory is of the order $\ln \|\xi_0\|$.

The map (3.6.1) on the stable manifold W^s_{loc}: $v = \psi^*(u)$ is given by

$$\bar{u} = A^- u + f(u, \psi^*(u)) \,. \tag{3.6.33}$$

On W^s_{loc} the point O is a stable fixed point. In general, this point is either a node (provided that only one coordinate is leading), or a focus (when there are two leading coordinates corresponding to a pair of complex-conjugate multipliers).

The map (3.6.1) on W^u_{loc} is given by

$$\bar{v} = A^+ v + g(\varphi^*(v), v) \,. \tag{3.6.34}$$

Here, the point O is a completely unstable fixed point and in the generic case, it is either a node or a focus.

We can now identify nine main types of saddle fixed points depending on the behavior of trajectories in the leading coordinates:

(1) *a saddle* $(+,+)$: a node $(+)$ on both W^s_{loc} and W^u_{loc};

(2) *a saddle* $(-,-)$: a node $(-)$ on both W^s_{loc} and W^u_{loc};

(3) *a saddle* $(+,-)$: a node $(+)$ on W^s_{loc} and a node $(-)$ on W^u_{loc};

(4) *a saddle* $(-,+)$: a node $(-)$ on W^s_{loc} and a node $(+)$ on W^u_{loc};

(5) *a saddle-focus* $(2, 1+)$: a focus on W^s_{loc} and a node $(+)$ on W^u_{loc};

(6) *a saddle-focus* $(2, 1-)$: a focus on W^s_{loc} and a node $(-)$ on W^u_{loc};

(7) *a saddle-focus* $(1+, 2)$: a node $(+)$ on W^s_{loc} and a focus on W^u_{loc};

(8) *a saddle-focus* $(1-, 2)$: a node $(-)$ on W^s_{loc} and a focus on W^u_{loc};

(9) *a saddle-focus* (2,2): a focus on both W^s_{loc} and W^u_{loc}.

Theorems 3.4 and 3.5 are valid for systems (3.6.33) and (3.6.34). This implies that in W^s_{loc} and W^u_{loc} there exists *a non-leading stable invariant submanifold* W^{ss}_{loc}, *a leading stable invariant submanifold* W^{sL}_{loc}, *a non-leading unstable invariant submanifold* W^{uu}_{loc}, and *a leading unstable invariant submanifold* W^{uL}_{loc}. We further select an extra three smooth invariant manifolds of a saddle fixed point. Introduce the notations:

$$r_{sL} = \left[\frac{\ln\hat{\lambda}}{\ln|\lambda_1|}\right], \tag{3.6.35}$$

$$r_{uL} = \left[\frac{\ln\hat{\gamma}}{\ln|\gamma_1|}\right], \tag{3.6.36}$$

where $\hat{\lambda}$ and $\hat{\gamma}$ respectively are the absolute values of non-leading stable and unstable multipliers nearest to the unit circle; $[x]$ denotes the largest integer strictly less than x.

Theorem 3.9. *In a neighborhood of a structurally stable fixed point of the saddle type of a* $\mathbb{C}^r$*-smooth map there exists the following invariant manifolds*:

1. *a* $\mathbb{C}^{\min(r,r_{uL})}$*-smooth extended stable manifold* W^{sE}_{loc} *which contains* W^s_{loc}, *and which is tangent at the point* O *to the extended stable eigen-subspace of the linearized system and transverse to* W^{uu}_{loc};
2. *a* $\mathbb{C}^{\min(r,r_{sL})}$*-smooth extended unstable manifold* W^{uE}_{loc} *which contains* W^u_{loc} *and which is tangent at* O *to the extended unstable eigen-subspace of the linearized system and transverse to* W^{ss}_{loc};
3. *a* $\mathbb{C}^{\min(r,r_{sL},r_{uL})}$*-smooth leading saddle manifold* $W^L_{loc} = W^{uE}_{loc} \cap W^{sE}_{loc}$.

See the proof in the Chap. 5. We note that, generally speaking, the manifold W^{sE}_{loc} is not unique but any two such manifolds are tangent to each other everywhere on W^s_{loc}. Analogously, any two manifolds W^{uE}_{loc} are tangent to each other on W^u_{loc}.

3.7. The boundary-value problem near a saddle fixed point

Let us consider the map T of class $\mathbb{C}^r$ $(r \geq 1)$

$$\begin{aligned} \bar{u} &= A^- u + f(u,v)\,, \\ \bar{v} &= A^+ v + g(u,v)\,, \end{aligned} \tag{3.7.1}$$

where $u \in \mathbb{R}^{m_1}$ and $v \in \mathbb{R}^{m_2}$. Let $O(0,0)$ be a saddle fixed point of the map T, *i.e.* $\mathrm{spectr}A^- = \{\lambda_1, \ldots, \lambda_{m_1}\}$ lies strictly inside and $\mathrm{spectr}A^+ = \{\gamma_1, \ldots, \gamma_{m_2}\}$ lies outside of the unit circle. Assume that the functions f and g vanish at the origin along with their first derivatives.

Just like in the case of a saddle equilibrium state which we have examined in Sec. 2.8, exponential instability near a saddle fixed point is a typical feature of the trajectories of the map (3.7.1). Therefore, in this case, instead of the initial-value problem it is quite reasonable to solve *the boundary-value problem* which can be formalized in the following way:

For any u^0 and v^1, and for an arbitrary $k > 0$ find a trajectory

$$\{(u_0, v_0), (u_1, v_1), \ldots, (u_k, v_k)\}$$

of the map (3.7.1) *in a neighborhood of the point* $O(0,0)$ *such that*

$$u_0 \equiv u^0 \,, \quad v_k \equiv v^1 \,, \tag{3.7.2}$$

where we assume that $\|u^0\| \leq \varepsilon$ and $\|v^1\| \leq \varepsilon$ for some sufficiently small $\varepsilon > 0$.

A trajectory $\{(u_j, v_j)\}_{j=0}^k$ of the map (3.7.1) is given by

$$\begin{aligned} u_{j+1} &= A^- u_j + f(u_j, v_j) \,, \\ v_{j+1} &= A^+ v_j + g(u_j, v_j) \,. \end{aligned} \tag{3.7.3}$$

In the linear case a solution of the boundary-value problem is trivially found:

$$u_j = (A^-)^j u^0 \,, \quad v_j = (A^+)^{-(k-j)} v^1 \,. \tag{3.7.4}$$

Since $\|(A^-)^j\|$ and $\|(A^+)^{-(k-j)}\|$ are bounded for all $0 \leq j \leq k$, the solution of the linear problem is stable with respect to perturbations of the initial conditions u^0 and v^1. The validity of this statement in the nonlinear case is established by the following theorem.

Theorem 3.10. *For sufficiently small $\varepsilon > 0$ and u^0, v^1 such that $\|u^0\| \leq \varepsilon$ and $\|v^1\| \leq \varepsilon$, a solution of the boundary-value problem* (3.7.2) *for the map* (3.7.1) *exists for any positive integer k. The solution is unique and depends continuously on* (u^0, v^1).

Proof. We shall seek for the solution

$$\{(u_0, v_0),\, (u_1, v_1), \ldots, (u_k, v_k)\}$$

of the boundary-value problem (3.7.2) and (3.7.3) as a solution of the following system of equations (see the analogous integral equations for the boundary-value problem near a saddle equilibrium state in Sec. 2.8)

$$
\begin{aligned}
u_j &= (A^-)^j u^0 + \sum_{s=0}^{j-1} (A^-)^{j-s-1} f(u_s, v_s)\,, \\
v_j &= (A^+)^{-(k-j)} v^1 - \sum_{s=j}^{k-1} (A^+)^{-(s+1-j)} g(u_s, v_s)
\end{aligned}
\tag{3.7.5}
$$

with respect to the variables $\{(u_j, v_j)\}$ $(j = 0, 1, \ldots, k)$. Observe that this system is derived directly from the relations (3.7.3); namely we have for u_j:

$$
\begin{aligned}
u_j &= A^- u_{j-1} + f(u_{j-1}, v_{j-1}) \\
&= (A^-)^2 u_{j-2} + A^- f(u_{j-2}, v_{j-2}) + f(u_{j-1}, v_{j-1}) \\
\cdots &= (A^-)^j u_0 + (A^-)^{j-1} f(u_0, v_0) + \cdots + f(u_{j-1}, v_{j-1})
\end{aligned}
$$

and for v_j:

$$
\begin{aligned}
v_j &= (A^+)^{-1} v_{j+1} - (A^+)^{-1} g(u_j, v_j) \\
&= (A^+)^{-2} v_{j+2} - (A^+)^{-2} g(u_{j+1}, v_{j+1}) - (A^+)^{-1} g(u_j, v_j) \\
\cdots &= (A^+)^{j-k} v_k - (A^+)^{j-k} g(u_{k-1}, v_{k-1}) - \cdots - (A^+)^{-1} g(u_j, v_j)\,.
\end{aligned}
$$

It is evident that $u_0 \equiv u^0$ and $v_k \equiv v^1$ for any solution (3.7.4). Thus, the sequence $\{(u_0, v_0), (u_1, v_1), \ldots, (u_k, v_k)\}$ is a solution of the boundary value problem if and only if it satisfies (3.7.5).

Let us construct a solution of the system (3.7.5) by the method of successive approximations. The first approximation is chosen as solution (3.7.4) of the linear boundary-value problem. Successive approximations will be calculated according to the formula

$$
\begin{aligned}
u_j^{(n+1)} &= (A^-)^j u^0 + \sum_{s=0}^{j-1} (A^-)^{j-s-1} f\Big(u_s^{(n)}, v_s^{(n)}\Big)\,, \\
v_j^{(n+1)} &= (A^+)^{j-k} v^1 - \sum_{s=j}^{k-1} (A^+)^{j-s-1} g\Big(u_s^{(n)}, v_s^{(n)}\Big)\,, \\
& (j = 0, 1, \ldots, k)\,.
\end{aligned}
\tag{3.7.6}
$$

Let us now show that the resulting sequence converges uniformly to some limit vector

$$z_0^* = \left\{(u_i^*, v_i^*)\right\}_{i=0}^{i=k}.$$

We first prove that

$$\|u_j^{(n)}\| \le 2\varepsilon\,, \quad \|v_j^{(n)}\| \le 2\varepsilon \tag{3.7.7}$$

for all n and $0 \le j \le k$.

When $n = 1$ it follows directly from the fact that $\|u^0\| \le \varepsilon$, $\|v^1\| \le \varepsilon$ and also from the inequalities

$$\|(A^-)^j\| \le \lambda^j\,, \quad \|(A^+)^{j-k}\| \le \gamma^{j-k}\,, \tag{3.7.8}$$

where $0 < \lambda < 1$ and $\gamma > 1$ are such numbers that spectrA^- lies strictly inside the circle of diameter λ, and spectrA^+ lies outside the circle of diameter γ.

We will prove inequality (3.7.7) for all n by induction. Since both functions f and g vanish at the point O along with their first derivatives, the inequalities[5]

$$\left\|\frac{\partial(f,g)}{\partial(u,v)}\right\| \le \delta\,, \quad \|f,g\| \le \delta\|u,v\| \tag{3.7.9}$$

are satisfied, where δ may be made arbitrarily small by decreasing the size of the neighborhood of the point O. Choose ε sufficiently small so that for any u and v in the 2ε-neighborhood of the saddle the inequality

$$2\delta \max\left(\frac{1}{1-\lambda}, \frac{1}{1-\gamma^{-1}}\right) \le 1 \tag{3.7.10}$$

holds. From (3.7.6), (3.7.8) and (3.7.9) we obtain

$$\|u_j^{(n+1)}\| \le \lambda^j\|u^0\| + \delta\sum_{s=0}^{j-1}\lambda^{j-s-1}\|u_s^{(n)}, v_s^{(n)}\|$$

$$\|v_j^{(n+1)}\| \le \gamma^{j-k}\|v^1\| + \delta\sum_{s=j}^{k-1}\gamma^{j-s-1}\|u_s^{(n)}, v_s^{(n)}\|$$

from which it follows that

$$\|u_j^{(n+1)}, v_j^{(n+1)}\| \le \varepsilon + \delta\,\max\left(\frac{1}{1-\lambda}, \frac{1}{1-\gamma^{-1}}\right)\max_{0\le s\le j}\|u_s^{(n)}, v_s^{(n)}\|\,.$$

[5] Hereafter $\|u,v\|$ denotes $\max\{\|u\|, \|v\|\}$.

By virtue of (3.7.10) we have that if $\|u_s^{(n)}, v_s^{(n)}\| \le 2\varepsilon$, then $\|u_j^{(n+1)}, v_j^{(n+1)}\| \le 2\varepsilon$, which implies that the inequalities (3.7.7) hold for all n.

We prove now that

$$\max_{0\le j\le k} \|u_j^{(n+1)} - u_j^{(n)}, v_j^{(n+1)} - v_j^{(n)}\| \le \frac{1}{2} \max_{0\le s\le k} \|u_s^{(n)} - u_s^{(n-1)}, v_s^{(n)} - v_s^{(n-1)}\| . \tag{3.7.11}$$

Since the variables $\left(u_j^{(n)}, v_j^{(n)}\right)$ lie in the 2ε-neighborhood of the saddle for all n, it follows that the estimates (3.7.9) are valid for values $f\left(u_j^{(n)}, v_j^{(n)}\right)$ and $g\left(u_j^{(n)}, v_j^{(n)}\right)$. Now, from (3.7.6) and (3.7.10) we obtain

$$\begin{aligned}
&\|u_j^{(n+1)} - u_j^{(n)}\| \\
&\quad \le \sum_{s=0}^{j-1} \lambda^{j-s-1} \|f\left(u_s^{(n)}, v_s^{(n)}\right) - f\left(u_s^{(n-1)}, v_s^{(n-1)}\right)\| \\
&\quad \le \frac{\delta}{1-\lambda} \max_{0\le s\le j} \|u_s^{(n)} - u_s^{(n-1)}, v_s^{(n)} - v_s^{(n-1)}\| \\
&\quad \le \frac{1}{2} \max_{0\le s\le j} \|u_s^{(n)} - u_s^{(n-1)}, v_s^{(n)} - v_s^{(n-1)}\| .
\end{aligned}$$

An analogous estimate applies to $\|v_j^{(n+1)} - v_j^{(n)}\|$.

It follows from (3.7.11) that the norms of the differences $\|u_j^{(n+1)} - u_j^{(n)}\|$ and $\|v_j^{(n+1)} - v_j^{(n)}\|$ decay in a geometric progression. Therefore, the series

$$\sum_{n=1}^{\infty} \left(u_j^{(n+1)} - u_j^{(n)}, v_j^{(n+1)} - v_j^{(n)}\right) \tag{3.7.12}$$

converges uniformly with respect to j as well as to u^0, v^1 and k in view of the obvious relation

$$\left(u_j^{(p)}, v_j^{(p)}\right) = \left(u_j^{(1)}, v_j^{(1)}\right) + \sum_{n=1}^{p-1} \left(u_j^{(n+1)} - u_j^{(n)}, v_j^{(n+1)} - v_j^{(n)}\right) .$$

The sequence $\left(u_j^{(n)}, v_j^{(n)}\right)$ converges to some vector $\{(u_j^*, v_j^*)\}_{i=0}^k$ as $n \to \infty$, which is a solution of the system (3.7.5) as well as a solution of the boundary-value problem. Since the convergence of the successive approximations is uniform, the solution (u_j^*, v_j^*) depends continuously on u^0 and v^1.

To prove the uniqueness of the solution, suppose that the system (3.7.5) has one more solution $\{(u_j^{**}, v_j^{**})\}_{i=0}^k$. Then, in the same manner as in the proof of inequality (3.7.11), one may show that

$$\|u_j^{**} - u_j^*, v_j^{**} - v_j^*\| \leq \frac{1}{2} \max_{0 \leq s \leq k} \|u_s^{**} - u_s^*, v_s^{**} - v_s^*\|$$

for all $j \in \{0, \ldots, k\}$. Hence the identities $u_j^{**} \equiv u_j^*$ and $v_j^{**} \equiv v_j^*$ hold. End of the proof.

By analogy with the proof of the smoothness of solutions of the boundary-value problem near a saddle equilibrium state presented in Sec. 2.8, it is possible to show that the solution of the boundary-value problem (3.7.2) and (3.7.3) near the saddle fixed point depends $\mathbb{C}^r$-smoothly on the initial condition (u^0, v^1). The derivatives with respect to u^0 and v^1 are determined as solutions (unique) of the boundary-values problems obtained by a formal differentiation of the relations (3.7.2) and (3.7.3). Hence, the derivatives $\partial u_j^*/\partial u^0$ and $\partial v_j^*/\partial u^0$ are solutions of the following system

$$\begin{aligned} U_{j+1} &= A^- U_j + f'_u\big(u_j^*, v_j^*\big) U_j + f'_v\big(u_j^*, v_j^*\big) V_j\,, \\ V_{j+1} &= A^+ V_j + g'_u\big(u_j^*, v_j^*\big) U_j + g'_v\big(u_j^*, v_j^*\big) V_j\,, \\ &\quad (j = 0, \ldots, k)\,, \end{aligned} \tag{3.7.13}$$

with the boundary conditions

$$U_0 = I_{m_1}\,, \quad V_k = 0\,, \tag{3.7.14}$$

where $U_j \equiv \partial u_j^*/\partial u^0$ and $V_j \equiv \partial v_j^*/\partial u^0$.

Just as we have found the solution of the boundary-value problem (3.7.2), (3.7.3) as a solution of the system (3.7.5), we will find a solution of the boundary-value problem (3.7.13), (3.7.14) as a solution of the system

$$\begin{aligned} U_j &= (A^-)^j + \sum_{s=0}^{j-1} (A^-)^{j-s-1} (f'_u(u_s^*, v_s^*) U_s + f'_v(u_s^*, v_s^*) V_s) \\ V_j &= -\sum_{s=j}^{k-1} (A^+)^{j-s-1} (g'_u(u_s^*, v_s^*) U_s + g'_v(u_s^*, v_s^*) V_s)\,. \end{aligned} \tag{3.7.15}$$

The convergence of successive approximations may be proven in the same manner as in Theorem 3.10, *i.e.* it may be shown that the norms of $\|U_j^{(n+1)} - U_j^{(n)}\|$ and $\|V_j^{(n+1)} - V_j^{(n)}\|$ decrease in a geometric progression.

The derivatives $\partial u_j^*/\partial v_1$ and $\partial v_j^*/\partial v_1$ are also found as solutions of the system (3.7.13) but with other boundary conditions:

$$V_k = I_{m_2}\,, \quad \text{and} \quad U_0 = 0\,. \tag{3.7.16}$$

The method of the boundary-value problem enables us to establish a very important geometrical result concerning the properties of the saddle map, called λ-*lemma*. For convenience, let us make a $\mathbb{C}^r$-smooth change of coordinates which straightens the stable and the unstable manifolds in some ε-neighborhood D_ε of the saddle fixed point (see Sec. 3.6). In terms of the new coordinates, the functions f, g in (3.7.1) vanish at the origin along with their first derivatives. Moreover, everywhere in D_ε

$$f(0,v) \equiv 0\,, \quad g(u,0) \equiv 0\,. \tag{3.7.17}$$

Thus, the equation of W^s_{loc} becomes $v = 0$ and the equation of W^u_{loc} becomes $u = 0$.

In the neighborhood D_ε we consider an arbitrary m_2-dimensional $\mathbb{C}^r$-smooth surface $H_0 : u = h_0(v)$ which intersects W^s_{loc} transversally at some point M.

Let us examine the sequence of the points $\{M, T(M), \ldots, T^k(M), \ldots\}$ which converges to O as $k \to +\infty$. The m_2-dimensional surfaces $T^k(H_0)$ pass through the corresponding points of this sequence. Let us denote by H_k a connected component of $T^k(H_0) \cap D_\varepsilon$ containing the point $T^k(M)$.

Lemma 3.2. (λ-*lemma*) *For all sufficiently large k the surface H_k is represented in the form $u = h_k(v)$, where the functions h_k tend uniformly to zero, along with all their derivatives, as $k \to +\infty$ (see Fig.* 3.7.1).[6]

Proof. Transversality of the surface H_0 with respect to the surface W^s_{loc}: $v = 0$ at the point $M(h_0(0), 0)$ implies that $\|h_0'(0)\|$ is bounded. Therefore, the norm $\|h_0'(v)\|$ is bounded for all sufficiently small v. Let us consider the surface H_k and choose an arbitrary point (u_k, v_k) on it. By construction, there always exists a point (u_0, v_0) on H_0 such that $T^k(u_0, v_0) = (u_k, v_k)$.

[6]In other words, the sequence H_k converges to W^u_{loc} in the $\mathbb{C}^r$-topology.

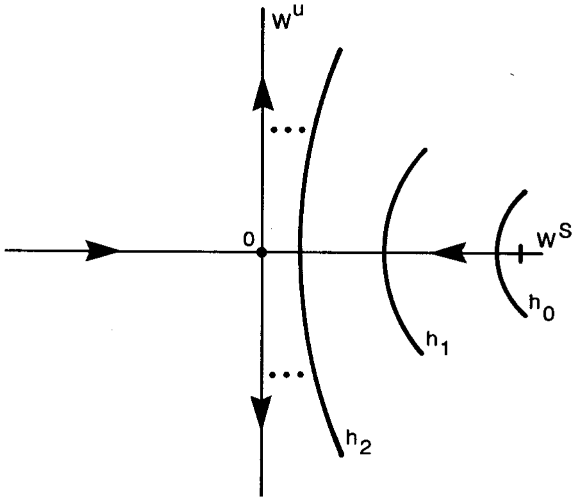


Fig. 3.7.1. A geometrical interpretation of the λ-lemma. With each successive iteration the graph of the surface $u = h_k(v)$ becomes flatter and flatter while approaching the unstable manifold W^u along the stable manifold W^s.

By virtue of Theorem 3.10 the map $T^k\colon (u_0, v_0) \mapsto (u_k, v_k)$ can be written for any positive k in the implicit form

$$u_k = \xi_k(u_0, v_k)\,, \quad v_0 = \eta_k(u_0, v_k)\,, \tag{3.7.18}$$

where ξ_k and η_k are $\mathbb{C}^r$-smooth functions. Below we show that as $k \to +\infty$, the norms of the functions ξ_k, η_k as well as the norms of their derivatives up to order r tend uniformly to zero.

Substituting $u_0 = h_0(v_0)$ into (3.7.18) gives that the points $(u_k, v_k) \in H_k$ and $(u_0, v_0) \in H_0$ are linked by the relations

$$u_k = \xi_k(h_0(v_0), v_k)\,, \quad v_0 = \eta_k(h_0(v_0), v_k)\,. \tag{3.7.19}$$

We have already noted that η_k along with all their derivatives up to order r converges to zero as $k \to +\infty$, whereas $\|h_0'\|$ remains bounded for small v_0. Hence, by virtue of the implicit function theorem, for sufficiently large k and any v_k whose norm does not exceed ε, the second equation in (3.7.19) can be uniquely resolved with respect to v_0: $v_0 = \varphi_k(v_k)$, where the function φ_k tends uniformly to zero along with all the derivatives up to order r as $k \to +\infty$.

The equation of the surface H_k can now be recast in the explicit form $u_k = \xi_k(h_0(\varphi_k(v_k)),\, v_k)$, what gives the lemma because the norms of the functions ξ_k and φ_k converge uniformly to 0 as $k \to +\infty$.

Thus, the proof of the λ-lemma is reduced to verifying that the norms $\|\xi_k\|$ and $\|\eta_k\|$ tend to 0 along with the norms of their derivatives. Let us prove this.

Lemma 3.3. *In the coordinate system where the stable and the unstable manifold are straightened, the norms of the functions $\|\xi_k\|$ and $\|\eta_k\|$ tend uniformly to zero as $k \to \infty$.*

Proof. Consider the system (3.7.5) which yields the solution of the boundary-value problem for the map T. We have $u_k \equiv \xi_k(u_0, v_k)$ and $v_0 \equiv \eta_k(u_0, v_k)$. We will show that solutions of system (3.7.5) satisfy the inequalities

$$\|u_j\| \leq K\bar{\lambda}^j\,, \quad \|v_j\| \leq K\bar{\gamma}^{j-k} \tag{3.7.20}$$

for some K and for some $\bar{\lambda} < 1$, $\bar{\gamma} > 1$. From the proof of Theorem 3.10 one can see that the solution of system (3.7.5) is found as the limit of the successive approximations $\left(u_j^{(n)}, v_j^{(n)}\right)$, which are calculated by formula (3.7.6). It is therefore sufficient to check that inequalities (3.7.20) hold for all steps of successive approximations with the same values of k, $\bar{\lambda}$, $\bar{\gamma}$.

For the first approximation

$$\left(u_j^{(1)} = (A^-)^j u^0\,, \quad v_j^{(1)} = (A^+)^{-(k-j)} v^1\right)$$

the validity of (3.7.20) follows from (3.7.8) provided that we choose $K > \varepsilon$ and $\bar{\lambda} > \lambda$, $\bar{\gamma} < \gamma$. Now let us show that if (3.7.20) holds for the n-th approximation, then it holds for the $(n+1)$-th approximation as well. Observe first that it follows from (3.7.17) and (3.7.9) that the functions f and g satisfy the following estimates

$$\|f(u,v)\| \leq \|f(0,v)\| + \left(\sup_{\|u,v\|\leq\varepsilon} \|f'_u\|\right)\|u\| \equiv \delta\|u\| \tag{3.7.21}$$

and

$$\|g(u,v)\| \leq \|g(u,0)\| + \left(\sup_{\|u,v\|\leq\varepsilon} \|g'_v\|\right)\|v\| \equiv \delta\|v\|\,. \tag{3.7.22}$$

Now from (3.7.7), (3.7.9) and (3.7.21), (3.7.22) we obtain

$$\|u_j^{(n+1)}\| \leq \lambda^j \varepsilon + \delta \sum_{s=0}^{j-1} \lambda^{j-s-1} \|u_s^{(n)}\| \,,$$

$$\|v_j^{(n+1)}\| \leq \gamma^{j-k} \varepsilon + \delta \sum_{s=j}^{k-1} \gamma^{j-s-1} \|v_s^{(n)}\| \,.$$

Hence, if $\left(u_s^{(n)}, v_s^{(n)}\right)$ satisfies (3.7.20), then for $u_j^{(n+1)}$, $v_j^{(n+1)}$ we have

$$\|u_j^{(n+1)}\| \leq \lambda^j \varepsilon + \delta \sum_{s=0}^{j-1} \lambda^{j-s-1} K \overline{\lambda}^s \leq \overline{\lambda}^j \left(\varepsilon + \frac{\delta K}{\overline{\lambda} - \lambda} \right) ,$$

$$\|v_j^{(n+1)}\| \leq \gamma^{j-k} \varepsilon + \delta \sum_{s=j}^{k-1} \gamma^{j-s-1} K \overline{\gamma}^{s-k} \leq \overline{\gamma}^{j-k} \left(\varepsilon + \frac{\delta K}{\gamma - \overline{\gamma}} \right) .$$

The inequalities (3.7.20) hold for $\left(u_j^{(n+1)}, v_j^{(n+1)}\right)$ provided that

$$K > \left(\varepsilon + \delta K \max \left(\frac{1}{\overline{\lambda} - \lambda}, \frac{1}{\gamma - \overline{\gamma}} \right) \right) .$$

Since δ may be made arbitrarily small for sufficiently small ε, such a constant K exists. Thus, one can select K, $\overline{\lambda}$ and $\overline{\gamma}$ such that the inequalities (3.7.20) hold for all approximations, and, consequently, for the solution of the boundary-value problem itself.

For the functions ξ_k and η_k we found

$$\|\xi_k\| \leq K \overline{\lambda}^{\,k} , \quad \|\eta_k\| \leq K \overline{\gamma}^{\,-k} ,$$

i.e. the norms of these functions tend uniformly and exponentially to zero as $k \to +\infty$. End of the proof.

Lemma 3.4. *The norms of the derivatives $\partial(\xi_k, \eta_k)/\partial(u_0, v_k)$ tend uniformly to zero as $k \to \infty$.*

Proof. Let us consider the derivatives $\partial\xi_k/\partial u_0$ and $\partial\eta_k/\partial u_0$. They are found from the solutions of the boundary-value problem (3.7.13), (3.7.14): $\partial\xi_k/\partial u_0 \equiv U_k$, $\partial\eta_k/\partial u_0 \equiv V_k$. Before we show that both $U_k(u_0, v_k)$ and

$V_0(u_0, v_k)$ tend to zero as $k \to +\infty$, we will prove that all U_j and V_j are bounded by a constant which depends neither on k nor on j.

The values U_j and V_j are found from the system (3.7.15) as a limit of the successive approximations

$$\begin{aligned} U_j^{(n+1)} &= (A^-)^j + \sum_{s=0}^{j-1} (A^-)^{j-s-1} \big(f'_u\big(u_s^*, v_s^*\big) U_s^{(n)} + f'_v\big(u_s^*, v_s^*\big) V_s^{(n)} \big)\,, \\ V_j^{(n+1)} &= -\sum_{s=j}^{k-1} (A^+)^{j-s-1} \big(g'_u\big(u_s^*, v_s^*\big) U_s^{(n)} + g'_v\big(u_s^*, v_s^*\big) V_s^{(n)} \big)\,, \end{aligned} \tag{3.7.23}$$

where u_s^* and v_s^* are the solutions of the boundary-value problem (3.7.2) and (3.7.3). We will prove that U_j and V_j are uniformly bounded if we show that all of the successive approximations $U_j^{(n)}$, $V_j^{(n)}$ are bounded by a constant which is independent of k, j and n. In order to verify this let us suppose that for all j

$$\|U_j^{(n)}, V_j^{(n)}\| \le 2\,. \tag{3.7.24}$$

It follows from (3.7.23), (3.7.9) and (3.7.10) that

$$\begin{aligned} \|U_j^{(n+1)}\| &\le \lambda^j + \sum_{s=0}^{j-1} \lambda^{j-s-1} \big(\|f'_u(u_s, v_s)\|\, \|U_s^{(n)}\| + \|f'_v(u_s, v_s)\|\, \|V_s^{(n)}\| \big) \\ &\le 1 + 2\delta \sum_{s=0}^{j-1} \lambda^{j-s-1} \le 1 + 2\delta/(1-\lambda) \le 2\,, \\ \|V_j^{(n+1)}\| &\le \sum_{s=j}^{k-1} \gamma^{j-s-1} \big(\|g'_u(u_s, v_s)\| \|U_s^{(n)}\| + \|g'_v(u_s, v_s)\|\, \|V_s^{(n)}\| \big) \\ &\le 2\delta \sum_{s=j}^{k-1} \gamma^{j-s-1} \le 2\delta/(\gamma-1) \le 1\,. \end{aligned}$$

which proves the claim.

Let us now show that U_k tends to zero as $k \to +\infty$. Since U_j satisfies (3.7.15), we obtain

$$\|U_j\| \le \lambda^j + \sum_{s=0}^{j-1} \lambda^{j-s-1} \big(\delta \|U_s\| + \|f'_{1v}(u_s^*, v_s^*)\|\, \|V_s\| \big)\,. \tag{3.7.25}$$

As $f(0,v) \equiv 0$, it follows that $f'_v(0,v) \equiv 0$, and therefore $f'_v(u,v) \to 0$ as $u \to 0$. Thus, by virtue of (3.7.20), $f'_v(u^*_s, v^*_s) \to 0$ as $s \to +\infty$. Now, since V_s remains bounded for all s, (3.7.25) gives

$$\|U_j\| \le \lambda^j + \sum_{s=0}^{j-1} \lambda^{j-s-1}(\delta\|U_s\| + \rho_s)\,, \tag{3.7.26}$$

where ρ_s is some sequence of positive numbers which converges to zero as $s \to +\infty$. Let us consider the sequence Z_j defined by the recurrent formula

$$Z_j = \lambda^j + \sum_{s=0}^{j-1} \lambda^{j-s-1}(\delta Z_s + \rho_s)\,. \tag{3.7.27}$$

It follows by induction from (3.7.26) that $\|U_k\| \le Z_k$. Therefore, to prove that $\partial\xi_k/\partial u_0 \equiv U_k \to 0$, it is sufficient to show that $Z_k \to 0$.

To prove this we first note that from (3.7.27)

$$Z_{j+1} - \lambda Z_j = \delta Z_j + \rho_j \tag{3.7.28}$$

whence

$$Z_{j+1} = (\lambda + \delta) Z_j + \rho_j\,. \tag{3.7.29}$$

Since δ may be chosen sufficiently small, we have $\lambda + \delta < 1$. Now, the convergence of Z_j to zero is proven in the same way as it was done for the sequence (3.5.14) (taking into account that $\rho_j \to 0$). Thus, $U_k = \frac{\partial\xi_k}{\partial u_0}$ tends to zero as $k \to +\infty$. The remaining derivatives $\partial\xi_k/\partial v_k$, $\partial\eta_k/\partial u_0$, and $\partial\eta_k/\partial v_k$ may be shown to tend to zero as $k \to +\infty$ in a similar fashion.

Lemma 3.5. *The norms of the first r derivatives of the functions ξ_k and η_k tend uniformly to zero as $k \to \infty$.*

Proof. Introduce the notations

$$U^i_{j(p,q)} \equiv \frac{\partial^i u_j}{\partial u_0^p \partial v_k^q}\,, \quad V^i_{j(p,q)} \equiv \frac{\partial^i v_j}{\partial u_0^p \partial v_k^q}\,,$$

where $p + q = i \le r$. The values U^i_j and V^i_j may be found by the successive approximations method as solutions of the system obtained from (3.7.5) by differentiating p-times with respect to u_0, and differentiating q-times with respect to v_k:

$$\begin{aligned} U_j^i &= \sum_{s=0}^{j-1} (A^-)^{j-s-1} \Big(f_u'(u_s, v_s) U_s^i + f_v'(u_s, v_s) V_s^i \\ &\qquad + P_i\big(u_s, v_s, \dots, U_s^{i-1}, V_s^{i-1}\big) \Big) \,, \\ V_j^i &= - \sum_{s=j}^{k-1} (A^+)^{j-s-1} \Big(g_u'(u_s, v_s) U_s^i + g_v'(u_s, v_s) V_s^i \\ &\qquad + Q_i\big(u_s, v_s, \dots, U_s^{i-1}, V_s^{i-1}\big) \Big) \,, \end{aligned} \tag{3.7.30}$$

where P_i and Q_i are certain polynomials of the variables $\big(U_s^1, V_s^1, \dots, U_s^{i-1}, V_s^{i-1}\big)$ and of the derivatives of the functions f and g computed at $u = u_s$ and $v = v_s$.

For example, for the derivatives

$$\Big(U_{j(2,0)}^2, V_{j(2,0)}^2\Big) \equiv \left(\frac{\partial^2 u_j}{\partial u_0^2} \,, \quad \frac{\partial^2 v_j}{\partial u_0^2} \right)$$

we have

$$\begin{aligned} U_{j(2,0)}^2 &= \sum_{s=0}^{j-1} (A^-)^{j-s-1} \Big(f_u'(u_s, v_s) U_{s(2,0)} + f_v'(u_s, v_s) V_{s(2,0)} \\ &\qquad + f_{uu}''(u_s, v_s) (U_{s(1,0)})^2 + 2 f_{uv}''(u_s, v_s) U_{s(1,0)} V_{s(1,0)} \\ &\qquad + f_{vv}''(u_s, v_s) (V_{s(1,0)})^2 \Big) \end{aligned}$$

and

$$\begin{aligned} V_{j(2,0)}^2 &= - \sum_{s=j}^{k-1} (A^+)^{j-s-1} \Big(g_u'(u_s, v_s) U_{s(2,0)} + g_v'(u_s, v_s) V_{s(2,0)} \\ &\qquad + g_{uu}''(u_s, v_s) (U_{s(1,0)})^2 + 2 g_{uv}''(u_s, v_s) U_{s(1,0)} V_{s(1,0)} \\ &\qquad + g_{vv}''(u_s, v_s) (V_{s(1,0)})^2 \Big) \,. \end{aligned}$$

In the manner previously employed for the first derivatives, one can show that the derivatives U_j^i and V_j^i of any higher order i are bounded by some constant which depends neither on j nor on k (but it may depend on the order i of the derivative).

In order to verify that the norm $\|U_k^i\| \to 0$ as $k \to +\infty$, we show that the norms $\|U_j^i\|$ are bounded by a sequence which is independent of k and which tends to zero as $j \to \infty$. We have already proven this statement for $i = 1$. We show by induction that it is valid for all i.

Let us assume that $\|U_j^i\| \to 0$ as $j \to \infty$ for all i less than some i_0. Consider Eq. (3.7.30) for $U_j^{i_0}$. Those terms in P_{i_0} which contain at least one of the values U_s^i $(i < i_0)$ tend to zero as $s \to +\infty$ by virtue of our inductive hypothesis. The remaining terms are products of certain values V_s^i with certain derivatives of $f(u_s, v_s)$ with respect to the variable v_s. Since all V_s^i are bounded uniformly and the derivatives of $f(u_s, v_s)$ with respect to v_s tends uniformly to zero as $u_s \to 0$ (because $f(0, v) \equiv 0$), it follows that all these terms, as well as $f(u_s, v_s)V_s^i$, tend to zero as $s \to +\infty$.

Thus, in complete analogy with the discussion concerning the first derivative, we obtain the estimate

$$\|U_j^i\| \le \lambda^j + \sum_{s=0}^{j-1} \lambda^{j-s-1} \left(\delta \|U_s^i\| + \rho_s^i\right),$$

where ρ_s^i is a sequence of positive numbers which converges to zero as $s \to +\infty$. Hence, similarly to estimate (3.7.26) for U_j^1, we obtain that the values U_j^i are majorized by some sequence Z_j^i which does not depend on k and which tends to zero as $j \to +\infty$. Thus, we may conclude now that all derivatives of ξ_k tend to zero as $k \to +\infty$.

For the values V_j^i we obtain the estimate

$$\|V_j^i\| \le \sum_{s=j}^{k-1} \gamma^{j-s-1} \left(\delta \|V_s^i\| + \sigma_{k-s}^i\right),$$

where $\sigma_{k-s}^i \to 0$ as $(k-s) \to +\infty$. Repeating the same arguments employed for U_j^i we can show that $\|V_{k-j}^i\| \to 0$ as $(k-j) \to +\infty$. Assuming $j = k$, we found that all derivatives of η_k tend to zero as $k \to +\infty$.

End of the proof.

3.8. Behavior of linear maps near saddle fixed points. Examples

In this section we will study some geometrical properties of the linear saddle maps. For suitable choice of coordinates a linear map T with a structurally stable fixed point O of the saddle type can be written in the form

$$\begin{aligned} \bar{x} &= A^{sL}x\,, & \bar{u} &= A^{ss}u\,, \\ \bar{y} &= A^{uL}y & \bar{v} &= A^{uu}v\,, \end{aligned} \tag{3.8.1}$$

where the absolute values of the eigenvalues of the matrix A^{sL} are equal to λ, $0 < \lambda < 1$, while those of the matrix A^{ss} are less than λ. The eigenvalues of the matrix A^{uL} are equal in absolute value to γ, $\gamma > 1$ and those of matrix A^{uu} are greater in absolute value than γ. Then, the equation of the stable invariant manifold W^s is $(y = 0, v = 0)$ and the equation of the non-leading (strongly) stable manifold W^{ss} is $(x = 0, y = 0,\, v = 0)$. The equation of the unstable manifold W^u is $(x = 0, u = 0)$, and the equation of thxe non-leading (strongly) unstable manifold W^{uu} is $(x = 0, u = 0, y = 0)$.

Let us choose two points on the stable and the unstable manifolds: $M^+(x^+, u^+, 0, 0) \in W^s/O$, and $M^-(0, 0, y^-, v^-) \in W^u/O$ and surround them by some small rectangular neighborhoods

$$\Pi^+ = \left\{\|x - x^+\| \leq \varepsilon_0, \|u - u^+\| \leq \varepsilon_0, \|y\| \leq \varepsilon_0, \|v\| \leq \varepsilon_0\right\}$$

$$\Pi^- = \left\{\|x\| \leq \varepsilon_1, \|u\| \leq \varepsilon_1, \|y - y^-\| \leq \varepsilon_1, \|v - v^-\| \leq \varepsilon_1\right\}$$

such that $T(\Pi^+) \cap \Pi^+ = \emptyset$ and $T(\Pi^-) \cap \Pi^- = \emptyset$. We assume also that the leading eigenvalues of the saddle fixed point O are simple, *i.e.* there is only one leading eigenvalue if it is real. Otherwise, there is a pair of leading eigenvalues if they are complex-conjugate. This implies that in the first case the vector x (or y) is one-dimensional, and A^{sL} (or A^{uL}) is a scalar. In the case where the leading eigenvalues are complex-conjugate, the vector x or y is two-dimensional, and the matrix A^{sL} or A^{uL} has the form

$$A^{sL} = \lambda \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \qquad \text{and} \qquad A^{uL} = \gamma \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix},$$

where $0 < \lambda < 1$, $\gamma > 1$, $(\varphi, \psi) \notin \{0, \pi\}$.

We consider the following question: are there any points in Π^+ whose trajectories reach Π^-? What is the set of such points in Π^- and what is the set of their images in Π^+?

We first consider the case where a saddle fixed point possesses leading eigenvalues only. It was established in Sec. 3.6 that there are nine main types of such saddle maps: four two-dimensional saddle points (all eigenvalues are real); four three-dimensional: two saddle-foci (2,1) and two saddle-foci (1,2) and one four-dimensional case: a saddle-focus (2,2).

Let us begin with two-dimensional maps. There may be four different (in the sense of topological conjugacy) situations depending on the signs of the eigenvalues of the saddle. The map may take one of the following forms:

$$\begin{array}{lll}
(1) & \bar{x} = \lambda x\,, & \bar{y} = \gamma y\,; \\
(2) & \bar{x} = -\lambda x\,, & \bar{y} = \gamma y\,; \\
(3) & \bar{x} = \lambda x\,, & \bar{y} = -\gamma y\,; \\
(4) & \bar{x} = -\lambda x\,, & \bar{y} = -\gamma y\,.
\end{array}$$

Without loss of generality we can assume that $\|x^+\| > 0$, $\|y^-\| > 0$.

For case (1) the map T forces the point $(x^+, 0)$ to jump to the point $(\lambda x^+, 0)$, then to $(\lambda^2 x^+, 0)$ and so on. Since $0 < \lambda < 1$, the points $T^k(M^+) = (\lambda^k x^+, 0)$ converge monotonically to the saddle O. Meanwhile, the map T expands the rectangle Π^+ by factor γ along the y-coordinate and compresses by factor λ along the x-coordinate. It is obvious that one can choose a large $\bar{k}$ ($\bar{k} \to +\infty$ as $\varepsilon_0, \varepsilon_1 \to 0$) such that for all $k \geq \bar{k}$ the following conditions hold

$$T^k(\Pi^+) \cap \Pi^- \neq \emptyset\,, \quad \left(\gamma^{\bar{k}} \xi_0 > y^- + \varepsilon_1\,, \quad \lambda^{\bar{k}}(x^+ + \varepsilon_0) < \varepsilon_1\right),$$

see Fig. 3.8.1.[7]

Denote $\sigma_k^1 = T^k(\Pi^+) \cap \Pi^-$, where $k \geq \bar{k}$. In the case under consideration, σ_k^1 is a strip on Π^- which is defined by the conditions

$$\sigma_k^1 = \left\{(x, y)\colon |x - \lambda^k x^+| \leq \lambda^k \varepsilon_0, |y - y^-| \leq \varepsilon_1\right\}.$$

As $k \to +\infty$ the strips σ_k^1 accumulate monotonically to the interval $W^u \cap \Pi^- = \{(x, y) : x = 0, |y - y^-| \leq \varepsilon_1\}$. For any sufficiently large k the map $T^k\colon \Pi^+ \to \Pi^-$ is defined. Its domain is the strip $\sigma_k^0 = T^{-k}(\Pi^-) \cap \Pi^+$ on Π^+ defined by the conditions

$$\sigma_k^0 = \left\{(x, y)\colon |x - x^+| \leq \varepsilon_0, |y - \gamma^{-k} y^-| \leq \gamma^{-k} \varepsilon_1\right\}.$$

[7] In the nonlinear case, the existence of such $\bar{k}$ follows from the λ-lemma.

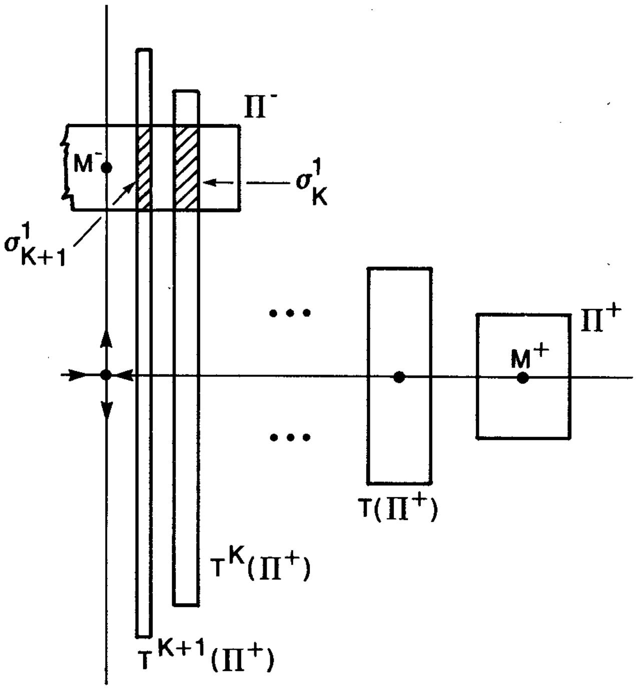


Fig. 3.8.1. The map T near a saddle fixed point. The initial rectangle Π^+ is expanded along the unstable direction y and compressed along the stable direction x. The range of the map $T' : \Pi^+ \to \Pi^-$ is composed of the strips σ_k^1 lying in the intersection between the images $T^k\Pi^+$ and the rectangle Π^-.

As $k \to +\infty$ the strips σ_k^0 accumulate monotonically to the interval $W^s \cap \Pi^+ = \{(x,y)\colon |x - x^+| \leq \varepsilon_0, y = 0\}$ (see Fig. 3.8.2). The location of the strips σ_k^0 and σ_k^1 is shown in Fig. 3.8.3(a).

For case (2) the point O is a node $(-)$ on W^s. Hence, the rectangles $T^k(\Pi^+)$ and $T^{k+1}(\Pi^+)$ are located on the opposite sides of the W^u. Thus, as $k \to +\infty$ the strips σ_k^1 converge to the interval $W^u \cap \Pi^-$ from the right for even k (from the side of positive values of x), and from the left for odd k. The location of the strips σ_k^0 and σ_k^1 is shown in Fig. 3.8.3(b).

For case (3) the "jumping direction" is the y-axis which is the unstable manifold W^u. The strips σ_k^0 accumulate to the interval $W^s \cap \Pi^+$ from both sides as shown in Fig. 3.8.3(c).

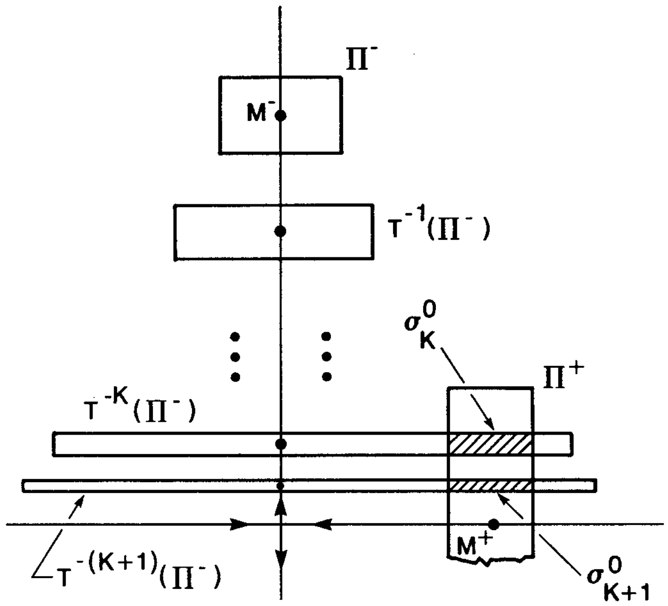


Fig. 3.8.2. The inverse map T^{-1} near the saddle. The rectangle Π^- is expanded along the stable direction x and compressed along the unstable direction y under the action of the inverse map T^{-1}. The strips σ_k^0 form the domain of definition of the map $T' : \Pi^+ \to \Pi^-$.

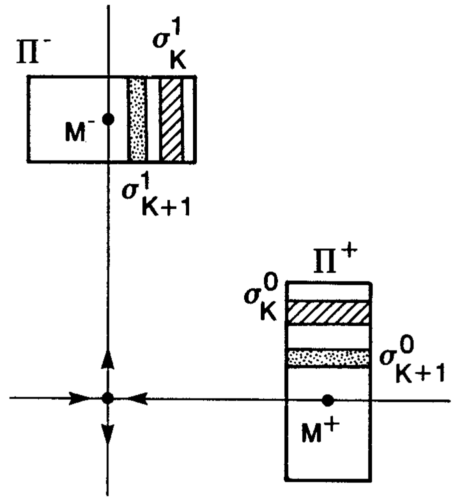
For case (4) the point O is a stable node $(-)$ on W^s and an unstable node $(-)$ on W^u. Therefore, the strips σ_k^0 converge to $W^s \cap \Pi^+$ from both sides. The strips σ_k^1 converge to $W^u \cap \Pi^-$ from both sides as well, see Fig. 3.8.3(d).

Let us now consider the cases where the leading eigenvalues comprise a complex-conjugate pair.

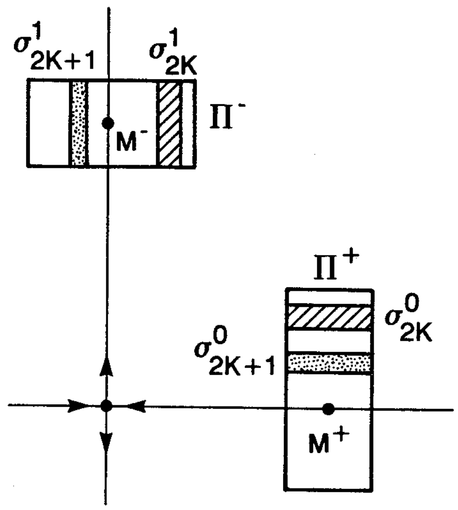
In the three-dimensional case where the point O is a saddle-focus (2,1), the linear map can be written in the form

$$\begin{aligned} \bar{x}_1 &= \lambda(\cos\varphi \cdot x_1 - \sin\varphi \cdot x_2)\,, \\ \bar{x}_2 &= \lambda(\sin\varphi \cdot x_1 + \cos\varphi \cdot x_2)\,, \\ \bar{y} &= \gamma y\,, \end{aligned} \tag{3.8.2}$$

where $\lambda_{1,2} = \lambda e^{\pm i\varphi}$ and γ are the eigenvalues of the saddle O, $\varphi \notin \{0, \pi\}$, $0 < \lambda < 1$, $|\gamma| > 1$. To be specific, let us consider the case of positive γ. The

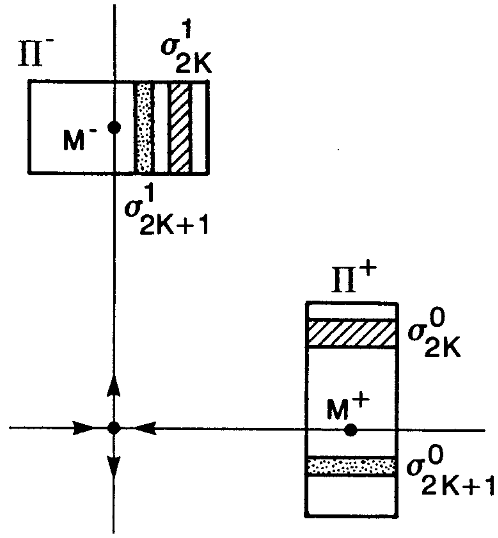


(a)

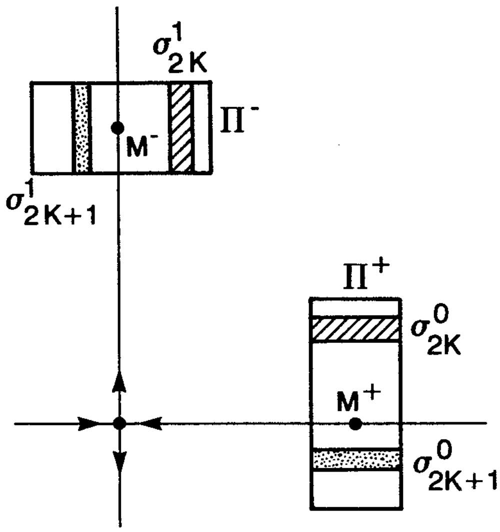


(b)

Fig. 3.8.3. The Poincaré map near saddle fixed points of different types. See captions to Figs. 3.8.1 and 3.8.2. (a) Near a saddle $(+,+)$, (b) near a saddle $(-,+)$. The even and odd iterations of Π^+ lie on the opposite sides from the unstable manifold $y(x)$, (c) near a saddle $(-,+)$, (d) near a saddle $(-,-)$.



(c)



(d)

Fig. 3.8.3. (*Continued*)

map T^k takes the form

$$\begin{aligned}\bar{x}_1 &= \lambda^k(\cos(k\varphi)\cdot x_1 - \sin(k\varphi)\cdot x_2)\,,\\ \bar{x}_2 &= \lambda^k(\sin(k\varphi)\cdot x_1 + \cos(k\varphi)\cdot x_2)\,,\\ \bar{y} &= \gamma^k y\,.\end{aligned} \tag{3.8.3}$$

Let us choose an arbitrary point $M^+(x_1^+, x_2^+, 0)$ on $W^s\backslash O$. By rotating the coordinate frame on the plane (x_1, x_2) one may always ensure that $x_2^+ = 0$, whereas formulae (3.8.2) and (3.8.3) remain unchanged. It follows from (3.8.3) that the domain σ_0 of the map $T'\colon \Pi^+ \to \Pi^-$ consists of a countable union of non-intersecting three-dimensional "plates"

$$\sigma_k^0 = \big\{(x_1, x_2, y)\colon |x - x_1^+| \leq \varepsilon_0, |x_2| \leq \varepsilon_0, |y - \gamma^{-k}y^-| \leq \gamma^{-k}\varepsilon_1\big\}$$

$(k \leq \bar{k})$ which converge to the square $W^s \cap \Pi^+$ as $k \to +\infty$. In order to describe the range σ_1 of the map $T'\colon \Pi^+ \to \Pi^-$, let us introduce the polar coordinates (r, θ) such that $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. The map (3.8.3) takes the form

$$\bar{r} = \lambda^k r\,, \quad \bar{\theta} = \theta + k\varphi\,, \quad \bar{y} = \gamma^k y\,.$$

It follows that $T^k(\Pi^+)$ is a parallelepiped of height $2\gamma^k\varepsilon_0$. Its base on W^s is a square with the side $2\varepsilon_0\lambda^k$, centered at the point $M_k^+ = (r_k = \lambda^k x_1^+, \theta_k = k\varphi)$. We remark that all points M_k^+ lie on the logarithmic spiral $\bar{r} = x_1^+\lambda^{\bar{\theta}/\varphi}$. Thus, σ_1 is the union of a countable number of three-dimensional vertical parallelepipeds σ_k^1 lying inside "the roulette" R^-

$$\big\{\big(|x_1^+| - \varepsilon_0\big)\lambda^{\bar{\theta}/\varphi} \leq \bar{r} \leq \big(|x_1^+| + \varepsilon_0\big)\lambda^{\bar{\theta}/\varphi}, |\bar{y} - y^-| \leq \varepsilon_1\big\}\,,$$

which winds onto the segment

$$W^u \cap \Pi^- = \{x_1 = x_2 = 0, |y - y^-| \leq \varepsilon_1\}$$

of the W^u-axis, see Fig. 3.8.4. The strip $\sigma_k^1 \subset R^-$ has a diameter of order $\varepsilon_0\lambda^k$ along the coordinates (x_1, x_2), and σ_k^1 is separated from σ_{k+1}^1 by an angle of order φ in the angular coordinate θ.

In the case where the fixed point is a saddle-focus (1,2) the map T can be written in the form

$$\begin{aligned}\bar{x} &= \lambda x\,,\\ \bar{y}_1 &= \gamma(\cos\psi\cdot y_1 - \sin\psi\cdot y_2)\,,\\ \bar{y}_2 &= \gamma(\sin\psi\cdot y_1 + \cos\psi\cdot y_2)\,,\end{aligned}$$

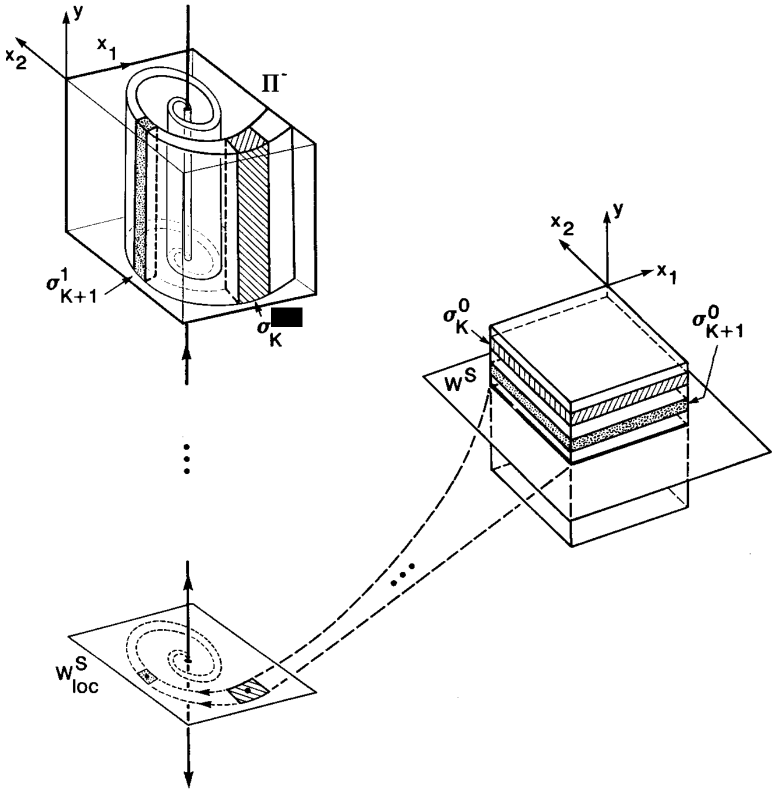


Fig. 3.8.4. The geometry of the Poincaré map near a saddle-focus (2,1+).

where $|\lambda| < 1$ and $\gamma > 1$, $\psi \neq \{0, \pi\}$. For definiteness, let us consider the case where $0 < \lambda < 1$. The map $T^{-k}\colon \Pi^- \to \Pi^+$ has the following form

$$\begin{aligned}
\bar{y}_2 &= \gamma^{-k}(\cos(k\psi) \cdot \bar{y}_2 - \sin(k\psi) \cdot \bar{y}_1)\,, \\
\bar{y}_1 &= \gamma^{-k}(\sin(k\psi) \cdot y_2 + \cos(k\psi) \cdot \bar{y}_1)\,, \\
\bar{x} &= \lambda^{-k} \cdot \bar{x}\,.
\end{aligned}$$

This formula is analogous to (3.8.3). Thus, by symmetry if we choose the points $M^+ \in W^s \backslash O$ and $M^-(0, 0, y_2^-) \in W^u \backslash O$, and their neighborhoods Π^+ and Π^-, the range of the map $T' : \Pi^+ \to \Pi^-$ consists of a countable union of non-intersecting three-dimensional plates σ_k^1 converging to the square

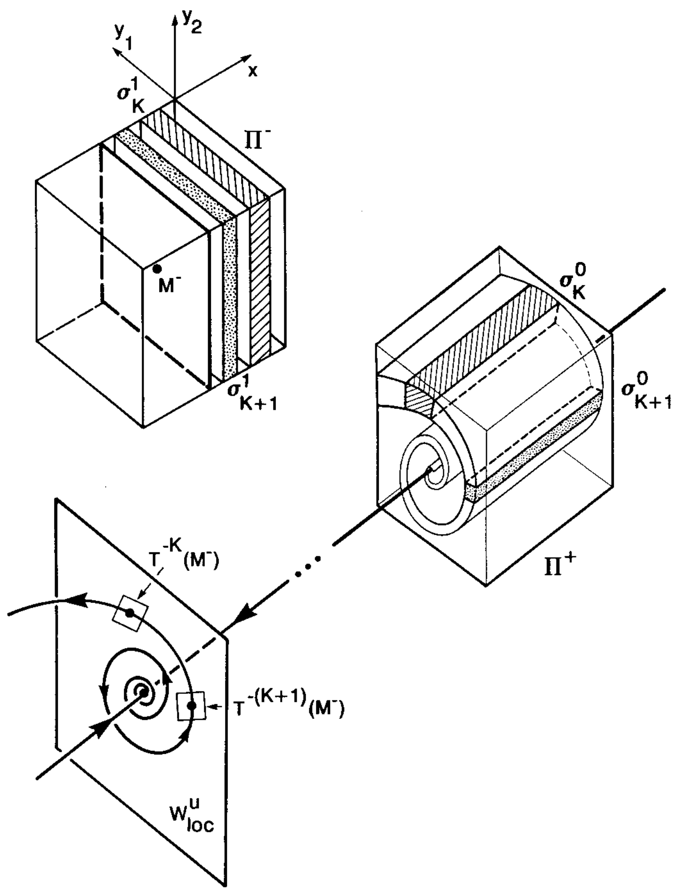


Fig. 3.8.5. The Poincaré map near a saddle-focus (1+,2). This is the inverse to the map in Fig. 3.8.4.

$W^u \cap \Pi^-$. The domain of the map $T'\colon \Pi^+ \to \Pi^-$ is the union of a countable set of three-dimensional horizontal parallelepipeds σ_k^0 lying within the roulette R^+ (see Fig. 3.8.5)

$$\left\{ \left(|y_2^-| - \varepsilon_1\right) \cdot \gamma^{-\theta/\psi} \leq r \leq \left(|y_2^-| + \varepsilon_1\right) \cdot \gamma^{-\theta/\psi}, |x - x^+| \leq \varepsilon_0 \right\},$$

which winds onto the segment

$$W^s \cap \Pi^+ = \left\{ |x - x^+| \leq \varepsilon_0, y_1 = y_2 = 0 \right\}$$

of the W^s-axis. The strip σ_k^0 has a diameter of order $\varepsilon_1 \cdot \gamma^{-k}$ along the coordinates (y_1, y_2). Moreover, σ_k^0 and σ_{k+1}^0 are separated by an angle of order ψ in the angular coordinate θ.

Let us consider next the case where the fixed point O is a saddle-focus (2,2). The corresponding linear map T can be written as

$$\begin{aligned}
\bar{x}_1 &= \lambda(\cos\varphi \cdot x_1 - \sin\varphi \cdot x_2)\,, \\
\bar{x}_2 &= \lambda(\sin\varphi \cdot x_1 + \cos\varphi \cdot x_2)\,, \\
\bar{y}_1 &= \gamma(\cos\psi \cdot y_1 - \sin\psi \cdot y_2)\,, \\
\bar{y}_2 &= \gamma(\sin\psi \cdot y_1 + \cos\psi \cdot y_2)\,,
\end{aligned} \tag{3.8.4}$$

where φ, $\psi \notin \{0, \pi\}$, $0 < \lambda < 1 < \gamma$. We choose two arbitrary points $M^+(x_1^+, x_2^+, 0, 0) \in W^s \backslash O$ and $M^-(0, 0, y_1^-, y_2^-) \in W^u \backslash O$. Without affecting formulae (3.8.4) we can always ensure that $x_1^+ = 0$ and $y_1^- = 0$ by the orthogonal rotation of the coordinate frames on the planes (x_1, x_2) and (y_1, y_2). Introducing the polar coordinates (r, θ) in the plane (x_1, x_2) and (ρ, α) in the plane (y_1, y_2), map (3.8.4) recasts in the following simple form:

$$\bar{r} = \lambda r\,, \quad \bar{\theta} = \theta + \varphi\,, \quad \bar{\rho} = \gamma\rho\,, \quad \bar{\alpha} = \alpha + \psi\,.$$

Hence the map T^k takes the form

$$\bar{r} = \lambda^k r\,, \quad \bar{\theta} = \theta + k\varphi\,, \quad \bar{\rho} = \gamma^k\rho\,, \quad \bar{\alpha} = \alpha + k\psi\,. \tag{3.8.5}$$

Since $0 < \lambda < 1 < \gamma$, it follows from (3.8.5) for sufficiently large k that $\gamma^k\varepsilon_0 > y_2^- + \varepsilon_1$ and $\lambda^k(x_1^+ + \varepsilon_0) < \varepsilon_1$, and hence $T^k(\Pi^+) \cap \Pi^- \neq \emptyset$. The four-dimensional strips $\sigma_k^1 \equiv T^k(\Pi^+) \cap \Pi^-$ converge to the two-dimensional square

$$W^u \cap \Pi^- = \big\{0, 0, |y_1| \leq \varepsilon_1, |y_2 - y_2^-| \leq \varepsilon_1\big\}$$

as $k \to +\infty$. In the plane $W^s\colon (x_1, x_2, 0, 0)$ the points $M_k^+ \equiv T^k(M^+) = (\lambda^k x_1^+, k\varphi)$ lie on the logarithmic spiral $\bar{r} = x_1^+ \cdot \lambda^{\bar{\theta}/\varphi}$. Thus, the range σ_1 of the map $T'\colon \Pi^+ \to \Pi^-$ is a union of a countable number of the strips σ_k^1 located inside the roulette R^- (Fig. 3.8.6)

$$\big\{(|x_1^+| - \varepsilon_0) \cdot \lambda^{\bar{\theta}/\varphi} \leq \bar{r} \leq (|x_1^+| + \varepsilon_0) \cdot \lambda^{\bar{\theta}/\varphi}, |\bar{y}_1| \leq \varepsilon_1, |\bar{y}_2 - y_2^-| \leq \varepsilon_1\big\}\,,$$

which winds towards the two-dimensional square $W^u \cap \Pi^-$. Along the variables (x_1, x_2) the strip σ_k^1 has a diameter of order $\varepsilon_0\lambda^k$, and along the angular coordinate θ the strips σ_k^1 and σ_{k+1}^1 are separated by an angle of order φ.

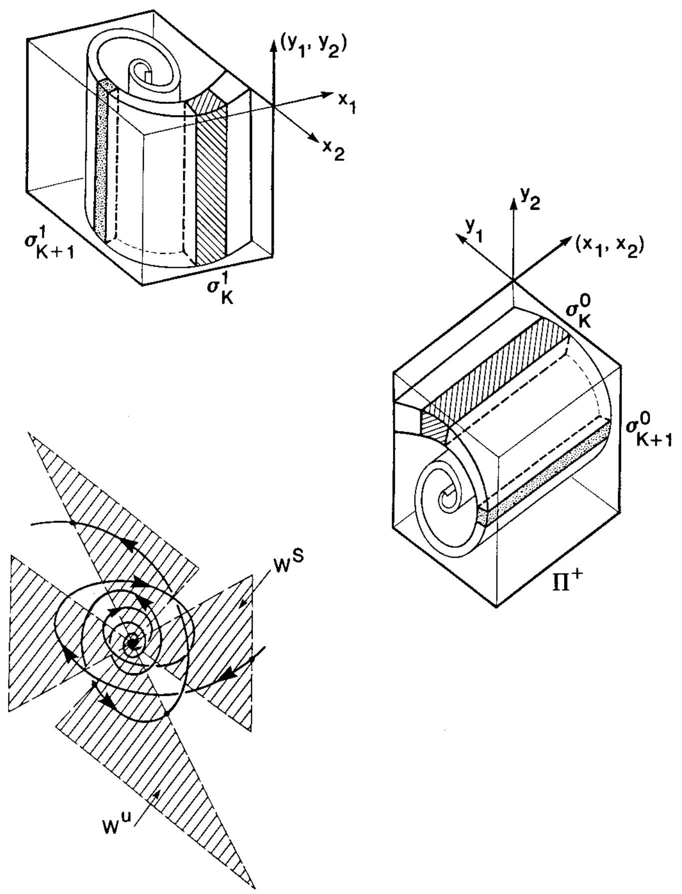


Fig. 3.8.6. The Poincaré map near a saddle-focus in $\mathbb{R}^4$. The original three-dimensional parallepiped Π^+ is transformed into a "roulette" within the parallepiped Π^- by the map T. The image of the parallepiped Π^+ under the inverse map T^{-1} has the same shape.

Let us now find the domain σ_0 of the map $T' : \Pi^+ \to \Pi^-$. The map $T^{-k}\colon \Pi^- \to \Pi^+$ can be written in the polar coordinates $(r, \theta, \rho, \alpha)$ as

$$r = \lambda^{-k}\bar{r}\,, \quad \theta = \bar{\theta} - k\varphi\,, \quad \rho = \gamma^{-k}\bar{\rho}\,, \quad \alpha = \bar{\alpha} - k\psi\,. \tag{3.8.6}$$

Because $0 < \lambda < 1$, $\gamma > 1$, the domain σ_0 consists of a countable number of four-dimensional strips σ_k^0 lying inside the roulette R^+ (see Fig. 3.8.6)

$$\begin{aligned}\big\{\big(|y_2^-| - \varepsilon_1\big)\cdot\gamma^{-\alpha/\psi} \le \rho \le \big(|y_2^-| + \varepsilon_1\big)\cdot\gamma^{-\alpha/\psi}\,,\\ |x_1 - x_1^+| \le \varepsilon_0, |x_2| \le \varepsilon_0,\big\}\end{aligned}$$

which winds towards the two-dimensional square $W^s \cap \Pi^+$. As $k \to +\infty$ the four-dimensional strips σ_k^0 converge to the square $W^s \cap \Pi^+$; in the coordinates (y_1, y_2) the strip σ_k^0 has a diameter $\sim \varepsilon_1\gamma^{-k}$, and in the angular coordinate α the angle between adjacent strips σ_k^0 and σ_{k+1}^0 is of order ψ.

We consider now the situation when the saddle fixed point has non-leading directions. Let us find the domain and the range of the map $T'\colon \Pi^+ \to \Pi^-$ for the three-dimensional cases. There are two cases to consider:

1. W^s is two-dimensional and W^u is one-dimensional;

2. W^s is one-dimensional and W^u is two-dimensional.

In the first case the linear map T is written as

$$\bar{x} = \lambda x\,, \quad \bar{u} = \lambda_2 u\,, \quad \bar{y} = \gamma y\,,$$

where λ and γ are assumed for more definiteness to be positive, and where $0 < |\lambda_2| < \lambda$. Since $M^+ \notin W^{ss}$, it follows that $x^+ \neq 0$ and therefore we can let $x^+ > 0$. The map $T^{-k}\colon \Pi^- \to \Pi^+$ is defined by

$$x = \lambda^{-k}\bar{x}\,, \quad u = \lambda_2^{-k}\bar{u}\,, \quad y = \gamma^{-k}\bar{y}\,,$$

where $(x, u, y) \in \Pi^+$ and $(\bar{x}, \bar{u}, \bar{y}) \in \Pi^-$. One observes that for sufficiently large k such that $\lambda^{-k}\varepsilon_1 > x^+ + \varepsilon_0$ and $|\lambda_2|^{-k}\varepsilon_1 > |u^+| + \varepsilon_0$, the strips $\sigma_k^0 \equiv T^{-k}(\Pi^-) \cap \Pi^+$ are given by

$$\sigma_k^0 = \big\{\big(x, u, y\big)\colon |x - x^+| \le \varepsilon_0, |u - u^+| \le \varepsilon_0, |y - \gamma^{-k}y^-| \le \gamma^{-k}\varepsilon_1\big\}\,,$$

i.e. they comprise certain three-dimensional plates of thickness $2\gamma^{-k}\varepsilon_1$ which converge to the square $W^s \cap \Pi^+$ as $k \to +\infty$, see Fig. 3.8.7.

The map T^k is written in the form

$$\bar{x} = \lambda^k x\,, \quad \bar{u} = \lambda_2^k u\,, \quad \bar{y} = \gamma^k y\,.$$

The strips $\sigma_k^1 \equiv T^k(\Pi^+) \cap \Pi^-$ are given by

$$\sigma_k^1 = \big\{(\bar{x}, \bar{u}, \bar{y})\colon \quad |\bar{x} - \lambda^k x^+| \le \lambda^k\varepsilon_0, |\bar{u} - \lambda_2^k u^+| \le \lambda_2^k\varepsilon_0, |\bar{y} - y^-| \le \varepsilon_1\big\}\,.$$

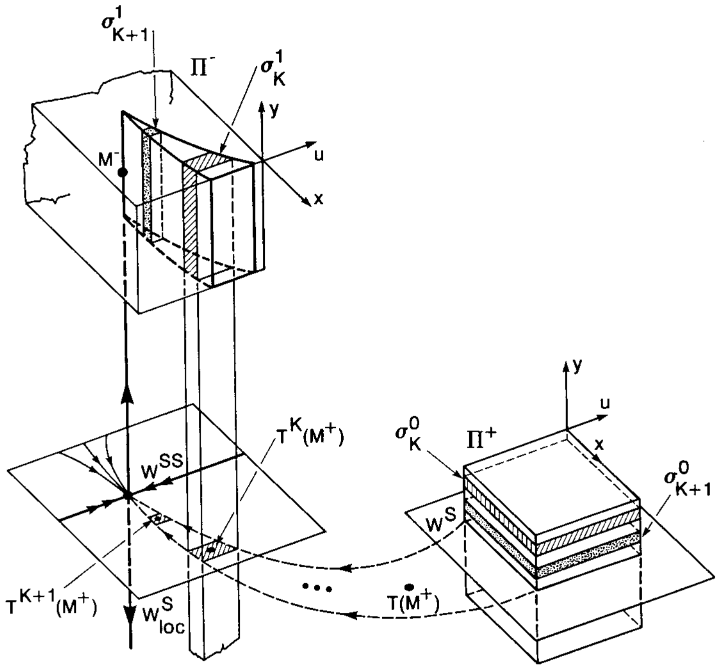


Fig. 3.8.7. The Poincaré map in a neighborhood of a saddle in $\mathbb{R}^3$. The images of the points lying in the intersection of the two-dimensional stable manifold W^s with the three-dimensional area Π^+ compose the edge of a wedge. The part of Π^+ above W^s is transformed into the wedge itself. The closer the dashed area of Π^+ is to W^s the thinner and closer to W^u is its image inside Π^-.

It follows that, first, as $k \to +\infty$ the strips σ_k^1 converge to the segment

$$W^u \cap \Pi^- = \{x = 0, u = 0, |y - y^-| \le \varepsilon_1\}$$

and that they have the shape of vertical "bars" located inside a three-dimensional wedge

$$\big\{\bar{x} > 0\,, \quad C_2\bar{x}^\alpha \le \bar{u} \le C_1\bar{x}^\alpha\,, \quad |\bar{y} - y^-| \le \varepsilon_1\,, \quad \alpha = \ln|\lambda_2|/\ln\lambda\,,$$
$$C_{1,2} = (u^+ \pm \varepsilon_0)/(x^+ \mp \varepsilon)^{-\alpha}\big\}\,.$$

This wedge adjoins to the segment

$$W^u \cap \Pi^- = \{x = 0, u = 0, |y - y^-| \le \varepsilon_1\}\,.$$

Since $\alpha > 1$ and $C_{1,2} \neq \infty$, the wedge is tangent to the extended unstable subspace $E^u \otimes E^{sL}$: $\{u = 0\}$ at the points of $W^u \cap \Pi^-$ as shown in Fig. 3.8.7.

In the case where W^s is one-dimensional and W^u is two-dimensional, the map T can be written as

$$\bar{x} = \lambda x\,, \quad \bar{y} = \gamma y\,, \quad \bar{v} = \gamma_2 v\,,$$

where $|\gamma_2| > |\gamma|$. This case is reduced to the previous one if we consider the inverse map T^{-1}. If we select the points $M^+ \in W^s$ and $M^- \in W^u/W^{uu}$ and select their neighborhoods Π^+ and Π^- respectively, then the range of the map $T' : \Pi^+ \to \Pi^-$ consists of a countable union of non-intersecting three-dimensional plates σ_k^1 converging to the square $W^u \cap \Pi^-$. At the same time, the domain of the map is a union of a countable number of three-dimensional horizontal bars σ_k^0 within the wedge

$$\big\{y > 0\,, \quad \tilde{C}_2 y^\alpha < v < \tilde{C}_1 y^\alpha\,, \quad |x - x^+| \leq \varepsilon_0\big\}$$

where $\alpha = \ln|\gamma_2|/\ln|\gamma|$. At the point of

$$W^s \cap \Pi^+ = \{y = 0, v = 0, |x - x^+| \leq \varepsilon_0\}$$

this wedge is tangent the extended stable subspace $E^s \oplus E^{uL}$: $\{v = 0\}$, see Fig. 3.8.8.

Let us now proceed by considering the general linear case, *i.e.* that of map (3.8.1), when $1 > \|A^{sL}\| = \lambda > \|A^{ss}\|$ and $1 < \|A^{uL}\| = \gamma < \|(A^{uu})^{-1}\|^{-1}$. We assume that neither the points M^+ nor M^- lie in the non-leading invariant manifolds of the saddle O, *i.e.* $M^+ \in W^s \backslash W^{ss}$ and $M^- \in W^u \backslash W^{uu}$. This condition implies that $\|x^+\| \neq 0$ and $\|y^-\| \neq 0$. Without loss of generality we may assume that $x^+ > 0$ and $y^- > 0$. One can easily show that the projections of the multi-dimensional strips σ_k^0 and σ_k^1 onto the leading directions (x, y) will be similar to the ones considered above. As far as the non-leading directions are concerned, the following relations hold: if $(x, u, y, v) \in \sigma_k^0$, then $\|v\|/\|y\| \to 0$ as $k \to +\infty$, and if $(\bar{x}, \bar{u}, \bar{u}, \bar{v}) \in \sigma_k^1$, then $\|\bar{u}\|/\|\bar{x}\| \to 0$ as $k \to +\infty$.

3.9. Geometrical properties of nonlinear saddle maps

The results of the previous section have a primarily illustrative character. It is, therefore, important that the geometrical structures considered in the linear case persist for generic nonlinear maps.

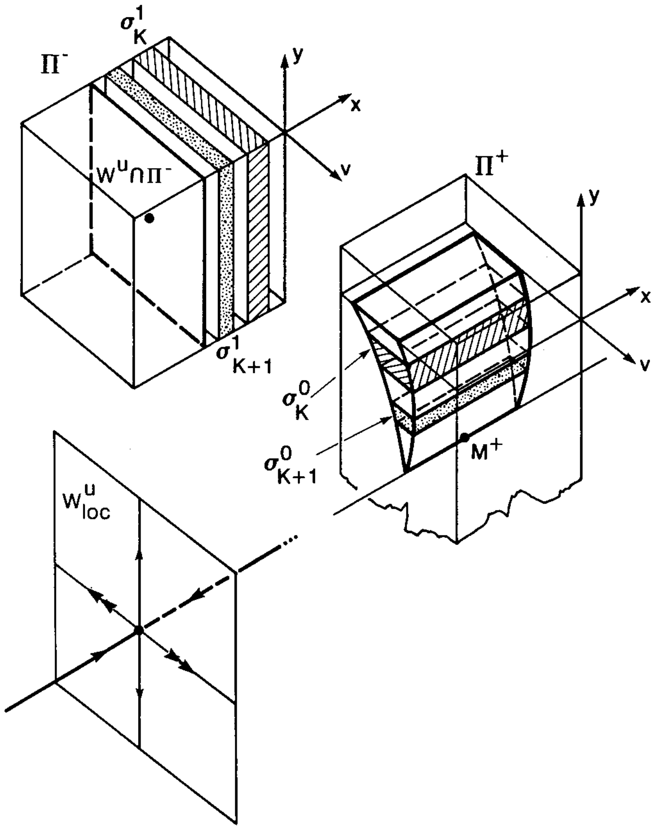


Fig. 3.8.8. The map near saddle of other topological type, *i.e.* with a one-dimensional stable manifold W^s and a two-dimensional unstable manifold W^u. This situation may be regarded as inverse to the map in Fig. 3.8.7.

Near a saddle fixed point a nonlinear map T can be written in the form

$$\begin{aligned}
\bar{x} &= A^{sL}x + f_1(x, u, y, v)\,, \\
\bar{u} &= A^{ss}u + f_2(x, u, y, v)\,, \\
\bar{y} &= A^{uL}y + g_1(x, u, y, v)\,, \\
\bar{v} &= A^{uu}v + g_2(x, u, y, v)\,,
\end{aligned} \tag{3.9.1}$$

where x and y are the leading coordinates, and u and v are the non-leading coordinates. The absolute values of the eigenvalues of the matrix A^{sL} are equal to λ $(0 < \lambda < 1)$, those of the matrix A^{ss} are less than λ, those of the matrix A^{uL} are equal to γ $(\gamma > 1)$ and those of the matrix A^{uu} are greater than γ. The functions f and g along with their first derivatives vanish at the origin. We suppose that in some sufficiently small neighborhood U of the saddle point O the invariant stable and unstable manifolds are straightened, *i.e.* $f(0,0,y,v) \equiv 0$ and $g(x,u,0,0) \equiv 0$. The equation of the manifold W^s_{loc} is then $(y = 0,\ v = 0)$ and that of W^u_{loc} is $(x = 0,\ u = 0)$.

We assume that the stable and unstable leading multipliers of the saddle fixed point O are simple (namely, a real leading eigenvalue, or a pair of complex-conjugate leading eigenvalues).

Let $M^+(x^+, u^+, 0, 0)$ and $M^-(0, 0, y^-, v^-)$ be arbitrary points on the stable and unstable manifolds of the saddle such that neither point lies in the corresponding non-leading manifolds. Let Π^+ and Π^- be sufficiently small rectangular neighborhoods of M^+ and M^- respectively:

$$\Pi^+ = \{\|x - x^+\| \le \varepsilon_0, \quad \|u - u^+\| \le \varepsilon_0, \quad \|y\| \le \varepsilon_0, \quad \|v\| \le \varepsilon_0\}$$

$$\Pi^- = \{\|x\| \le \varepsilon_1, \quad \|u\| \le \varepsilon_1, \quad \|y - y^-\| \le \varepsilon_1, \quad \|v - v^-\| \le \varepsilon_1\}$$

such that $T(\Pi^+) \cap \Pi^+ = \emptyset$ and $T(\Pi^-) \cap \Pi^- = \emptyset$.

What can we say about the map $T' : \Pi^+ \to \Pi^-$ in this case? It is not hard to show that just like in the linear case there exists a countable set of "strips" $\sigma^0_k = T^{-k}(\Pi^-) \cap \Pi^+$ and $\sigma^1_k = T^k(\Pi^+) \cap \Pi^-$ converging, respectively, to $W^s_{loc} \cap \Pi^+$ and $W^u_{loc} \cap \Pi^-$ as $k \to +\infty$, and for these strips $T^k(\sigma^0_k) \equiv \sigma^1_k$.

Indeed, it follows from the existence of the solution of the boundary-value problem (see Sec. 3.7) that

$$(x_k, u_k) = \{\xi^1_k(x_0, u_0, y_k, v_k), \quad \xi^2_k(x_0, u_0, y_k, v_k)\}\,, \tag{3.9.2}$$

$$(y_0, v_0) = \{\eta^1_k(x_0, u_0, y_k, v_k), \quad \eta^2_k(x_0, u_0, y_k, v_k)\}\,, \tag{3.9.3}$$

where $\|\xi_k\| \to 0$ and $\|\eta_k\| \to 0$ as $k \to +\infty$ (Lemma 3.3). Therefore, for k sufficiently large (*i.e.* such that the inequalities $\|\xi_k\| \le \varepsilon_1$ and $\|\eta_k\| \le \varepsilon_0$ are satisfied) the strips σ^0_k and σ^1_k are defined by the following conditions:

— σ^0_k is the set of all points on Π^+ whose coordinates (x_0, u_0, y_0, v_0) satisfy (3.9.3) with $\|y_k - y^-\| \le \varepsilon_1$, $\|v_k - v^-\| \le \varepsilon_1$;
— σ^1_k is the set of all points on Π^- whose coordinates (x_k, u_k, y_k, v_k) satisfy (3.9.2) with $\|x_0 - x^+\| \le \varepsilon_0$ and $\|u_0 - u^+\| \le \varepsilon_0$.

Note that since $T(\Pi^+) \cap \Pi^+ = \emptyset$, $T(\Pi^-) \cap \Pi^- = \emptyset$, and since the mapping T is a diffeomorphism, the strips σ_k^j ($j = 0, 1$) have no intersections for different k.

In order to describe the geometrical properties of the domain $\cup\sigma_k^0$ and the range $\cup\sigma_k^1$ of the map $T' : \Pi^+ \to \Pi^-$ in the nonlinear case, we need some additional estimates on the solutions (3.9.2) and (3.9.3) of the boundary-value problem. This can be achieved in the following way. Let us introduce the coordinates (x, u, y, v) such that the following conditions hold for the system (3.9.1):

$$\begin{aligned} f_i &= f_{i1}(x, y, v)x + f_{i2}(x, u, y, v)u \\ g_i &= g_{i1}(x, u, y)y + g_{i2}(x, u, y, v)v \\ f_{1j}|_{y=0,v=0} &\equiv 0, \quad g_{1j}|_{x=0,u=0} \equiv 0\,, \\ f_{i1}|_{x=0} &\equiv 0, \quad g_{i1}|_{y=0} \equiv 0 \quad (i, j = 1, 2)\,. \end{aligned} \tag{3.9.4}$$

The existence of such $\mathbb{C}^{r-1}$-coordinates ($r \geq 2$) is the result of Theorem 3.22 (the proof repeats the proof of Theorem 2.20 in Appendix A). In these coordinates the non-leading invariant manifolds are also straightened: The equation of W^{ss}_{loc} is ($y = 0\,v = 0\,x = 0$), and that of W^{uu}_{loc} is ($x = 0\,u = 0\,y = 0$). Moreover, the equations for the leading coordinates are linear on both W^s and W^u. Note also that all terms of the kind $x \cdot p(y, v)$ are eliminated in the right-hand sides of equations for $\bar{x}$ and $\bar{u}$ and the terms of the kind $y \cdot q(x, u)$ are eliminated in the right-hand sides of equations for $\bar{y}$ and $\bar{v}$.

Lemma 3.6. *If identities* (3.9.4) *hold and if the leading eigenvalues are simple* (*real or complex*), *then*

$$\xi_k^1 = (A^{sL})^k x_0 + o(\lambda^k), \quad \eta_k^1 = (A^{uL})^{-k} y_k + o(\gamma^{-k})\,, \tag{3.9.5}$$

$$\xi_k^2 = o(\lambda^k), \qquad \eta_k^2 = o(\gamma^{-k})\,. \tag{3.9.6}$$

where the terms $o(\lambda^k)$ and $o(\gamma^{-k})$ are $\mathbb{C}^{r-1}$-smooth and all their derivatives are also of order, *respectively*, *$o(\lambda^k)$ and $o(\gamma^{-k})$.*

The proof is in Appendix B.

It is immediately seen from Lemma 3.6 that the geometrical structure of the sets of strips σ_k^0 and σ_k^1 is, in essence, the same as in the linear case. Indeed, estimates (3.9.6) imply that the strips lie within wedges along the non-leading coordinates (because contraction and expansion in the non-leading coordinates are asymptotically much stronger than that in the leading coordinates). Estimates (3.9.5) imply that in the leading coordinates the geometrical structure is determined mainly by the linear terms of the map T: the strips belong to the roulettes if the leading multipliers are complex; if the stable or unstable leading multiplier is real, then the corresponding strips accumulate, respectively, to W^u_{loc} or W^s_{loc} from one side if the multiplier is positive and from the both sides if it is negative.

Note that we derive here this picture based on Lemma 3.6 which is valid only for maps of class $\mathbb{C}^r$ with $r \geq 2$. To prove that the same geometry persists in the $\mathbb{C}^1$-case, one may use the modified boundary-value problem introduced in Sec. 5.2.

We must note that Lemma 3.6 may not hold unless one performs the preliminary reduction of the map T to the special form where the functions f and g satisfy condition (3.9.4). Let us show this on the following example.

Consider a three-dimensional map T_0 of the following form

$$\bar{x} = \lambda x, \quad \bar{u} = \lambda_2 u + xy, \quad \bar{y} = \gamma y\,,$$

where $0 < \lambda_2 < \lambda < 1 < \gamma$. Here $O(0,0,0)$ is a saddle fixed point. The equation of the two-dimensional stable invariant manifold W^s is $y = 0$, and that of the one-dimensional unstable invariant manifold W^u is $x = u = 0$. The equation of the non-leading stable invariant manifold $W^{ss} \in W^s$ is $y = x = 0$.

The boundary-value problem for the map T_0 reads: *Given the initial data* (x_0, u_0, y_k) *and given* k, *find* (x_k, u_k, y_0) *such that* $T_0^k(x_0, u_0, y_0) = (x_k, u_k, y_k)$.

We can recast the system in the form

$$\begin{aligned}
x_j &= \lambda^j x_0\,, \\
u_j &= \lambda_2^j u_0 + \sum_{s=0}^{j-1} \lambda_2^{j-s-1} \cdot \lambda^s x_0 \cdot \gamma^{s-k} y_k\,, \\
y_j &= \gamma^{j-k} y_k\,, \\
(j &= 0, 1, \ldots, k)\,.
\end{aligned} \tag{3.9.7}$$

From (3.9.7) we see that $x_k = \lambda^k x_0$ and $y_0 = \gamma^{-k} y_k$. Yet

$$\begin{aligned}
u_k &= \lambda_2^k u_0 + \sum_{s=0}^{k-1} \lambda_2^{k-s-1} \cdot \lambda^s x_0 \cdot \gamma^{s-k} y_k \\
&= \lambda_2^k u_0 + \lambda_2^{k-1} \gamma^{-k} x_0 y_k \sum_{s=0}^{k-1} \left(\frac{\lambda\gamma}{\lambda_2}\right)^s .
\end{aligned}$$

Since

$$\delta = \frac{\lambda\gamma}{\lambda_2} > 1\,,$$

the coefficient

$$\sum_{s=0}^{k-1}\Big(\frac{\lambda\gamma}{\lambda_2}\Big)^s = \frac{\delta^k-1}{\delta-1} \ \sim\ \frac{\lambda^{k-1}}{\lambda_2^{k-1}}\gamma^{k-1}\,.$$

We see that if $x_0 y_k \neq 0$, then $u_k \sim \lambda^k$ for sufficiently large k. Thus, although $\lambda_2 < \lambda$, the velocity of the convergence in both the leading and the non-leading coordinates is the same, in contrast to Lemma 3.6.

3.10. Normal coordinates in a neighborhood of a periodic trajectory

In the following sections we will focus on an approach for studying periodic trajectories which is based on the reduction to a system of non-autonomous periodic equations whose dimension is one less than the dimension of the original system. We shall also examine the problem of constructing a Poincaré map, and of calculating the multipliers of a periodic trajectory.

Let us consider an $(n+1)$-dimensional $\mathbb{C}^r$-smooth ($r \geq 1$) system of differential equations

$$\dot{x} = X(x)\,, \tag{3.10.1}$$

which has a periodic trajectory L: $x = \varphi(t)$ of period τ.

Theorem 3.11. *There exists a $\mathbb{C}^r$-smooth coordinate transformation and rescaling of time such that in a small neighborhood of a periodic trajectory L the system takes the form*

$$\begin{aligned} \dot{y} &= A(\theta)y + F(\theta, y)\,, \\ \dot{\theta} &= 1\,, \end{aligned} \tag{3.10.2}$$

where $y \in \mathbb{R}^n$, $\theta \in \mathbb{S}^1$. Here, $A(\theta)$ is a $\mathbb{C}^r$-smooth $(n \times n)$-matrix of period τ with respect to θ. The $\mathbb{C}^r$-smooth function F is also periodic with respect to θ and has period τ. Moreover,

$$F(\theta, 0) = 0, \qquad F'_\theta(\theta, 0) = 0\,. \tag{3.10.3}$$

Remark. The rescaling of time is equivalent to multiplication of the right-hand side by a scalar function. Thus, without the rescaling, the system in normal coordinates is written as

$$\begin{aligned} \dot{y} &= A(\theta)y + \tilde{F}(\theta, y)\,, \\ \dot{\theta} &= 1 + b(\theta, y)\,, \end{aligned} \tag{3.10.4}$$

where

$$\tilde{F}(\theta, 0) = 0, \quad \tilde{F}'_y(\theta, 0) = 0, \quad b(\theta, 0) = 0\,.$$

Proof of the theorem. The original system may be reduced to the form (3.10.2) in the following way. At each point $M_\theta(x = \varphi(\theta))$ we choose $(n+1)$ linearly independent vectors $(N_0(\theta), N_1(\theta), \ldots, N_n(\theta))$, where $N_0(\theta) \equiv \varphi'(\theta) = X(\varphi(\theta))$ is the velocity vector which is tangent to the periodic trajectory L at M_θ. Assume that $N_i(\theta)$ $(i = 1, \ldots, n)$ are smooth functions of θ. Let $\mathcal{M}_\theta$ be the space spanned on $(N_1(\theta), \ldots, N_n(\theta))$, *i.e.* the space $\mathcal{M}_\theta$ is transverse to L.

Let us denote the coordinates in the space $\mathcal{M}_\theta$ with the basis $(N_1(\theta), \ldots, N_n(\theta))$ by $(y_1, \ldots, y_n)$. If a point $M \in \mathcal{M}_\theta$ has coordinates the $(y_1, \ldots, y_n)$, then the vector connecting the points M_θ and M (see Fig. 3.10.1) is given by

$$\overline{M_\theta\, M} = y_1 N_1(\theta) + \cdots + y_n N_n(\theta)\,.$$

Thus, the original coordinates x of the point M are given by the formula

$$x = \varphi(\theta) + y_1 N_1(\theta) + \cdots + y_n N_n(\theta) \tag{3.10.5}$$

or by

$$\begin{aligned} x_1 &= \varphi_1(\theta) + y_1\, N_{11}(\theta) + \cdots + y_n\, N_{n,1}(\theta)\,, \\ x_2 &= \varphi_2(\theta) + y_1\, N_{12}(\theta) + \cdots + y_n N_{n,2}(\theta)\,, \\ &\;\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\ x_{n+1} &= \varphi_{n+1}(\theta) + y_1\, N_{1,n+1}(\theta) + \cdots + y_n\, N_{n,n+1}(\theta)\,, \end{aligned} \tag{3.10.6}$$

where N_{ij} is the j-th component of the vector N_i, and φ_j is the j-th coordinate of the point M on the periodic trajectory L.

Formula (3.10.5) can be viewed as a smooth change of variables $(\theta, y_1, \ldots, y_n) \leftrightarrow (x_1, \ldots, x_{n+1})$. In order to show that this is really a good change of

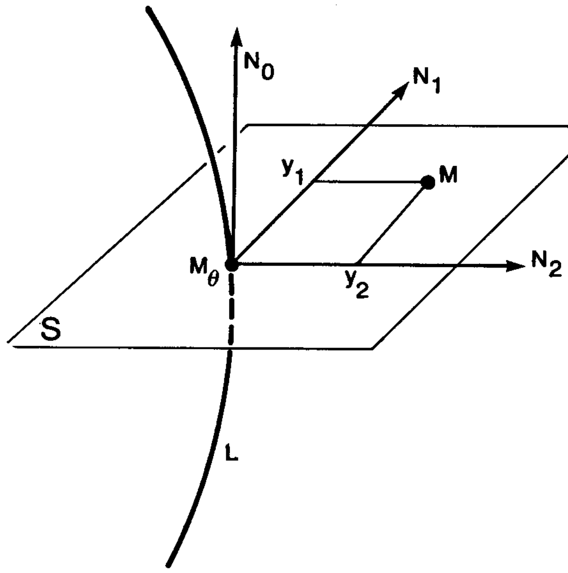


Fig. 3.10.1. The normal coordinates near a periodic trajectory. The vectors $N_i(\theta)$ in the cross-section S are orthogonal to the velocity vector N_0.

variables, one must check the non-singularity of the Jacobian matrix J. The value of vector y for the points on L is equal to zero, *i.e.* $(y_1, y_2, \ldots, y_n) = (0, 0, \ldots, 0)$, and since we are concerned with a small neighborhood of the periodic trajectory, it is sufficient to verify that J does not vanish at $y = 0$.

From (3.10.6) we obtain

$$J(\theta, y) = \det \begin{pmatrix} \varphi_1'(\theta) + \sum_{i=1}^{n} y_i N_{i1}'(\theta) & N_{11}(\theta) & \cdots & N_{n,1}(\theta) \\ \varphi_2'(\theta) + \sum_{i=1}^{n} y_i N_{i2}'(\theta) & N_{12}(\theta) & \cdots & N_{n,2}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n+1}'(\theta) + \sum_{i=1}^{n} y_i N_{i,n+1}'(\theta) & N_{1,n+1}(\theta) & \cdots & N_{n,n+1}(\theta) \end{pmatrix}.$$

Upon substituting $y = 0$ the Jacobian matrix becomes

$$J(\theta, 0) = \begin{vmatrix} \varphi_1^{'}(\theta) & N_{11}(\theta) & \cdots & N_{n,1}(\theta) \\ \varphi_2^{'}(\theta) & N_{12}(\theta) & \cdots & N_{n,2}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{n+1}^{'}(\theta) & N_{1,n+1}(\theta) & \cdots & N_{n,n+1}(\theta] \end{vmatrix}.$$

The first column of $J(\theta, 0)$ is composed of the components of the vector $N_0(\theta)$; the remaining columns are the components of the vectors $N_i(\theta)$. By construction, these vectors must be linearly independent for all θ's, *i.e.* the Jacobian matrix $J(\theta, 0)$ is non-singular.

Let us write system (3.10.1) in the new variables. Substitution of (3.10.5) into (3.10.1) gives

$$\begin{aligned}(X(\varphi(\theta)) + y_1\, N_1'(\theta) + \cdots + y_n\, N_n'(\theta))\, \dot{\theta} + \dot{y}_1\, N_1(\theta) + \cdots \\ + \dot{y}_n\, N_n(\theta) = X(\varphi(\theta) + y_1 N_1(\theta) + \cdots + y_n N_n(\theta))\,.\end{aligned} \tag{3.10.7}$$

It follows directly from (3.10.7) that $\dot{\theta} = 1$ and $\dot{y} = 0$ at $y = 0$. Thus, the system takes the form (3.10.4).

Reduction to the form (3.10.2) can be achieved by changing the time variable as follows:

$$dt = \frac{d\tilde{t}}{1 + b(\theta,\, y)}\,. \tag{3.10.8}$$

The missing point in this proof is a method for constructing a family of sufficiently smooth vectors $(N_1(\theta), \ldots, N_n(\theta))$ which along with the vector $N_0(\theta)$ comprise a linearly-independent system. Instead of describing this algorithm we present another proof which is based on an approach we will use further.

Choose a small cross-section S through L. Let $y = (y_1, \ldots, y_n)$ be the coordinates on S. Let $\mathcal{X}(x, t)$ be the time t shift of a point x along the corresponding trajectory of system (3.10.1). For each point x in a small neighborhood of L there is a uniquely defined $t(x) \geq 0$ such that $y = \mathcal{X}(-t(x), x)$ is the first point of the intersection of the backward trajectory of the point x and S. By definition $t(x) \leq \bar{t}(y)$ where $\bar{t}(y)$ is the Poincaré return time. We may rescale time so that this return time is constant for all small y: $\bar{t}(y) \equiv \tau$. To do this we define a new time $\tilde{t}$ as

$$d\tilde{t} = \left(1 + \alpha(y)\xi\left(\frac{t(x)}{\bar{t}(y)}\right)\right) dt\,, \tag{3.10.9}$$

where $y \equiv \mathcal{X}(-t(x), x)$ is constant along the segment of the trajectory of the point x between two its sequential intersections with the cross-section S. The value of y jumps when x crosses S. To make the transformation (3.10.9) continuous we choose the function ξ identically equal to zero when its argument is close to 0 or to 1 (this corresponds to the situation where $t(x)$ is close to 0 or to $\bar{t}(y)$, *i.e.* to the moments of intersection of the trajectory of x with S). Apart from these values, ξ must be a smooth non-zero function. The existence of such functions ξ is a standard fact. Moreover, we may require that $\int_0^1 \xi(s)ds = 1$.

Thus, we have a smooth transformation of time provided that α is sufficiently small. A new return time is found as

$$\int_0^{\bar{t}(y)} \frac{d\tilde{t}}{dt} dt = \bar{t}(y) + \alpha(y) \int_0^{\bar{t}(y)} \xi\left(\frac{t}{\bar{t}(y)}\right) dt = \bar{t}(y)(1+\alpha(y)) \,. \tag{3.10.10}$$

Hence, if we let $\alpha(y) = \frac{\tau}{\bar{t}(y)} - 1$, we find that the new return time is constant indeed. Observe that $\bar{t}(y)$ was close to τ for small y, whence $\alpha(y)$ is small and, therefore, the factor in (3.10.9) is non-zero. Thus, formula (3.10.9) does give a good rescaling of time.

Let

$$\bar{y} = By + F_0(y), \tag{3.10.11}$$

be the Poincaré map $T : S \to S$, where $F_0(0) = 0$ and $F_0'(0) = 0$. By definition,

$$T(y) = \mathcal{X}(\tau, y) \,. \tag{3.10.12}$$

As we mentioned earlier, the product of the eigenvalues of the matrix B (equiv. of the multipliers of L) is positive. Therefore, one can prove that there is a family of non-singular matrices $\tilde{A}(\theta)$ $(0 \leq \theta \leq \tau)$ such that $\tilde{A}(0) = I$, $\tilde{A}(\tau) = B$, $\tilde{A}'(0) = \tilde{A}'(\tau) = 0$ and $\tilde{A}(\theta)$ depends $\mathbb{C}^{r+1}$-smoothly on θ (we leave this quite standard fact without proof).

Consider a family of diffeomorphisms $Y_\theta : y_0 \mapsto y_\theta$ $(0 \leq \theta \leq \tau)$

$$y_\theta = \tilde{A}(\theta) y_0 + \eta(\theta) F_0(y_0) \,, \tag{3.10.13}$$

where $\eta(\theta)$ is a $\mathbb{C}^{r+1}$-smooth scalar function which is equal identically to zero in a small neighborhood of $\theta = 0$, and is equal identically to 1 in a small neighborhood of $\theta = \tau$ (this also implies that $\eta'(\theta) \equiv 0$ for values of θ close to 0, and to τ). We assume, as well, that $\tilde{A}'(\theta) = 0$ for θ close to 0 and to

τ. Thus, Y_θ is an identity map at all θ close to zero and it coincides with the Poincaré map T at all θ close to τ.

Since $F_0'(0) = 0$ and $\tilde{A}(\theta)$ is non-degenerate for all θ, the map (3.10.13) is invertible, *i.e.*

$$y_0 = \tilde{A}^{-1}(\theta)y_\theta + F_1(\theta, y_\theta)\,, \tag{3.10.14}$$

where $F_1(0,0) = 0$, $F_{1y}'(\theta, 0) = 0$ and $F_{1\theta}' \equiv 0$ for all θ close to 0 and τ.

Let us make a coordinate transformation $(\theta, y_1, \ldots, y_n) \to (x_1, \ldots, x_{n+1})$ by the following rule

$$x = \mathcal{X}(\theta, Y_\theta^{-1}(y))\,. \tag{3.10.15}$$

In other words, we identify the time θ shift of the point $y_0 \in S$ along the trajectories of system (3.10.1) and the time θ shift of y_0 which is given by (3.10.13). When θ is close to 0, Eq. (3.10.15) reads as

$$x = \mathcal{X}(\theta, y)\,, \tag{3.10.16}$$

and when θ is close to τ, it reads as

$$x = \mathcal{X}(\theta, T^{-1}(y))\,. \tag{3.10.17}$$

By definition (3.10.12), formula (3.10.17) coincides with

$$x = \mathcal{X}(\theta - \tau, y)\,.$$

Comparing the latter with (3.10.16) we get that the $\mathbb{C}^r$-smooth transformation of coordinates (3.10.15) is τ-periodic.

The evolution of the new y-coordinates is given by (3.10.13), where $\dot{\theta} = 1$. From (3.10.14) we have

$$\begin{aligned}\dot{y} &= \tilde{A}'(\theta)y_0 + \eta'(\theta)F_0(y_0)\\ &= \tilde{A}'(\theta)\tilde{A}(\theta)^{-1}y + \tilde{A}'(\theta)F_1(\theta, y) + \eta'(\theta)F_0(\tilde{A}(\theta)^{-1}y + F_1(\theta, y))\,.\end{aligned} \tag{3.10.18}$$

Denoting

$$\begin{aligned}A(\theta) &= \tilde{A}'(\theta)A(\theta)^{-1}\\ F(\theta, y) &= \tilde{A}'(\theta)F_1(\theta, y) + \eta'(\theta)F_0(\tilde{A}(\theta)^{-1}y + F_1(\theta, y))\end{aligned} \tag{3.10.19}$$

this completes the proof.

The coordinates in which the representation (3.10.2) is valid are called *normal coordinates*. In the normal coordinates, $y = 0$ is the equation of the periodic trajectory L. The phase θ parametrizes the point on L. Observe that the normal coordinates are not unique: A different base $N_i(\theta)$ $(i = 1, \dots, n)$ gives a different coordinate transformation. Nevertheless, the advantage of our construction is that it increases the smoothness of the system in the normal coordinates. Namely, the following statement holds:

Lemma 3.7. *All derivatives of the right-hand side of* (3.10.2) *with respect to* y *are* $\mathbb{C}^r$*-smooth functions of* θ *at* $y = 0$.

Proof. It is immediately seen from (3.10.13) that all the derivatives $\left.\frac{d^k y_0}{dy_\theta^k}\right|_{y_0=0}$ $(k = 1, \dots, r)$ are $\mathbb{C}^{r+1}$-smooth functions of θ. According to (3.10.13), it means that all derivatives of F_1 with respect to y_θ at $y_\theta = 0$ are also $\mathbb{C}^{r+1}$-smooth functions of θ. Now, the lemma follows from (3.10.19).

Observe that our construction does not work in the case where the system is analytic (the functions ξ, η cannot be analytic because they are identically zero on some intervals). To resolve the analytic case, one may first make a $\mathbb{C}^\infty$-smooth transformation by formula (3.10.15), then take $\mathbb{C}^\infty$-vectors $\tilde{N}_i(\theta)$ as images of the basis vectors in the y-space by (3.10.15). Since the system of the vectors $\{N_0(\theta), \tilde{N}_1(\theta), \dots, \tilde{N}_n(\theta)\}$ is linearly independent, for a sufficiently close analytical approximation $(N_1(\theta), \dots, N_n(\theta))$ the condition of linear independence will not be destroyed. Now, after we have found the linear independent system of vectors $\{N_0(\theta), N_1(\theta), \dots, N_n(\theta)\}$ which depend analytically on θ, the sought coordinate transformation is given by (3.10.6) and (3.10.8).

Note that once knowing the solution $L : \{x = \varphi(t)\}$ explicitly, one can find the system of the normal vectors $(N_1(\theta), \dots, N_n(\theta))$, and hence find the coordinate transformation that gives the right-hand side of (3.10.2) explicitly too.

The form (3.10.2) is very convenient as any plane $\theta =$ const. is the cross-section and the return time of any point on the cross-section is always the same and is equal to τ. Let us choose the plane $S : \theta = 0$ as such a cross-section and determine the Poincaré map $S \to S$. For trajectories starting from S, we have $\theta = t$ from the second equation in (3.10.2), and hence the problem is reduced to integrating the system

$$\dot{y} = A(t)\, y + F(t, y)\,. \tag{3.10.20}$$

The solution of this equation subject to the initial condition y_0 may be determined in the form of a series in powers of y_0 with time-dependent coefficients, by applying the method of successive approximations. For the first approximation we choose a solution of the linearized system

$$\dot{y}^{(1)} = A(t)\, y^{(1)} \,. \tag{3.10.21}$$

The m-th approximation is given by

$$\dot{y}^{(m)} = A(t)\, y^{(m)} + F(t, y^{(m-1)}) \,. \tag{3.10.22}$$

Let $\Phi(t)$ be the fundamental matrix of solutions of the system (3.10.21), *i.e.* the solution of system (3.10.21) has the form

$$y^{(1)}\,(t) = \Phi(t)\, y^0 \,.$$

Then, the solution of system (3.10.22) is given by the formula

$$y^{(m)}\,(t) = \Phi(t)\left(y^0 + \int_0^t \Phi^{-1}(s)\, F(s, y^{(m-1)}(s))\, ds\right) . \tag{3.10.23}$$

It is seen from (3.10.23) that each successive approximation differs from the previous one by terms of higher orders; namely

$$y^{(m)}(t) - y^{(m-1)}(t) = \Phi(t) \int_0^t \Phi^{-1}(s)(F(s, y^{(m-1)}(s)) - F(s, y^{(m-2)}(s)))\, ds$$

whence

$$\|y^{(m)} - y^{(m-1)}\| \sim \|F'\|\, \|y^{(m-1)} - y^{(m-2)}\| = o(\|y^{(m-1)} - y^{(m-2)}\|)$$

(because $F' = 0$ at $y = 0$). By using formula (3.10.23) one can find terms of any order in the Taylor expansion of the solution of system (3.10.20). Plugging $t = \tau$ (the period) into the resulting expansion, one obtains the Taylor expansion of the Poincaré map.

We note also that the Poincaré map is represented in the form

$$\bar{y} = \Phi(\tau)\, y + \Psi(y) \,, \tag{3.10.24}$$

where the function $\Psi(y)$ vanishes together with its first derivatives when $y = 0$. The linear part of the Poincaré map has the form

$$\bar{y} = \Phi(\tau)\, y \,.$$

whence the multipliers $(\rho_1, \ldots, \rho_n)$ of the periodic trajectory L can be found as the eigenvalues of the matrix $\Phi(\tau)$. Thus, when the periodic solution $\{x = \varphi(t)\}$ and the fundamental matrix $\Phi(t)$ of the linear system

$$\dot{y} = A(t)\, y\,, \tag{3.10.25}$$

are known, then there exists a standard procedure for constructing the Poincaré map and for computing the multipliers of the periodic trajectory.

3.11. The variational equations

The problem of the stability of a periodic trajectory does not differ essentially from the corresponding problem for equilibrium states. In both cases the stability conditions are determined by the equations of the first approximation, *i.e.* by the associated linearized system for an equilibrium state, or the so-called variational equation for a periodic trajectory .

Let $x = \varphi(t)$ be a periodic solution of period τ of an $(n+1)$-dimensional autonomous system

$$\dot{x} = X(x)\,. \tag{3.11.1}$$

Introduce a new variable ξ such that

$$x = \xi + \varphi(t)\,.$$

In terms of the new variable the system takes the following form

$$\dot{\xi} = D(t)\xi + \cdots\,,$$

where

$$D(t) = \left.\frac{\partial X}{\partial x}\right|_{x=\varphi(t)}$$

and the ellipsis denotes terms of a higher order with respect to ξ. Observe that this change of variables reduced an $(n+1)$-dimensional autonomous system to an $(n+1)$-dimensional non-autonomous system.

The linear periodic system

$$\dot{\xi} = D(t)\xi \tag{3.11.2}$$

is called *a variational equation.* Obviously, if $\xi(t)$ is a solution of (3.11.2), then $\xi(t+\tau)$ is also a solution. Indeed, after the shift of time $t \to t+\tau$ we obtain

$$\frac{d\xi(t+\tau)}{d(t+\tau)} = D(t+\tau)\xi(t+\tau)\,,$$

and, consequently,

$$\frac{d\xi(t+\tau)}{dt} = D(t)\xi(t+\tau)\,.$$

The general solution of (3.11.2) is

$$\xi(t) = \Psi(t)\xi(0) \tag{3.11.3}$$

where $\Psi(t)$ is *the fundamental matrix*, whose columns $\Psi^{(i)}(t)$ $(i = 1, \ldots, n+1)$ are the solutions of (3.11.2) which start at $t = 0$ with basis vectors. Since $\Psi^{(i)}(t+\tau)$ is a solution as well, it follows from (3.11.3) that $\Psi^{(i)}(t+\tau) = \Psi(t)\Psi^{(i)}(\tau)$, or

$$\Psi(t+\tau) = \Psi(t)\Psi(\tau)\,. \tag{3.11.4}$$

The equation

$$|\Psi(\tau) - \rho I| = 0 \tag{3.11.5}$$

is called *a characteristic equation*. The roots $(\rho_1, \ldots, \rho_{n+1})$ of (3.11.5) are called *the characteristic roots* or *Floquet multipliers*.

The characteristic equation is invariant with respect to any change of variables

$$\eta = Q(t)\xi\,, \tag{3.11.6}$$

where the matrix $Q(t)$ is non-singular for all t, depends smoothly on time and is periodic of period τ. Indeed, after this change of variables system (3.11.2) remains linear periodic system. Denote its fundamental matrix as $\tilde{\Psi}(t)$; *i.e.* the general solution is $\eta(t) = \tilde{\Psi}(t)\eta(0)$. By (3.11.3), (3.11.6) we have $\tilde{\Psi}(t) = Q(t)\Psi(t)Q(0)^{-1}$. Thus, by virtue of τ-periodicity of $Q(t)$, the matrix $\tilde{\Psi}(\tau)$ is similar to $\Psi(\tau)$:

$$\tilde{\Psi}(\tau) = Q(0)\Psi(\tau)Q(0)^{-1}\,.$$

Thus,

$$|\tilde{\Psi}(\tau) - \rho I| = |Q(0)\Psi(\tau)Q(0)^{-1} - \rho I| = |Q(0)(\Psi(\tau) - \rho I)Q(0)^{-1}| = |\Psi(\tau) - \rho I|$$

which proves the claim.

It follows that *the Floquet multipliers of a periodic trajectory do not depend on the specific choice of coordinates* x. Indeed, let $y = h(x)$ be a diffeomorphism transforming the system (3.11.1) into $\dot{y} = Y(y)$ in some small neighborhood of the periodic trajectory L. In the new variables the equation of L is $y = h(\varphi(t)) = \psi(t)$. The variational equation for $\psi(t)$, which is now given by

$$\dot{\eta} = \left.\frac{\partial Y}{\partial y}\right|_{y=\psi(t)} \eta\,,$$

is obtained from (3.11.2) by the change of variables (3.11.6) with $Q(t) = h'(\varphi(t))$. Hence, the characteristic equation and the characteristic roots remain unchanged indeed.

In particular, in the normal coordinates, when the system has the form (3.10.4), the linearization in a neighborhood of the periodic solution ($y = 0$, $\theta = t \bmod \tau$) gives the following variational equation

$$\dot{\xi} = \begin{pmatrix} A(t) & 0 \\ b'_y(t,0) & 0 \end{pmatrix} \xi \,.$$

It is easy to see that the fundamental matrix of this system has the form

$$\begin{pmatrix} \Phi(t) & 0 \\ \beta(t) & 1 \end{pmatrix} ,$$

where $\Phi(t)$ is the fundamental matrix of the system

$$\dot{\eta} = A(t)\eta \,,$$

with $\Phi(0) = I$. Hence, the characteristic equation can be represented in the form

$$|\Phi(\tau) - \rho I|(\rho - 1) = 0 \,.$$

On the other hand, we showed in the previous section that multipliers of the Poincaré map near L are the roots of the equation $|\Phi(\tau) - \rho I| = 0$. Therefore, *the Floquet multipliers* $(\rho_1, \ldots, \rho_n)$ *coincide with the multipliers of the fixed point of the Poincaré map and the last Floquet multiplier* ρ_{n+1} *is always trivial*: $\rho_{n+1} = 1$.

The existence of the trivial characteristic root is a peculiarity of the variational equations near a periodic trajectory of an autonomous system. Since $x = \varphi(t)$ is a solution, *i.e.* since

$$\dot{\varphi}(t) = X(\varphi(t)) \,,$$

we obtain by differentiating with respect to t

$$\frac{d\dot{\varphi}(t)}{dt} = \frac{\partial X(\varphi(t))}{\partial x} \, \dot{\varphi}(t) \,.$$

This implies that $\xi(t) = \dot{\varphi}(t)$ is a solution of the variational equation (3.11.2). Thus, by (3.11.3), $\dot{\varphi}(t) = \Psi(t)\dot{\varphi}(0)$. Since $\varphi(t)$ is a periodic function of period τ, it follows that $\dot{\varphi}(0) = \Psi(\tau)\dot{\varphi}(0)$. This means that $\dot{\varphi}(0)$ is always the eigenvector of $\Psi(\tau)$ and the corresponding characteristic root (ρ_{n+1}) is always equal to 1. This observation is due to Poincaré.

Let us introduce the values

$$\lambda_k = \frac{\ln \rho_k}{\tau} = \frac{1}{\tau}[\ln|\rho_k| + i(\arg\rho_k + 2\pi m_k)]\,, \qquad k = 1, \dots, n+1 \tag{3.11.7}$$

which are called *the characteristic exponents*. Observe from (3.11.7) that λ_k is defined modulo $i2\pi m_k/\tau$, where m_k is an integer. However, ($\mathrm{Re}\lambda_1, \dots, \mathrm{Re}\lambda_{n+1}$) are uniquely defined. They are called *the Lyapunov exponents* of the periodic trajectory $x = \varphi(t)$.

These quantities have sense for any linear periodic system of type (3.11.2). Recall that in case the variational equation is obtained from an autonomous system there is always a trivial characteristic root, hence one Lyapunov exponent is always zero in this case.

The columns of the fundamental matrix satisfy (3.11.2), *i.e.* $\frac{d}{dt}\Psi(t) = D(t)\Psi(t)$. Hence,

$$\frac{d}{dt}\det\Psi(t) = \mathrm{tr}\, D(t)\cdot \det\Psi(t)$$

which gives the *Wronsky formula*

$$\det\Psi(t) = e^{\int_0^t \mathrm{tr}\, D(s)ds}\,.$$

At $t = \tau$ we obtain

$$\rho_1\rho_2\cdots\rho_{n+1} = e^{\int_0^\tau \mathrm{tr}\, D(s)ds}\,, \tag{3.11.8}$$

where ($\rho_1, \dots, \rho_{n+1}$) are the characteristic roots. It is clear that all $\rho_1, \dots, \rho_{n+1}$ are different from zero and that $\Psi(\tau)$ is non-singular.

When the linear system (3.11.2) is obtained from the autonomous system (3.11.1) this formula reads as

$$\rho_1\rho_2\cdots\rho_n = e^{\int_0^\tau \mathrm{div}\, X|_{x=\varphi(s)}ds}\,, \tag{3.11.9}$$

or

$$\lambda_1 + \cdots + \lambda_n = \frac{1}{\tau}\int_0^\tau \mathrm{div}\, X|_{x=\varphi(s)}ds\,. \tag{3.11.10}$$

Note that in the general case, finding the fundamental matrix of the variational equation or its characteristic roots in the explicit form is not possible. The two-dimensional case is the only exception. In this case the formula

(3.11.10) gives the single non-trivial Lyapunov characteristic exponent as

$$\lambda = \frac{1}{\tau}\int_0^\tau \left[\frac{\partial X_1(\varphi_1(t),\varphi_2(t))}{\partial x_1} + \frac{\partial X_2(\varphi_1(t),\varphi_2(t))}{\partial x_2}\right] dt\,.$$

Let $\xi^{(k)}(0)$ be an eigenvector of $\Psi(\tau)$ corresponding to a multiplier ρ_k. The solution starting with $\xi^{(k)}(0)$ is $\xi^{(k)}(t) = \Psi(t)\xi^{(k)}(0)$. Since $\Psi(\tau)\xi^{(k)}(0) = \rho_k\xi^{(k)}(0)$, it follows from (3.11.4) that

$$\xi^{(k)}(t+\tau) = \rho_k\xi^{(k)}(t)$$

for all t. It follows that the function

$$\phi_k(t) = e^{-\lambda_k t}\xi^{(k)}(t)$$

is τ-periodic: since $e^{\lambda_k\tau} = \rho_k$, we have

$$\phi_k(t+\tau) = e^{-\lambda_k(t+\tau)}\xi^{(k)}(t+\tau) = e^{-\lambda_k t}e^{-\lambda_k\tau}\rho_k\xi^{(k)}(t) = \phi_k(t)\,.$$

Therefore,

$$\xi^{(k)}(t) = \phi_k(t)e^{\lambda_k t}\,, \tag{3.11.11}$$

where $\phi_k(t)$ is a periodic function.

A more general statement also holds, known as *Floquet theorem* [24]: the fundamental matrix Ψ of a linear time periodic system satisfies

$$\Psi(t) = \Phi(t)e^{\Lambda t} \tag{3.11.12}$$

where $\Phi(t)$ is a τ-periodic matrix and Λ is a constant matrix whose eigenvalues are the characteristic exponents $(\lambda_1,\ldots,\lambda_{n+1})$.

For a proof note that by (3.11.4) the matrix $\Phi(t) = \Psi(t)e^{-\Lambda t}$ is τ-periodic if

$$\Psi(\tau) = e^{\Lambda\tau}\,, \tag{3.11.13}$$

i.e. if $\tau\Lambda$ is a logarithm of $\Psi(\tau)$. The existence of a logarithm of a non-singular matrix is a well-known fact. For example, if all $(\rho_1,\rho_2,\ldots,\rho_{n+1})$ are different, then $\Phi(t)$ is similar to a diagonal matrix:

$$\Psi(\tau) = P\begin{pmatrix}\rho_1 & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \rho_{n+1}\end{pmatrix}P^{-1}\,,$$

and the matrix Λ is simply

$$\Lambda = P \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{n+1} \end{pmatrix} P^{-1}\,.$$

It follows immediately from (3.11.12) that the periodic change of variables $\xi = \Phi(t)y$ brings the system (3.11.2) to the autonomous form

$$\dot{y} = \Lambda y\,.$$

Note that Eq. (3.11.13) defines, in general, a complex-valued matrix Λ, even if $\Psi(\tau)$ is real. Thus, the matrix $\Phi(t)$ is complex and a real τ-periodic transformation which brings the system to the autonomous form does not always exists. Nevertheless, the following theorem is valid.

Theorem 3.12. (Lyapunov) *There exists a change of variables of the form* $\xi = \tilde{\Phi}(t)y$, *where* $\tilde{\Phi}(t)$ *is a real periodic matrix of period* 2τ, *which transforms system* (3.11.3) *into*

$$\dot{y} = \tilde{\Lambda} y\,,$$

where $\tilde{\Lambda}$ *is a real constant matrix whose eigenvalues satisfy* $e^{2\tau\tilde{\lambda}_k} = \rho_k^2$.

It is seen that $\operatorname{Re}\tilde{\lambda}_k = \operatorname{Re}\lambda_k$ where λ_k are the characteristic exponents of (3.11.2). Thus,

- If all of the Lyapunov exponents are negative, then the solution $\xi = 0$ of the linear system (3.11.2) is exponentially stable as $t \to +\infty$;
- If there is at least one positive Lyapunov exponent, then the trivial solution is unstable.

For a proof of the Lyapunov theorem, let us denote as U the eigenspace of the matrix $\Psi(\tau)$ corresponding to all real negative ρ_k and denote as V the eigenspace corresponding to the rest of eigenvalues of $\Psi(\tau)$. So, $\xi = (u, v)$ where $u \in U$, $v \in V$. For the linear transformation

$$\sigma : (u, v) \mapsto (-u, v) \qquad (3.11.14)$$

the matrix $\tilde{A} = \sigma\Psi(\tau) = \Psi(\tau)\sigma$ does not have real negative eigenvalues. By construction, its eigenvalues $\tilde{\rho}_k$ satisfy $\tilde{\rho}_k^2 = \rho_k^2$. The matrix $\tilde{A}$ is similar to a matrix in the real Jordan form:

$$\tilde{A} = P(A^\circ + \Delta A)P^{-1}$$

where

$$\begin{aligned}
A^\circ_{kk} &= \tilde\rho_k \qquad \text{if } \tilde\rho_k \text{ is real} \\
\begin{pmatrix} A^\circ_{kk} & A^\circ_{k,k+1} \\ A^\circ_{k+1,k} & A^\circ_{k+1,k+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} re^{i\phi} & 0 \\ 0 & re^{-i\phi} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} \\
&= \begin{pmatrix} r\cos\phi & -r\sin\phi \\ r\sin\phi & r\cos\phi \end{pmatrix} \text{ if } (\tilde\rho_k = re^{i\phi}, \tilde\rho_{k+1} = re^{-i\phi}),
\end{aligned} \tag{3.11.15}$$

all the other entries of A° are zero; the only non-zero entries in ΔA may be

$$\begin{aligned}
&(\Delta A)_{k,k+1} = 1 \quad \text{if } \tilde\rho_k = \tilde\rho_{k+1} \text{ are real multiple eigenvalues} \\
\text{and}& \\
&(\Delta A)_{k,k+2} = 1 \quad \text{if } \tilde\rho_k = \tilde\rho_{k+2} \text{ are complex multiple eigenvalues.}
\end{aligned} \tag{3.11.16}$$

One can check that the *real* logarithm of $\tilde A$ is given by the following formula:

$$\begin{aligned}
\ln \tilde A &= P \ln(A^\circ + \Delta A) P^{-1} = P[\ln A^\circ + \ln(I + (A^\circ)^{-1}\Delta A)]P^{-1} \\
&= P\left[\ln A^\circ + \sum_{s=1}^{\infty} \frac{(-1)^s}{s} (A^\circ)^{-s} (\Delta A)^s\right] P^{-1},
\end{aligned} \tag{3.11.17}$$

where

$$\begin{aligned}
(\ln A^\circ)_{kk} &= \ln\tilde\rho_k \qquad \text{if } \tilde\rho_k \text{ is real,} \\
\begin{pmatrix} (\ln A^\circ)_{kk} & (\ln A^\circ)_{k,k+1} \\ (\ln A^\circ)_{k+1,k} & (\ln A^\circ)_{k+1,k+1} \end{pmatrix} &= \begin{pmatrix} \ln r & -\phi \\ \phi & \ln r \end{pmatrix} \quad \text{if } \tilde\rho_k = \tilde\rho^*_{k+1} = re^{i\phi}.
\end{aligned} \tag{3.11.18}$$

The formula (3.11.18) gives a real-valued matrix $\ln A^\circ$ because all real $\tilde\rho_k$ are positive by construction; the matrix $(A^\circ)^{-1}$ exists because all ρ_k are non-zero by (3.11.8); the series in (3.11.17) is convergent because $(\Delta A)^s \equiv 0$ for sufficiently large s. The expansion for $\ln(A^\circ + \Delta A)$ in (3.11.17) is a calque of the Taylor expansion for a scalar logarithm: the scalar arithmetic is applied here because the matrices $\ln A^\circ$, A° and ΔA commute, *i.e.* $A^\circ \cdot \ln A^\circ = \ln A^\circ \cdot A^\circ$, $A^\circ \cdot \Delta A = \Delta A \cdot A^\circ$, $\Delta A \cdot \ln A^\circ = \ln A^\circ \cdot \Delta A$ (see (3.11.15), (3.11.16),(3.11.18)).

Let us now take

$$\tilde\Lambda = \frac{1}{\tau} \ln \tilde A,$$

so

$$\Psi(\tau) = \sigma e^{\tilde\Lambda\tau}.$$

If follows that the matrix $\tilde{\Phi}(t) = \Psi(t)e^{-\tilde{\Lambda}t}$ satisfies

$$\tilde{\Phi}(t+\tau) = \tilde{\Phi}(t)\sigma \tag{3.11.19}$$

(see (3.11.4)). In particular, $\tilde{\Phi}(t)$ is 2τ-periodic. By construction, the general solution (3.11.3) of (3.11.2) is written as

$$\xi(t) = \tilde{\Phi}(t)e^{\tilde{\Lambda}t}\xi(0)$$

which means that, indeed, the transformation $y = \tilde{\Phi}(t)^{-1}\xi$ brings the system to the linear autonomous form.

Remark. If it is seen from (3.11.19) that there is no real negative Floquet multipliers, then the reduction to the real autonomous form is done by a periodic transformation of period τ. In case real negative multipliers exist, the involution σ is not identical; in this case we will call the functions satisfying (3.11.19) τ-*antiperiodic.*

Theorem 3.12 holds for any real linear system with time-periodic coefficients. In particular, applying the theorem to the linear part of y-equation of either (3.10.4) or (3.10.2) we get that *the normal coordinates can be introduced in such a way that the system near a periodic orbit is written as*

$$\begin{aligned} \dot{y} &= \Lambda y + F(\theta, y)\,, \\ \dot{\theta} &= 1 + b(\theta, y)\,; \end{aligned} \tag{3.11.20}$$

or, after rescaling the time, as

$$\begin{aligned} \dot{y} &= \Lambda y + F(\theta, y)\,, \\ \dot{\theta} &= 1\,. \end{aligned} \tag{3.11.21}$$

Here, the right-hand sides satisfy the τ-(anti)periodicity conditions

$$\begin{aligned} F(\theta+\tau, \sigma y) &= \sigma F(\theta, y)\,, \\ b(\theta+\tau, \sigma y) &= b(\theta, y)\,, \end{aligned} \tag{3.11.22}$$

where σ is the involution (3.11.14) changing sign of some of the variables y — those which correspond to the real negative multipliers.

3.12. Stability of periodic trajectories. Saddle periodic trajectories

It is well known that Lyapunov solved the problem of stability of a periodic trajectory of the system

$$\dot{x} = F(x, t)\,, \tag{3.12.1}$$

where $x = (x_1, \dots, x_n)$, and where F is a periodic function with respect to the time variable t. He gave the following definition.

Definition 3.2. *A solution $x = \varphi(t)$ of system* (3.12.1) *is called stable* (*in the sense of Lyapunov*) *if given arbitrary small $\varepsilon > 0$ there exists δ such that if $\|x_0 - \varphi(t_0)\| \leq \delta$, then $\|x(t) - \varphi(t)\| \leq \varepsilon$ for all $t \geq t_0$, where $x(t)$ is the solution with the initial condition x_0.*

Let system (3.12.1) have a periodic solution $x = \varphi(t)$ of period τ being either equal to the period of the function $F(x,t)$ or being divisible by it. Let us denote by $(\lambda_1, \dots, \lambda_n)$ *the characteristic exponents* of the associated variational equation

$$\dot{\xi} = F'_x(\varphi(t), t)\xi \,.$$

Theorem 3.13. (Lyapunov) *Let* $\operatorname{Re}\lambda_i < 0$ $(i = 1, \dots, n)$. *Then the solution $x = \varphi(t)$ is stable. Moreover, it is exponentially stable, i.e. any solution with initial conditions close to $\varphi(t_0)$ at $t = t_0$ tends to $\varphi(t)$ exponentially as $t \to +\infty$.*

Lyapunov proved this theorem for the case where the system (3.12.1) has an analytic right-hand side, though it also holds when the function F is only of $\mathbb{C}^1$-smoothness with respect to x and continuous with respect to t.

In a small neighborhood of $x = \varphi(t)$ system (3.12.1) can be brought to the form (see Sec. 3.11)

$$\dot{y} = \Lambda y + G(y, t)\,, \tag{3.12.2}$$

where Λ is a constant matrix such that the real parts of its eigenvalues are the Lyapunov exponents $(\operatorname{Re}\lambda_1, \dots, \operatorname{Re}\lambda_n)$. The function $G(y,t)$ is periodic of period τ or 2τ with respect to t. Moreover, $G(0,t) \equiv 0$, $G'_y(0,t) \equiv 0$. It follows that $\|G'_y\|$ is uniformly bounded by a small constant for all t and for all small y. After reducing the system to the form (3.12.2) the proof of Theorem 3.13 repeats the proof of the theorem on the validity of the linearization near a stable equilibrium state (Theorem 2.4).

Let us now consider an $(n+1)$-dimensional autonomous system

$$\dot{x} = X(x) \tag{3.12.3}$$

having a periodic solution $x = \varphi(t)$ of period τ. We have learned in Sec. 3.11 that one of the characteristic exponents of the variational equation of (3.12.3) is always equal to zero. Therefore, from the point of view of Lyapunov stability this situation corresponds to the critical case. Nevertheless, the following theorem is valid.

Theorem 3.14. (Andronov-Vitt) *If all n non-trivial characteristic exponents of a periodic solution of the system* (3.12.3) *have negative real parts, the periodic solution is stable in the sense of Lyapunov.*

This theorem justifies the linearization but only for a very weak form of stability. The problem is as follows. Let L be the corresponding periodic trajectory: $L = \{x : x = \varphi(\theta), 0 \leq \theta \leq \tau)\}$. Then for any two neighboring points on L, the associated solutions of (3.12.3) cannot approach asymptotically each other as $t \to +\infty$. It is easily seen when the system is written in the normal coordinates near L. Recall that in the normal coordinates (y, θ) near L where $\|y\|$ measures the distance to L and $\theta \in \mathbb{S}^1$ is the angular variable, the system recasts in the form (3.11.20) where $\|F'_y\|$ is uniformly bounded by a small constant for all t and for all small y. The real parts of the eigenvalues of Λ are the non-trivial Lyapunov exponents of L. Like in the Lyapunov Theorem 3.13, if all the eigenvalues have negative real parts, then

$$\|y(t)\| < \|y_0\| e^{-\lambda\, t} \tag{3.12.4}$$

where $\lambda > 0$. At the same time, since $b(\theta, 0) \equiv 0$, we have $\dot{\theta} = 1 + O(y) = 1 + O(e^{-\lambda t})$. Therefore,

$$\theta(t) = \theta_0 + t + \psi(t; \theta_0, y_0) \tag{3.12.5}$$

where $\psi(t)$ has a finite limit as $t \to +\infty$. Obviously, ψ vanishes at $y = 0$. Thus, at $y = 0$, the solutions with different values of initial phase θ_0 stay at a finite distance from each other for all times.

In the original coordinates x, the periodic trajectory L corresponds to a family of periodic solutions $x = \varphi(t + \theta_0)$ parametrized by the initial phase θ_0. Under the assumption that all non-trivial characteristic exponents lie in the open left half plane, Lyapunov posed the question: for which initial conditions $x(0) = x_0$ does the solution $x(t)$ approach $\varphi(t + \theta_0)$ in the limit $t \to +\infty$? He showed that the locus of such initial points is a surface S_{θ_0} passing through the point $\varphi(\theta_0)$. This implies that a small neighborhood of the stable periodic trajectory L is foliated into a family of the surfaces $\{S_\theta : 0 \leq \theta \leq \tau\}$, called *the Lyapunov surfaces*. Observe, that in normal coordinates the equation of the Lyapunov surfaces is given by $S_{\theta_0} = \{\theta + \psi(+\infty; \theta, y_0) = \theta_0\}$.

Finally, we come to the following theorem (see formulae (3.12.4), (3.12.5))

Theorem 3.15. (*On the asymptotical phase*) *Let all non-trivial characteristic exponents of a periodic trajectory L lie to the left of the imaginary axis. Then, given sufficiently small ε there exists δ such that if $\|x_0 - \varphi(\theta_0)\| < \delta$, then there exists ψ, $|\psi| < \varepsilon$ such that the solution $x(t)$, $x(0) = x_0$, satisfies the inequality*

$$\|x(t) - \varphi(t + \theta_0 + \psi)\| < Ke^{-\lambda t},$$

where K and λ are positive constants.

We refer the reader to a detailed proof of this theorem in the book by Codington and Levinson.

Another important concept, introduced by Poincaré, is the notion of orbital stability.

Definition 3.3. *A periodic trajectory L is orbitally stable as $t \to +\infty$ ($t \to -\infty$) if given any $\varepsilon > 0$ there exists δ such that any semi-trajectory $x(t)$, $0 \leq t < +\infty$ ($-\infty < t \leq 0$), such that $\|x(0) - \varphi(\theta_0)\| < \delta$ for some θ_0, lies in the ε-neighborhood of L.*

We say that the periodic trajectory L is *asymptotically orbitally stable* as $t \to +\infty$ if

$$\lim_{t \to +\infty} \operatorname{dist}(x(t), L) = 0\,.$$

where

$$\operatorname{dist}(x, L) = \inf_{0 \leq \theta \leq \tau} \|x - \varphi(\theta)\|\,.$$

Theorem 3.16. *If all multipliers of the periodic trajectory L lie inside the unit circle, then L is orbitally stable as $t \to +\infty$, and satisfies the following estimate*

$$\operatorname{dist}(x(t), L) \leq Ke^{-\lambda t},$$

where K and λ are positive constants.

This theorem follows directly from the theorem on the stability of the fixed point of the Poincaré map and the theorem on the continuous dependence of the solution on the initial conditions (or it immediately follows from (3.12.4)) because the multipliers of L lie inside the unit circle if and only if the non-trivial characteristic exponents of L lie to the left of the imaginary axis (see Sec. 3.11).

The case where all the multipliers of the periodic trajectory L are outside of the unit circle is reduced to the above case by means of reversion $t \to -t$. All trajectories in a small neighborhood of such an unstable periodic trajectory leave the neighborhood as t increases. The time, over which a trajectory escapes from the neighborhood, depends on the position of the initial point of the trajectory with respect to L, the closer the point is to the periodic trajectory the larger the escaping time is. Such unstable periodic trajectories are called *completely unstable* or *repelling*.

Let us consider next the case where some multipliers of L, $(\rho_1, \dots, \rho_k)$ lie inside and the rest $(\rho_{k+1}, \dots, \rho_n)$ lie outside of the unit circle.

In a neighborhood of the periodic trajectory L of period τ the system is written in the form (3.10.2). Dividing the first equation of (3.10.2) by the second we obtain the non-autonomous system with the periodic right-hand side

$$\frac{dy}{d\theta} = A(\theta)y + F(y, \theta) , \tag{3.12.6}$$

where $F(0, \theta) \equiv 0$, $F'_y(0, \theta) \equiv 0$. Just like in Sec. 3.10 we can integrate (3.12.6) with the initial data $(y^0, 0)$ and find a solution $y = y(\theta; y^0)$ which is $\mathbb{C}^r$-smooth with respect to both arguments. If we let $\theta = \tau$, we obtain the Poincaré map

$$\bar{y} = \Phi(\tau)y + \Psi(y) , \tag{3.12.7}$$

where $\Psi(0) = 0$, and $\Psi'(0) = 0$. The roots of the equation $|\Phi(\tau) - \rho I| = 0$ are the multipliers $(\rho_1, \dots, \rho_n)$ of L.

It follows from Hadamard's theorem (see Sec. 3.6) that two smooth invariant manifolds, stable $W^s_{loc}(O)$ and unstable $W^u_{loc}(O)$, pass through the point O at the origin. These manifolds are tangent, respectively, to the stable k-dimensional subspace and the $(n-k)$-dimensional unstable subspace of the associated linearized map $\bar{y} = \Phi y$ at the point O which we denote as E^s and as E^u. Let $y = (y_1, y_2)$ where $y_1 \in \mathbb{R}^k$ and $y_2 \in \mathbb{R}^{n-k}$. Let $y_2 = C^s y_1$ be the equation of E^s, and let $y_1 = C^u y_2$ be the equation of E^u, where C^s and C^u are some matrices.[8] Thus, the equation of $W^s_{loc}(O)$ is given by

$$y_2 = \psi(y_1), \qquad \text{where} \quad \frac{\partial \psi(0)}{\partial y_1} = C^s ,$$

[8]To obtain the equations of the stable and unstable manifolds of the saddle periodic trajectory L there is no necessity to reduce the Poincaré map to the special form (3.6.1), *i.e.* we do not assume that the linear part of the Poincaré map decouples into two equations.

and the equation of $W^u_{loc}(O)$ is given by

$$y_1 = \phi(y_2), \qquad \text{where} \quad \frac{\partial\phi(0)}{\partial y_2} = C^u.$$

Both ψ and ϕ are $\mathbb{C}^r$-smooth functions. The condition of invariance of $W^s_{loc}(O)$ with respect to the Poincaré map reads as follows. If we choose the point $(y_1^0, \psi(y_1^0)) \in W^s_{loc}(O)$ to be an initial point at $\theta = 0$ and from this point we begin a trajectory, then at $\theta = \tau$ this trajectory returns on the cross-section at a point of $W^s_{loc}(O)$. The set of all forward trajectories starting from $W^s_{loc}(O)$ is a $(k+1)$-dimensional invariant surface which is the local stable manifold $W^s_{loc}(L)$ of the saddle periodic trajectory L. The equation of $W^s_{loc}(L)$ is given by

$$y(\theta) = (y_1(\theta; y_1^0, \psi(y_1^0)), y_2(\theta; y_1^0, \psi(y_1)^0)) \, . \tag{3.12.8}$$

In a similar manner we can define the invariant unstable manifold $W^u_{loc}(L)$ of dimension $(n-k+1)$. Its equation is given by

$$y(\theta) = (y_1(\theta; \phi(y_2^0), y_2^0), y_2(\theta; \phi(y_2^0), y_2^0)) \, . \tag{3.12.9}$$

Observe that $W^s_{loc}(L)$ and $W^u_{loc}(L)$ have the same smoothness as $W^s_{loc}(O)$ and $W^u_{loc}(O)$. The subindex *loc* here means that both manifolds are defined in $\mathbb{D}^n \times \mathbb{S}^1$, where $\mathbb{D}^n : \{y, \|y\| < \varepsilon\}$ is a disk of some sufficiently small radius ε.

Let us pause to consider the three-dimensional examples, *i.e.* when $n = 2$ in (3.12.3) and the Poincaré map is two-dimensional.

1. Let $0 < \rho_1 < 1$ and $\rho_2 > 1$. The saddle fixed point O breaks the stable and the unstable manifolds into two components each, such that

$$W^s_{loc}(O) = \Gamma_1 \cup O \cup \Gamma_2$$

and

$$W^u_{loc}(O) = \Gamma_3 \cup O \cup \Gamma_4 \, .$$

Moreover, each Γ_i, (i=1,. . . ,4) is invariant, *i.e.* it is taken into itself by the Poincaré map. Hence, $W^s_{loc}(L)$ and $W^u_{loc}(L)$ are smooth two-dimensional surfaces which are homeomorphic to a cylinder, see Fig. 3.12.1. Observe that the three-dimensional system (3.12.2) near the periodic trajectory L may then be reduced to the following form (see Sec. 3.11)

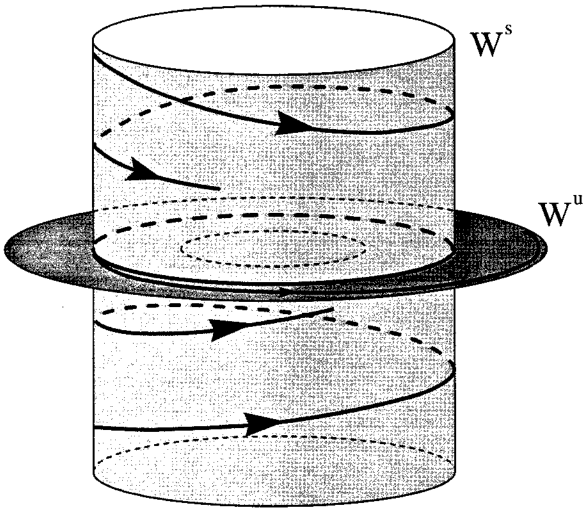


Fig. 3.12.1. A saddle periodic trajectory with two-dimensional stable W^s and unstable W^u manifolds which are homeomorphic to a cylinder.

$$\begin{aligned}
\dot{y}_1 &= \lambda_1 y_1 + f_1(y_1, y_2, \theta)\,, \\
\dot{y}_2 &= \lambda_2 y_2 + f_2(y_1, y_2, \theta)\,, \\
\dot{\theta} &= 1\,,
\end{aligned} \tag{3.12.10}$$

where $\lambda_{1,2} = \frac{\ln \rho_{1,2}}{\tau}$.

2. Let $|\rho_1| < 1$ and $|\rho_2| > 1$, moreover, $\rho_1 < 0$ and $\rho_2 < 0$. In this case the Poincaré map takes Γ_1 into Γ_2, and Γ_2 into Γ_1. The manifold $W^s_{loc}(L)$ will be then diffeomorphic to a two-dimensional Möbius band. The same is true for $W^u_{loc}(L)$, see Fig. 3.12.2. In this case L is a middle line of the Möbius band.

Thus, we can see that in the three-dimensional case saddle periodic trajectories may be of two different topological types because there is no homeomorphism between cylinders and Möbius bands. An analogous situation holds in the high-dimensional case. If sign $(\rho_1 \times \cdots \times \rho_k) = 1$, which implies also that sign $(\rho_{k+1} \times \cdots \times \rho_n) = 1$, then $W^s_{loc}(L)$ is homeomorphic to the multidimensional cylinder $\mathbb{D}^k \times \mathbb{S}^1$, and $W^u_{loc}(L)$ is homeomorphic to $\mathbb{D}^{n-k} \times \mathbb{S}^1$.

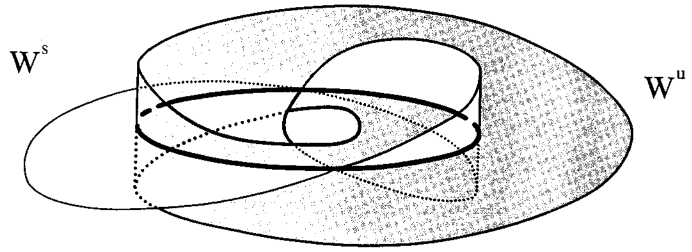


Fig. 3.12.2. A saddle periodic trajectory with two-dimensional stable W^s and unstable W^u manifolds which are homeomorphic to a Möbius band.

If sign $(\rho_1 \times \cdots \times \rho_k) = -1$ and sign $(\rho_{k+1} \times \cdots \times \rho_n) = -1$, then $W^s_{loc}(L)$ and $W^u_{loc}(L)$ are non-orientable manifolds of the type of multi-dimensional Möbius bands (*i.e.* they are represented as a fiber bundle of $\mathbb{D}^k$ and, respectively, $\mathbb{D}^{n-k}$ by $\mathbb{S}^1$). Similarly to the classification of structurally stable fixed points, we can distinguish structurally stable periodic trajectories by introducing the invariants $\delta_s = \text{sign} \prod_{i=1}^{k} \rho_i$ and $\delta_u = \text{sign} \prod_{i=k+1}^{n} \rho_i$.

Up to now we have been talking about the local manifolds of periodic trajectories. However, we can define these manifolds globally. Let $x = X(t; x_0)$ denote the trajectory with initial condition x_0.

Definition 3.4. *The stable manifold of the periodic trajectory L is the set*

$$W^s_L = \{x \in \mathbb{R}^n | \quad X(t; x) \to L \quad \text{as } t \to +\infty\}.$$

The unstable manifold W^u_L is defined in a similar way with the difference that $t \to -\infty$.

By that definition, for any point $x \in W^s(L)$ there is a moment of time at which the trajectory of x enters a small neighborhood of L, so some time shift of x belongs to the local stable manifold. Thus,

$$W^s(L) = \bigcup_{t^* \leq 0} W^s_L(t^*),$$

where

$$W^s_L(t^*) = \{x \in \mathbb{R}^n | \quad x = X(t^*; x^*) \quad \text{for some } x^* \in W^s_{loc}(L)\}.$$

Since $W^s_L(t^*)$ is a smooth image of $W^s_{L_{loc}}$ by $X(t^*; \cdot)$, it follows that W^s_L is a smooth image of either a cylinder $\mathbb{R}^k \times \mathbb{S}^1$ or a Möbius band.

The same holds true for $W^u(L)$:

$$W^u(L) = \bigcup_{t^* \geq 0} W^u_L(t^*)\,,$$

where

$$W^u_L(t^*) = \{x \in \mathbb{R}^n | \quad x = X(t^*; x^*), \quad \text{for some } x \in W^u_{loc}(L)\}\,.$$

3.13. Smooth equivalence and resonances

The problem of reduction of a nonlinear diffeomorphism to a linear form in a neighborhood of a fixed point is essentially identical to the corresponding problem in the case of vector fields (see Sec. 2.9). The major obstacle in both situations is the resonances. However, in contrast to the resonances of vector fields the resonances of diffeomorphisms have a multiplicative character.

Consider an n-dimensional diffeomorphism

$$\bar{x} = Ax + f(x)\,, \tag{3.13.1}$$

where $f(0) = 0$, $f'(0) = 0$. Let us denote by $(\rho_1, \ldots, \rho_n)$ the eigenvalues of the matrix A. Then, *a resonance* is defined by the relation

$$\rho_k = \rho^m\,, \tag{3.13.2}$$

where $\rho^m = (\rho_1^{m_1} \rho_2^{m_2} \cdots \rho_n^{m_n})$, m_k $(k = 1, \ldots, n)$ are some non-negative integers such that $|m| = \sum\limits_{k=1}^{n} m_k \geq 2$. The number $|m|$ is called *the order of the resonance.*

Lemma 3.8. *Let the function $f(x) \in \mathbb{C}^N$ and let there be no resonances of the order $|m| \leq N$. Then, the change of variables*

$$y = x + \varphi_2(x) + \cdots + \varphi_N(x)\,, \tag{3.13.3}$$

where φ_l $(l = 2, \ldots, N)$ is a homogeneous polynomial of degree l, transforms diffeomorphism (3.13.1) *into*

$$\bar{y} = Ay + o_N(y), \tag{3.13.4}$$

where $o_N(y)$ vanishes at the origin along with its derivatives up to order N.

It is obvious that the changes of variables above and below, are local, namely they are valid only in some small neighborhood of a fixed point of diffeomorphism (3.13.1)

Lemma 3.8 is well known and it is valid even when A has multiple eigenvalues. Here, we will discuss only the case of simple eigenvalues (the extension onto the general case is made in the same way as in Lemma 2.2). The matrix A can then be represented in the form

$$A = \begin{pmatrix} \rho_1 & & & \mathbf{0} \\ & \rho_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \rho_n \end{pmatrix}.$$

Let us recast the function $f(x)$ into the following form

$$f(x) = f_2(x) + f_3(x) + \cdots + f_N(x) + o_N(x)\,,$$

where $f_l(x)$ ($l = 2, \ldots, N$) are homogeneous polynomials of degree l. We have

$$f_l(x) = \sum_k \sum_{l_1+\cdots+l_n=l} d_{k,l_1\cdots l_n} x_1^{l_1} \cdots x_n^{l_n} e_k\,,$$

where $e_k = (\underbrace{0, \ldots, 0, 1}_{k}, 0, \ldots, 0)$ is the k-th basis vector; *i.e.* $f_l(x)$ is a sum of monomials $d_{kl} x^l e_k$. By the change of variables (3.13.3) we obtain

$$\begin{aligned} \bar{y} &= \bar{x} + \sum_{l=2}^{N} \varphi_l(\bar{x}) \\ &= Ax + \sum_{l=2}^{N} f_l(x) + \sum_{l=2}^{N} \varphi_l \left(Ax + \sum_{j=2}^{N} f_j(x) \right) + \cdots \qquad (3.13.5) \\ &= Ay - \sum_{l=2}^{N} A\varphi_l(x) + \sum_{l=2}^{N} f_l(x) + \sum_{l=2}^{N} \varphi_l \left(Ax + \sum_{j=2}^{N} f_j(x) \right) + \cdots\,, \end{aligned}$$

where the ellipsis denotes the terms of degree higher than N (observe that other summands above also contain the terms of degree $(N+1)$ and higher). The process of eliminating the redundant terms begins with the quadratic terms. In order to find $\varphi_2(x)$ we write the following equation

$$-A\varphi_2(x) + f_2(x) + \varphi_2(Ax) = 0\,. \qquad (3.13.6)$$

By representing $\varphi_2(x)$ in the form

$$\varphi_2(x) = \sum_k \sum_{m_1+\cdots+m_n=2} c_{km} x^m e_k, \qquad x^m = x_1^{m_1} \cdots x_n^{m_n}$$

we obtain the following equation for the unknown coefficients c_{km}

$$(-\rho_k + \rho^m)\, c_{km} + d_{km} = 0\,, \tag{3.13.7}$$

where $m = 2$ and $k = (1, \ldots, n)$. Since there are no resonances, we find

$$c_{km} = \frac{d_{km}}{\rho_k - \rho^m}\,. \tag{3.13.8}$$

In order to obtain $\varphi_3(x)$ we write the following equation

$$-A\varphi_3(x) + \varphi_3(Ax) + f_3(x) + \{\varphi_2(Ax + f_2(x))\} = 0\,,$$

where "{}" denotes the cubic terms. Since we have already found $\varphi_2(x)$, we have the following equation for the unknown coefficients of $\varphi_3(x)$:

$$(-\rho_k + \rho^m)c_{km} + \tilde{d}_{km} = 0\,. \tag{3.13.9}$$

Here, $\tilde{d}_{km} = d_{km} + d'_{km}$, where d'_{km} is the coefficient of x^m in the k-th component of the vector polynomial $\varphi_2(Ax + f_2(x))$. By repeating this procedure we eliminate all terms up to degree N.

In the case where there are resonances of the form $\rho_k = \rho^m$ one cannot kill the monomials of the type $\tilde{d}_{km} x^m e_k$. For this case we have the following well-known lemma.

Lemma 3.9. *Let $f(x) \in \mathbb{C}^N$. Then, the change of variables*

$$y = x + \varphi(x)\,,$$

where $\varphi(x)$ is a polynomial such that $\varphi(0) = 0$, $\varphi'(0) = 0$, transforms the diffeomorphism (3.13.1) *into the form*

$$\bar{x} = Ax + R_N(x) + o_N(x)\,, \tag{3.13.10}$$

where

$$R_N(x) = \sum_{\rho^m = \rho_k}^{|m| \le N} b_{km} x^m e_k\,. \tag{3.13.11}$$

Now, let the function $f(x)$ be analytic. By taking the limit $N \to \infty$ we reduce the original diffeomorphism either to a linear form or to the following form

$$\bar{y} = Ay + R(y)\,, \tag{3.13.12}$$

where

$$R(y) = \sum_{\rho^m = \rho_k} b_{km} y^m e_k\,. \tag{3.13.13}$$

However, the change of variables

$$y = x + \varphi_2(x) + \cdots + \varphi_m(x) + \cdots \tag{3.13.14}$$

is, in general, merely a formal series, as well as the right-hand side of (3.13.13).

For diffeomorphisms we have a theorem analogous to the Poincaré-Dulac theorem for vector fields (see Sec. 2.9).

Theorem 3.17. *Let $|\rho_i| < 1$ ($|\rho_i| > 1$), $i = 1, \ldots, n$. Then there exists an analytic change of variables which transforms diffeomorphism* (3.13.1) *into*

$$\bar{y} = Ay + R(y)\,, \tag{3.13.15}$$

where $R(y)$ is a polynomial composed of resonant monomials. In the absence of resonances $R(y) \equiv 0$.

The situation when the fixed point 0 is of the saddle type, *i.e.* its multipliers lie both inside and outside of the unit circle, is much more complicated. The reason is that even when the collection $\{\rho_1, \rho_2, \ldots, \rho_n\}$ is not resonant, the zero point is a limit point for the set

$$\{\rho^m - \rho_k\}_{m=2}^{\infty}, \qquad k = 1, \ldots, n \tag{3.13.16}$$

Here, the problem of convergence of normalizing series (3.13.14) becomes much more uncertain because of the "small denominators" appearing in (3.13.8). It is established in the works of Siegel and Bruno that in the saddle case both possibilities are realized: changes of variables in the form of the series may converge and diverge as well.

The situation becomes more definite in the $\mathbb{C}^\infty$-smooth case.

Theorem 3.18. (Sternberg) *A $\mathbb{C}^\infty$-smooth change of variables reduces the n-dimensional diffeomorphism*

$$\bar{x} = Ax + f(x)\,,$$

into a linear form if the function $f(x) \in \mathbb{C}^\infty$ and if there are no resonances.

In the case where there are resonances the following theorem is valid.

Theorem 3.19. *If $f \in \mathbb{C}^\infty$, then a $\mathbb{C}^\infty$-smooth change of variables transforms diffeomorphism* (3.13.1) *into*

$$\bar{y} = Ay + R(y)\,, \tag{3.13.17}$$

where $R(y)$ is a $\mathbb{C}^\infty$-smooth function whose formal Taylor series is comprised of resonant monomials.

It follows from these two theorems that the dependence of the normal forms on the collection $\rho = \{\rho_1, \dots, \rho_n\}$ has a discontinuous character.

Just like in the case of vector fields we may pose a question concerning the reduction of diffeomorphisms to a linear form by changes of variables of only finite smoothness.

Theorem 3.20. (Belitskii) *Let $f \in \mathbb{C}^{N+1}$ and let q be the number of different in absolute values multipliers $(\rho_1, \dots, \rho_n)$. Assume also that there are no resonances of orders less than or equal to N. Then, there exists a $\mathbb{C}^{[\frac{N}{q}]}$-smooth change of variables which transforms system* (3.13.1) *into the linear form.*

It follows from this theorem that a nonlinear diffeomorphism can be transformed into the linear form by a $\mathbb{C}^r$-smooth change of variables provided that there are no resonances of order $|m| \le N$, where $N \ge r\,n$.

Theorem 3.21. (Belitskii) *Let $f \in \mathbb{C}^2$, $|\rho_k| \neq 1$ $(k = 1, \dots, n)$ and assume the following conditions hold*

$$|\rho_i| \neq |\rho_j|\,|\rho_k|, \qquad |\rho_i| < 1 < |\rho_k|\,, \tag{3.13.18}$$

where $\{i, j, k\} \in (1, \dots, n)$. Then system (3.13.1) *may be transformed into linear form by a $\mathbb{C}^1$-smooth change of variables.*

Conclusion 1. When $|\rho_i| < 1$ $(i = 1, \dots, n)$ and when $f \in \mathbb{C}^2$, diffeomorphism (3.13.1) can be transformed into the linear form by a $\mathbb{C}^1$-smooth change of variables.

Conclusion 2. If $n = 2$, $|\rho_1| < 1$ and $|\rho_2| > 1$, and $f \in \mathbb{C}^2$, then the two-dimensional diffeomorphism can be reduced to a linear form by a $\mathbb{C}^1$-smooth change of variables.

From the viewpoint of dynamics the problem of the reduction of a nonlinear diffeomorphism to a linear form in a neighborhood of a saddle fixed point does not seem to be very significant on its own. Of course, all necessary information about the behavior of trajectories near a saddle can be derived by means of standard methods discussed in this chapter. However, if we are interested in global (far away from the saddle) features of the behavior of trajectories, then the situation becomes more intriguing.

For example, the description of the trajectories in a neighborhood of a homoclinic Poincaré trajectory[9] (*i.e.* a trajectory bi-asymptotic to a saddle fixed point as $t \to \pm\infty$) requires the description of the properties of the trajectories staying for a long time near a saddle fixed point. Of course, such description can be easily done when the diffeomorphism is reduced to a linear form, and such reduction in the $\mathbb{C}^\infty$-smooth case was used by Smale in his study of the homoclinic. However, this approach does not always work; for instance, in the Hamiltonian case there always exist resonances.

Moreover, in order to study bifurcations of homoclinic trajectories one must imbed the diffeomorphism under consideration into a finite-parameter family. Therefore, the local reduction to a suitable form must depend continuously on parameters.

Let us consider a finite-parameter family of diffeomorphisms X_μ. We assume that $X_\mu \in \mathbb{C}^r$ $(r \geq 2)$ with respect to all variables and parameters, and that it is represented in the form

$$\begin{aligned} \bar{x} &= A_1(\mu)x + f_1(x, y, u, v, \mu)\,, \\ \bar{u} &= A_2(\mu)u + f_2(x, y, u, v, \mu)\,, \\ \bar{y} &= B_1(\mu)y + g_1(x, y, u, v, \mu)\,, \\ \bar{v} &= B_2(\mu)v + g_2(x, y, u, v, \mu)\,, \end{aligned} \tag{3.13.19}$$

where $f_i(x, y, u, v, \mu)$ and $g_i(x, y, u, v, \mu)$ $(i = 1, 2)$ vanish at the origin along with their first derivatives with respect to the variables (x, y, u, v) for sufficiently small μ.

We assume also that the eigenvalues of the matrix

$$A(0) = \begin{pmatrix} A_1(0) & 0 \\ 0 & A_2(0) \end{pmatrix}$$

[9] Poincaré was the first to discover the existence of such trajectories in problems of Hamiltonian dynamics.

lie strictly inside the unit circle, and that the eigenvalues of the matrix

$$B(0) = \begin{pmatrix} B_1(0) & 0 \\ 0 & B_2(0) \end{pmatrix}$$

lie outside of the unit circle. Assume also that the eigenvalues $(\rho_1, \ldots, \rho_{m_1})$ of the matrix $A_1(0)$ satisfy the conditions $|\rho_i| = \rho < 1$ $(i = 1, \ldots, m_1)$, and the eigenvalues $(\gamma_1, \ldots, \gamma_{p_1})$ of the matrix $B_1(0)$ satisfy $|\gamma_i| = \gamma > 1$ $(i = 1, \ldots, p_1)$. With regard to the eigenvalues $(\rho_{m_1+1}, \ldots, \rho_m)$ of the matrix $A_2(0)$ and the eigenvalues $(\gamma_{p_1+1}, \ldots, \gamma_p)$ of the matrix $B_2(0)$ we will assume that

$$\begin{aligned} |\rho_i| &< \rho, \qquad i = m_1 + 1, \ldots, m \\ |\gamma_i| &> \gamma, \qquad i = p_1 + 1, \ldots, p\,. \end{aligned} \tag{3.13.20}$$

Hence, the fixed point O is of the saddle type, the x and y coordinates are, respectively, the leading stable and the leading unstable coordinates.

Theorem 3.22. *Under the above assumptions there exists a $\mathbb{C}^{r-1}$-smooth change of variables which transforms the family* (3.13.19) *into*

$$\begin{aligned} \bar{x} &= A_1(\mu)x + f_{11}x + f_{12}u\,, \\ \bar{u} &= A_2(\mu)u + f_{21}x + f_{22}u\,, \\ \bar{y} &= B_1(\mu)y + g_{11}y + g_{12}v\,, \\ \bar{v} &= B_2(\mu)v + g_{21}x + g_{22}v\,, \end{aligned} \tag{3.13.21}$$

where $f_{ij}(x, y, u, v, \mu)$ and $g_{ij}(x, y, u, v, \mu)$ $(i, j = 1, 2)$ are $\mathbb{C}^{r-1}$-functions which vanish at the origin and satisfy

$$\begin{aligned} f_{1j}(x, u, 0, 0, \mu) &\equiv 0, \qquad f_{i1}(0, u, y, v, \mu) \equiv 0\,, \\ g_{1j}(0, 0, y, v, \mu) &\equiv 0, \qquad g_{i1}(x, u, 0, v, \mu) \equiv 0\,. \end{aligned}$$

The smoothness with respect to parameters is the same as in Theorem 2.20.

Note that reduction to the form (3.13.21) proved to be sufficient to the study of main homoclinic bifurcations (see Gonchenko & Shilnikov and Gonchenko *et al.* [1996]) via estimates of the type we obtained in Lemma 3.6.

Observe also that the basic idea of the proof of this theorem is to get rid of some "non-resonant functions". The proof itself repeats completely the proof of Theorem 2.20 in the case of vector field in the Appendix A.

In the case where the eigenvalues of the matrix A of diffeomorphism (3.13.1) lie on the unit circle there are always a finite number of resonances, namely:

$$\rho_k = \rho_k^m, \qquad m \geq 2 \tag{3.13.22}$$

when $\rho_k = 1$; and

$$\rho_k = \rho_k^{2m+1}, \qquad m \geq 1 \tag{3.13.23}$$

when $\rho_k = -1$; and

$$\rho_k = \rho_k \left(\rho_k \rho_{k+1}\right)^m, \qquad m \geq 1 \tag{3.13.24}$$

if $\rho_{k,\,k+1} = e^{\pm i\varphi}$, where $\varphi \neq 0$.

The theory of normal forms is especially valuable here. This is, first of all, related to the problem of stability in the critical cases, as well as to the study of the associated bifurcation phenomena. In the latter case it is natural to consider not only the diffeomorphism itself but a sufficiently close smooth finite-parameter family. It is clear that the reduction of the family to the simplest form is the primary problem.

Assume now that only the eigenvalues $(\rho_1, \ldots, \rho_p)$ lie on the unit circle. If $p < n$, then it is convenient to use the theorem on the center manifold (see Chap. 5) which allows the n-dimensional original family to be reduced to a p-dimensional finite-parameter family of the form

$$\bar{x} = Ax + g(x) + h(x, \varepsilon)\,, \tag{3.13.25}$$

where the eigenvalues of the matrix A are $(\rho_1, \ldots, \rho_p)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_q)$, and where $g(x)$ and $h(x, \varepsilon)$ are sufficiently smooth functions. Moreover,

$$g(0) = 0, \quad g'(0) = 0, \quad h(x,0) \equiv 0, \quad h'_x(x,0) \equiv 0\,.$$

Let us now consider a $(p+q)$-dimensional diffeomorphism in the triangular form

$$\begin{aligned} \bar{x} &= Ax + g(x) + h(x, \varepsilon)\,, \\ \bar{\varepsilon} &= \varepsilon\,. \end{aligned} \tag{3.13.26}$$

This diffeomorphism has a fixed point $O(0,0)$ with the Jacobian matrix

$$\tilde{A} = \begin{pmatrix} A & h'_\varepsilon(0,0) \\ 0 & I \end{pmatrix},$$

where I is the identity matrix. The eigenvalues of $\tilde{A}$ are $\rho_1, \dots, \rho_p$ and $\gamma_1 = \dots = \gamma_q = 1$, In this case, besides the resonances of the type

$$\rho_k = \rho^m ,$$

where

$$\rho^m = \rho_1^{m_1} \cdots \rho_p^{m_p}, \qquad \sum_{i=1}^{p} m_i \geq 2 ,$$

which exist when $\varepsilon = 0$, there are also the following resonances:

$$\rho_k = \rho_k \gamma^l , \tag{3.13.27}$$

$$\rho_k = \rho^m \gamma^l , \tag{3.13.28}$$

$$\gamma_k = \gamma^l , \tag{3.13.29}$$

where

$$\gamma^l = \gamma_1^{l_1} \cdots \gamma_q^{l_q}, \qquad \sum_{j=1}^{q} l_j \geq 2 .$$

The reduction of system (3.13.26) to normal form can be achieved via the change of variables

$$\begin{aligned} y &= x + \varphi(x, \varepsilon) \\ \varepsilon &= \varepsilon \end{aligned} \tag{3.13.30}$$

which leaves the second equation in (3.13.26) unchanged (the latter means that we do not need to consider the resonances of the kind (3.13.29)). Similar to the case in Lemma 3.9, the original family may be transformed into

$$\bar{y} = Ay + R_0(\varepsilon) + R_1(\varepsilon)y + R_N(y, \varepsilon) + o_N(y, \varepsilon) , \tag{3.13.31}$$

where $R_1(\varepsilon)$ is a polynomial of degree not higher than $N - 1$, $R_1(0) = 0$ and

$$R_N(y, \varepsilon) = \sum_{\rho_k = \rho^m}^{|m| \leq N} b_{km}(\varepsilon) y^m e_k , \tag{3.13.32}$$

where $b_{km}(\varepsilon)$ are certain polynomials of degree not exceeding $(N - |m|)$. Moreover, $R_0(\varepsilon) \equiv 0$ if, among the eigenvalues $(\rho_1, \dots, \rho_p)$, there is none equal to one. Otherwise, $R_0(\varepsilon)$ is a polynomial of degree not higher than N, and

$R_0(0) = 0$. The appearance of the term $R_0(\varepsilon)$ in (3.13.31) is due to the existence of resonances of the kind

$$\rho_k = \gamma^l . \tag{3.13.33}$$

In many cases, to describe the behavior of the trajectories in a small fixed neighborhood of the fixed point O, as well as to construct the bifurcation diagram, it is sufficient to restrict our considerations to the finite normal form

$$\bar{y} = Ay + R_0(\varepsilon) + R_1(\varepsilon)y + R_N(y, \varepsilon) \tag{3.13.34}$$

for some suitable choice of N and p. Just like the case of vector field, the information extracted from the analysis of the truncated normal form (3.13.34) must be substantiated before it can be applied to the original family of diffeomorphisms. This is the method used in our study of the main cases of bifurcations of periodic trajectories in the second part of this book.

3.14. Autonomous normal forms

In this section we discuss a different kind of normal forms near a periodic trajectory. We saw in Sec. 3.11 that a linear non-autonomous system, periodic in time, can always be recast in an autonomous form by a periodic coordinate transformation. Here we extend this result and show that by a *formal* change of variables all non-autonomous terms in an arbitrary *non-linear* system can be reduced to an autonomous form near a periodic trajectory.

Consider a $\mathbb{C}^r$-smooth system in normal coordinates

$$\begin{cases} \dot{y} = A(\theta)y + F(\theta, y) \\ \dot{\theta} = 1 \end{cases} \tag{3.14.1}$$

near a periodic trajectory $\{y = 0\}$ of period τ (so we assume A and F to be τ-periodic in θ). For the sake of simplicity we consider the case where y is a vector of complex variables ($y \in C^n$). The difficulties in the case where y is real can be overcome in the same manner as in Lemma 2.2.

Let $\{e_1, \ldots, e_n\}$ be the Jordan base in C^n relative to the matrix of the linear part of the Poincaré map of system (3.14.1) and let $\{y_1, \ldots, y_n\}$ be the coordinates in this basis. We have shown in Sec. 3.11 that the system may be reduced to

$$\dot{y}_k = \lambda_k y_k + \delta_k y_{k+1} + \sum_{2\le|m|\le r} F_{km}(\theta) y^m + o(\|y\|^r) \qquad (k = 1, \dots, n) \tag{3.14.2}$$

where λ_k are the (non-trivial) characteristic exponents, the coefficients δ_k are either 0 or 1; moreover, δ_k may be non-zero only in the case $\lambda_k = \lambda_{k+1}$. The functions F_{km} are τ-periodic in θ, and it follows from Lemma 3.7 that they are $\mathbb{C}^r$-smooth with respect to θ. Recall that the characteristic exponents are defined in terms of the multipliers of the periodic trajectory:

$$\lambda_k = \frac{1}{\tau} \ln \rho_k \tag{3.14.3}$$

where $\rho_1, \dots, \rho_n$ are the multipliers. In the previous section we introduced the notion of a resonant relation

$$\rho_1^{m_1} \cdot \dots \cdot \rho_n^{m_n} = \rho_k \, ,$$

(where $m_1, \dots, m_n$ are non-negative integers) which we can recast as

$$m_1\lambda_1 + \dots + m_n\lambda_n = \lambda_k + \frac{2\pi i}{\tau} m_{n+1} \, , \tag{3.14.4}$$

where m_{n+1} is also an integer, and may possibly be negative.

Theorem 3.23. *There is an integer S defined by the values of the multipliers $\rho_1, \dots, \rho_n$ only, such that for any finite r there exists a local transformation of coordinates y, $(S\tau)$-periodic in θ, which makes all coefficients F_{km} in* (3.14.2) *independent of θ.*[10] *Moreover, if the monomial $y^m e_k$ is non-resonant for some k, then $F_{km} \equiv 0$ in the new coordinates.*

Proof. Let us make a sequence of coordinate transformations of the form

$$y_k^{\text{new}} = y_k + f_{km}(\theta) y^m \, , \tag{3.14.5}$$

each of which will make a coefficient of $(y^{\text{new}})^m e_k$ independent of θ; here $e_k = (\underbrace{0, \dots, 0, 1}_{k}, 0, \dots, 0)$ is the k-th basis vector. Such a transformation does not

[10]Note that the terms $o(\|y\|^r)$ will remain non-autonomous.

change the coefficients of the monomials $y^{m'}e_{k'}$ of orders lower than the order of $y^m e_k$ (in the sense of Lemma 2.2; see (2.9.18)). Therefore, increasing (in the above sense) multiindex (k, m) in (3.14.5) will finally give us the theorem.

Equating the coefficients of y^m in the identity

$$\frac{d}{dt}y_k^{\mathrm{new}} = \frac{d}{dt}[y_k + f_{km}(\theta)y^m]$$

we obtain

$$F_{km}^{\mathrm{new}}(\theta) = F_{km}^{\mathrm{old}}(\theta) + (f'_{km}(\theta) + f_{km}(\theta)[(m,\lambda) - \lambda_k])\,.$$

We may consider the last expression as a differential equation on f_{km}. Its solution is given by

$$f_{km}(\theta) = e^{-\gamma_{km}\theta}\left(C + \int_0^\theta e^{\gamma_{km}t}\{F_{km}^{\mathrm{new}}(t) - F_{km}^{\mathrm{old}}(t)\}dt\right) \tag{3.14.6}$$

where

$$\gamma_{km} = (m,\lambda) - \lambda_k\,.$$

One can see that if the monomial y^m is non-resonant (*i.e.* $\gamma_{km} \neq 2\pi i\frac{j}{\tau}$), then the constant C in (3.14.6) may be taken such that f_{km} is a τ-periodic function of θ with $F_{km}^{\mathrm{new}} \equiv 0$. Indeed, the condition of periodicity of f_{km} is

$$f_{km}(\theta + \tau) = f_{km}(\theta)\,,$$

or, when $F_{km}^{\mathrm{new}} = 0$, we have

$$\begin{aligned}(e^{\gamma_{km}\tau} - 1)C &= e^{\gamma_{km}\tau}\int_0^\theta e^{\gamma_{km}t}F_{km}(t)dt - \int_0^{\theta+\tau} e^{\gamma_{km}t}F_{km}(t)dt\\ &= \int_0^\theta e^{\gamma_{km}(t+\tau)}F_{km}(t+\tau)dt - \int_0^{\theta+\tau} e^{\gamma_{km}t}F_{km}(t)dt\\ &= -\int_0^\tau e^{\gamma_{km}t}F_{km}(t)dt\end{aligned}$$

(here we used the τ-periodicity of $F_{km}(t)$). If $\gamma_{km} \neq 2\pi i\frac{j}{\tau}$, then the coefficient of C is non-zero, and the required C is immediately found.

For the resonant case we have two possibilities: $\gamma_{km} = 0$ and $\gamma_{km} \neq 0$. If $\gamma_{km} = 0$ ($m_{n+1} = 0$ in terms of (3.14.4)), Eq. (3.14.6) takes the form

$$f_{km}(\theta) = C + \int_0^\theta \{F_{km}^{\mathrm{new}}(t) - F_{km}^{\mathrm{old}}(t)\}dt\,. \tag{3.14.7}$$

One can see immediately that $F_{km}^{\text{new}} = \text{const.} = \frac{1}{\tau}\int_0^\tau F_{km}^{\text{old}}(t)dt$ gives the τ-periodic function f_{km}. Thus, we could reduce the system to the autonomous normal form by a τ-periodic transformation, if the values γ_{km} vanished for all resonant monomials.

Although this is not the case in general, we will however prove that this can be achieved if we consider the system as $(S\tau)$-periodic with some integer $S \geq 1$. The idea is that the characteristic exponents λ_k are not defined uniquely by expression (3.14.3) because the logarithm is not a single-valued function. In fact, we may write

$$\lambda_k = \frac{1}{\tau}\ln\rho_k + 2\pi i\frac{j_k}{\tau}$$

where j_k are arbitrary integers (if $\rho_k = \rho_{k+1}$, chose $j_{k+1} = j_k$ to obtain $\lambda_{k+1} = \lambda_k$). If we consider the system as $(S\tau)$-periodic, we have

$$\lambda_k^{\text{new}} = \frac{1}{S\tau}\ln(\rho_k^S) + 2\pi i\frac{j_k}{S\tau} = \lambda_k^{\text{old}} + 2\pi i\frac{j_k}{S\tau}\,. \tag{3.14.8}$$

We will prove now that there exist integers S and $j_1, \ldots, j_n$ such that in all resonant relations the imaginary part[11] will vanish simultaneously, when one proceed to the new λ_k defined by formula (3.14.8) (the theorem then follow immediately from the above discussion).

The resonant relation (3.14.4) is a particular case of the relation

$$m_1\lambda_1 + \cdots + m_n\lambda_n + m_{n+1}i\omega = 0 \tag{3.14.9}$$

where $\omega = 2\pi/\tau$. It can be regarded as a linear equation in the variables $(m_1, \ldots, m_{n+1})$ with the given coefficients $(\lambda_1, \ldots, \lambda_n, i\omega)$. This equation can have only a finite number of linearly independent integer-valued solutions which we denote by $(m_1^{(1)}, \ldots, m_{n+1}^{(1)}), \ldots, (m_1^{(q)}, \ldots, m_{n+1}^{(q)})$ $(q \leq n)$. Any other integer-valued solution can be expressed by a linear combination $m = \sigma_1 m^{(1)} + \cdots + \sigma_q m^{(q)}$ with some coefficients σ. We must change the characteristic exponents $\lambda_1, \ldots, \lambda_n$ so that in all resonant monomials the value m_{n+1} is equal to zero simultaneously. Since any solution of (3.14.9) is a linear combination of a finite number of basic solutions $m^{(1)}, \ldots, m^{(q)}$, it is sufficient to satisfy $m_{n+1}^{(1)} = 0, \ldots, m_{n+1}^{(q)} = 0$.

[11] This is $\frac{2\pi i}{\tau}m_{n+1}$ in (3.14.4).

We have

$$\begin{array}{l} m_1^{(1)}\lambda_1 + \cdots + m_n^{(1)}\lambda_n + m_{n+1}^{(1)} i\omega = 0\,, \\ \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ m_1^{(q)}\lambda_1 + \cdots + m_n^{(q)}\lambda_n + m_{n+1}^{(q)} i\omega = 0\,, \end{array}$$

Let us consider this as a linear homogeneous system on $(\lambda_1, \ldots, \lambda_n, i\omega)$ with the integer coefficients $m_\beta^{(\alpha)}$. It is well known that if a linear homogeneous system with integer coefficients has a non-trivial solution, it necessarily has an integer-valued non-trivial solution as well. Therefore, there exist integers $j_1, \ldots, j_n$ and S such that

$$\begin{array}{l} m_1^{(1)} j_1 + \cdots + m_n^{(1)} j_n + m_{n+1}^{(1)} S = 0\,, \\ \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ m_1^{(q)} j_1 + \cdots + m_n^{(q)} j_n + m_{n+1}^{(q)} S = 0\,. \end{array} \tag{3.14.10}$$

Substituting the integer solutions $j_1, \ldots, j_n$ and S of (3.14.10) into the formula (3.14.8) we obtain a set of new characteristic exponents such that

$$\begin{array}{l} m_1^{(1)}\lambda_1^{new} + \cdots + m_n^{(1)}\lambda_n^{\text{new}} = 0\,, \\ \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ m_1^{(q)}\lambda_1^{\text{new}} + \cdots + m_n^{(q)}\lambda_n^{\text{new}} = 0\,. \end{array}$$

This implies that in terms of these newly defined $\lambda_1, \ldots, \lambda_n$, for the case where the system is considered as $(S\tau)$-periodic with S calculated from (3.14.10), the value γ_{km} equals to zero for all resonant monomials. This completes the proof of the theorem.

The general meaning of this result is that the behavior of solutions which remain in a small neighborhood of a periodic trajectory very much resembles the behavior of the solutions in a small neighborhood of an equilibrium state of an autonomous system. More precisely, if we reduce a system in the normal coordinates to the autonomous form up to the terms of some order r, then the trajectories in a sufficiently small ε-neighborhood of a periodic trajectory will be close to the trajectories of the truncated autonomous system for a very long period of time (of order, say, $\frac{1}{\varepsilon^r}$). However, we must be cautious because as r increases, the coordinate transformation which we have constructed may not converge (since it is a special case of a *formal* normal form transformation), and in general, the behavior of the original and truncated autonomous systems over the *infinite* time interval will be rather different.

Finally, we remark that it follows from the proof of the above theorem that in the important special case where all the characteristic exponents λ_k are zero (*i.e.* all the multipliers are equal to 1) the autonomous normal form has the same period τ as the original system (because the quantity $\gamma_{km} \equiv (m, \lambda) - \lambda_k$ is zero for all resonant monomials).

3.15. The principle of contraction mappings. Saddle maps

In this section we give a simple criterion for the existence of fixed points which is based on the principle of the contraction mappings. This criterion, when applied to the Poincaré map, gives conditions which guarantee the existence of periodic trajectories. The principle of the contraction mappings is a rather general mathematical result and its applicability is not restricted to the problem of establishing the existence of periodic trajectories. In the following chapters we will use an infinite-dimensional version of this principle (on the space of continuous functions) while proving theorems on invariant manifolds.

Definition 3.5. *The map $T : D \to D$ of a closed set $D \subseteq \mathbb{R}^n$ is called a contraction mapping, or simply a contraction, if there exists a constant $K < 1$ such that for any two points M_1 and M_2 in D the distance between their images $T(M_1)$ and $T(M_2)$ does not exceed the distance between the points M_1 and M_2 multiplied by K:*

$$\|TM_1 - TM_2\| \leq K\|M_1 - M_2\| \tag{3.15.1}$$

Theorem 3.24. (Banach principle of contraction mappings) *A contraction mapping T has a unique fixed point M^* in D. Moreover, the trajectory $T^i M$ of any point $M \in D$ tends exponentially to M^* as $i \to \infty$.*

Proof. Let us choose an arbitrary point $M \in D$. Since $TD \subseteq D$, the trajectory $\{T^i M\}_{i=0}^{\infty}$ of the point M lies entirely in D. It follows from (3.15.1) that for any i

$$\|T^{i+1}M - T^i M\| \leq K^i \|TM - M\| .$$

Therefore, for any m and j, we have

$$\begin{aligned}\|T^{m+j}M - T^m M\| &\leq \sum_{i=0}^{j-1} \|T^{m+i+1}M - T^{m+i}M\| \\ &\leq \left(\sum_{i=0}^{j-1} K^{m+i}\right)\|TM - M\| \leq \frac{K^m}{1-K}\|TM - M\| \,.\end{aligned}$$

Hence, the sequence of the points $\{T^i M\}_{i=0}^{\infty}$ is a fundamental (or Cauchy) sequence, *i.e.* for any $\varepsilon > 0$ one can find an m such that the inequality $\|T^{m+j}M - T^m M\| \leq \varepsilon$ is satisfied for any j. In our case,

$$m > \frac{1}{|\ln K|} \cdot \left| \ln \frac{\varepsilon(1-K)}{\|TM - M\|} \right| .$$

Since any fundamental sequence converges,[12] there exists a limit $M^* = \lim\limits_{i\to\infty} T^i M$. Since the mapping T is continuous (it follows from (3.15.1)) we have that

$$TM^* = T \lim_{i\to\infty} T^i M = \lim_{i\to\infty} T^{i+1} M = M^* \,,$$

i.e. M^* is a fixed point of T.

If T possesses another fixed point M^{**}, then

$$\|M^* - M^{**}\| = \|TM^* - TM^{**}\| \leq K\|M^* - M^{**}\|$$

whence $\|M^* - M^{**}\| = 0$, *i.e.* $M^* = M^{**}$. Thus, the mapping T has a unique fixed point M^*.

We have shown that the trajectory of any point M tends exponentially to some fixed point of the mapping T. Since this point is unique, trajectories starting from all points in D tend exponentially to M^*.

Theorem 3.25. *Let the mapping $T : D \to D$ depend continuously on some parameter μ, and let the mappings T_μ be contracting with the same constant K in* (3.15.1) *for all μ, then the fixed point M^*_μ depends continuously on μ.*

Proof. Let M^*_μ and $M^*_{\mu+\Delta\mu}$ be fixed points of the mappings T_μ and $T_{\mu+\Delta\mu}$, respectively. By definition

$$T_\mu M^*_\mu = M^*_\mu$$

[12]The space $\mathbb{R}^n$ is complete.

and

$$T_{\mu+\Delta\mu} M^*_{\mu+\Delta\mu} = M^*_{\mu+\Delta\mu} \,.$$

Hence

$$\begin{aligned}\|M^*_\mu - M^*_{\mu+\Delta\mu}\| &= \|T_\mu M^*_\mu - T_{\mu+\Delta\mu} M^*_{\mu+\Delta\mu}\| \\ &\le \|T_\mu M^*_\mu - T_{\mu+\Delta\mu} M^*_\mu\| + \|T_{\mu+\Delta\mu} M^*_\mu - T_{\mu+\Delta\mu} M^*_{\mu+\Delta\mu}\| \\ &\le \|T_\mu M^*_\mu - T_{\mu+\Delta\mu} M^*_\mu\| + K\|M^*_\mu - M^*_{\mu+\Delta\mu}\| \,,\end{aligned}$$

whence

$$\|M^*_\mu - M^*_{\mu+\Delta\mu}\| \le \frac{1}{1-K}\|T_\mu M^*_\mu - T_{\mu+\Delta\mu} M^*_\mu\| \,.$$

Since T_μ depends continuously on μ, the right-hand side of this last inequality tends to zero as $\Delta\mu \to 0$, and, therefore, $M^*_{\mu+\Delta\mu} \to M^*_\mu$ as $\Delta\mu \to 0$. End of the proof.

The following criterion on the existence of fixed points of smooth mappings follows immediately from the Banach principle.

Theorem 3.26. *Let a mapping $\bar{x} = F(x)$ be defined on a closed convex set $D \subseteq \mathbb{R}^n$ such that*

$$F(D) \subseteq D \tag{3.15.2}$$

$$\|F'\| \le K < 1 \,. \tag{3.15.3}$$

Then, $F(x)$ has a unique fixed point $x^* \in D$ such that all trajectories of F converge to x^*.

Proof. In order to prove this theorem it is sufficient to verify that F is a contraction mapping. Select two points x_1 and x_2 in D, and examine their images $\bar{x}_1$ and $\bar{x}_2$ under F. Since D is a convex set, the interval $I = \{x_1 + s(x_2 - x_1)\}_{s\in[0,1]}$ connecting the points x_1 and x_2 lies entirely in D. Consider the function $\varphi(s) = F(x_1 + s(x_2 - x_1))$. This function maps I into D so that $\varphi(0) = \bar{x}_1$, $\varphi(1) = \bar{x}_2$. Since

$$\varphi(1) = \varphi(0) + \int_0^1 \varphi'(s)ds \,,$$

we have

$$\bar{x}_2 = \bar{x}_1 + \int_0^1 F'(x_1 + s(x_2 - x_1))(x_2 - x_1)ds$$

and

$$\|\bar{x}_2 - \bar{x}_1\| \le \int_0^1 \|F'\| ds \cdot \|x_2 - x_1\| \,.$$

Hence,

$$\|\bar{x}_2 - \bar{x}_1\| \le K\|x_2 - x_1\|$$

i.e. F is a contraction mapping and it follows from Theorem 3.24 that it has a unique fixed point in D.

Remark. Here, we have re-proved the well-known inequality

$$\|F(x_2) - F(x_1)\| \le \left(\sup_D \|F'\|\right) \cdot \|x_2 - x_1\| \,, \tag{3.15.4}$$

where x_1 and x_2 are arbitrary points in a convex set D, and F is a smooth function. We will frequently use this estimate. We remark that, generally speaking, it is not satisfied for non-convex sets.

When the function F depends continuously on some parameter μ, x^* also depends continuously on μ by virtue of Theorem 3.25. If the dependence of F on μ is smooth, then the following theorem holds.

Theorem 3.27. *Let the function F of Theorem 3.15.3 depend $\mathbb{C}^r$-smoothly on $x \in D$ and on a parameter μ. Then, the fixed point x^* also depends $\mathbb{C}^r$-smoothly on μ.*

Proof. Let us compute the first derivative $dx^*/d\mu$. Since x^* is a fixed point,

$$x^* = F(x^*, \mu) \,.$$

Consider an increment $\Delta\mu$ of μ. The corresponding increment Δx^* of x^* is given by

$$\Delta x^* = F'_x \Delta x^* + F'_\mu \Delta\mu + o(\|\Delta x^*\|) + o(\|\Delta\mu\|) \,,$$

i.e.

$$(I - F'_x)\Delta x^* = F'_\mu \Delta\mu + o(\|\Delta x^*\|) + o(\|\Delta\mu\|) \,,$$

where I is the identity matrix. Since $\|F'_x\| \le K < 1$, it follows that $(I - F'_x)$ is invertible. Therefore,

$$\Delta x^* = (I - F'_x)^{-1} F'_\mu \Delta\mu + o(\|\Delta x^*\|) + o(\|\Delta\mu\|) \,,$$

i.e. x^* depends smoothly on μ, and

$$\frac{dx^*_\mu}{d\mu} = (I - F'_x)^{-1} F'_\mu \Big|_{x=x^*(\mu)} . \tag{3.15.5}$$

We can now show that x^* depends $\mathbb{C}^r$-smoothly on μ. To do this it is sufficient to differentiate (3.15.5) $(r-1)$ times in accordance with the following rule:

$$\frac{d}{d\mu} = \frac{\partial}{\partial\mu} + \left(\frac{\partial}{\partial x}\right) \cdot \frac{dx^*}{d\mu} = \frac{\partial}{\partial\mu} + \left(\frac{\partial}{\partial x}\right) \cdot [(I - F'_x)^{-1} F'_\mu] .$$

End of the proof.

Theorem 3.26 yields a sufficient condition for the existence of a stable fixed point. In order to obtain a sufficient condition for the existence of a completely unstable fixed point we simply require that formulae (3.15.2) and (3.15.3) hold for the inverse mapping F^{-1}.

For saddle fixed points the problem is that close to such a point it is impossible to select a region which would be mapped onto itself by F. Similarly, there is no region that is mapped into itself by F^{-1}. This is readily seen in the following example:

$$\bar{x} = \lambda x, \qquad \bar{y} = \gamma y, \qquad 0 < \lambda < 1 < \gamma . \tag{3.15.6}$$

In order to overcome this difficulty we consider the map in the so-called *cross form*.

Definition 3.6. *Let D_1 and D_2 be certain sets, and let $P : D_1 \times D_2 \to D_1$, $Q : D_1 \times D_2 \to D_2$ be certain functions, and let T be a mapping defined on certain subset of the direct product $D_1 \otimes D_2$. We shall say that P and Q define the mapping T in cross form when the point $(\bar{x}, \bar{y}) \in D_1 \otimes D_2$ is the image of the point $(x, y) \in D_1 \otimes D_2$ under the mapping T if and only if*

$$\begin{aligned} \bar{x} &= P(x, \bar{y}) , \\ y &= Q(x, \bar{y}) . \end{aligned} \tag{3.15.7}$$

The map $T^\times$ defined by formulae (3.15.7) is called the *cross-map*. By construction, $T^\times(D_1 \otimes D_2) \subseteq D_1 \otimes D_2$.

In the forward form the map T is given by formulae

$$\begin{aligned} \bar{x} &= F(x, y) , \\ \bar{y} &= G(x, y) . \end{aligned}$$

It follows from (3.15.7) that

$$\begin{aligned} F(x,y) &= P(x, G(x,y))\,, \\ y &= Q(x, G(x,y))\,, \end{aligned}$$

whence

$$\begin{aligned} F'_x dx + F'_y dy &= (P'_x + P'_y G'_x)dx + P'_y G'_y dy\,, \\ dy &= (Q'_x + Q'_y G'_x)dx + Q'_y G'_y dy\,. \end{aligned}$$

Here, the derivatives of the functions F and G are taken with respect to (x, y), and those of P and Q are taken with respect to $(x, \bar{y})$. Equaling the coefficients of dy and dx, we obtain

$$\begin{aligned} G'_y &= (Q'_y)^{-1}\,, \\ G'_x &= -(Q'_y)^{-1}Q'_x\,, \\ F'_y &= P'_y(Q'_y)^{-1}\,, \\ F'_x &= P'_x - P'_y(Q'_y)^{-1}Q'_x \end{aligned} \tag{3.15.8}$$

and

$$\begin{aligned} Q'_y &= (G'_y)^{-1}\,, \\ Q'_x &= -(G'_y)^{-1}G'_x\,, \\ P'_y &= F'_y(G'_y)^{-1}\,, \\ P'_x &= F'_x - F'_y(G'_y)^{-1}G'_x\,. \end{aligned} \tag{3.15.9}$$

Observe that a smooth forward map does not always correspond to a smooth cross-map. In the case where $(Q'_y)^{-1}$ is not defined, the smoothness of the map T may be violated, or the map T may not even be a one-to-one map. However, the results below remain valid for such map.

Definition 3.7. *The map T defined in the cross form* (3.15.7) *by the smooth functions P and Q on the direct product of the closed convex sets D_1 and D_2 ($D_1 \subseteq \mathbb{R}^n$, $D_2 \subseteq \mathbb{R}^m$) is called a saddle map if:*

$$\begin{gathered} \|P'_x\|_\circ < 1, \qquad \|Q'_y\|_\circ < 1\,, \\ \|P'_y\|_\circ \|Q'_x\|_\circ < (1 - \|P'_x\|_\circ)(1 - \|Q'_y\|_\circ)\,, \end{gathered} \tag{3.15.10}$$

where $\|\cdot\|_\circ = \sup\limits_{(x,y)\in D_1\times D_2} \|\cdot\|$.

Example: The cross-map corresponding to the map (3.15.6) is trivially computed:

$$\bar{x} = \lambda x, \qquad y = \gamma^{-1}\bar{y}\,.$$

Since $0 \le \lambda < 1$ and $\gamma^{-1} < 1$, we can assign the subsets D_1 and D_2 to be the intervals $[-\varepsilon, \varepsilon]$ of the x and y axes respectively. The region $D_1 \otimes D_2$ is now mapped into itself under the action of the mapping $T^\times$. Here, we have $P'_x = \lambda$, $P'_y = 0$, $Q'_x = 0$, $Q'_y = \gamma^{-1}$. Therefore, since $\max\{\lambda, \gamma^{-1}\} < 1$, it follows that the conditions (3.15.10) hold, *i.e.* this map is of the saddle type. Analogously, an arbitrary linear map

$$\bar{x} = A^- x, \qquad y = (A^+)^{-1}\bar{y}$$

such that the Spec A^- lies strictly inside the unit circle and Spec A^+ lies strictly outside of it, is also of the saddle type. Here $\max\{\|A^-\|, \|(A^+)^{-1}\|\} < 1$, and D_1 and D_2 can be chosen to be certain balls in the x-space and the y-space, respectively.

When the mapping T is written in the forward form, conditions (3.15.10) are no longer symmetric.

Statement 3.1. *In order that condition* (3.15.10) *holds it is sufficient that*:

$$\begin{gathered} \|F'_x\|_\circ < 1, \qquad \|(G'_y)^{-1}\|_\circ < 1\,, \\ \|F'_y(G'_y)^{-1}\|_\circ \cdot \|G'_x\|_\circ < (1 - \|F'_x\|_\circ) \cdot (1 - \|(G'_y)^{-1}\|_\circ)\,. \end{gathered} \tag{3.15.11}$$

To prove this statement note that (3.15.10) follows from (3.15.9) if

$$\|(G'_y)^{-1}\|_\circ < 1$$

and

$$\begin{aligned} &\|F'_y(G'_y)^{-1}\|_\circ \|G'_x\|_\circ \|(G'_y)^{-1}\|_\circ \\ &\quad \le (1 - \|F'_x\|_\circ - \|F'_y(G'_y)^{-1}\|_\circ \|G'_x\|_\circ) \cdot (1 - \|(G'_y)^{-1}\|_\circ)\,. \end{aligned}$$

Observe now that these inequalities follow from conditions (3.15.11).

The first two inequalities in conditions (3.15.11) mean that the mapping T is expanding along the y-variables and contracting along the x-variables. If the derivatives F'_y and G'_x were equal to zero as in the linear map considered previously, then it would be sufficient for the map to be of the saddle type. The last inequality in (3.15.11) simply means that the distortion induced by F'_y and G'_x is not essential.

Theorem 3.28. *A saddle map T has a unique fixed point in $D_1 \times D_2$.*

Proof. First of all observe that the fixed points of the forward map T and those of the cross-map $T^\times$ coincide. Therefore, we need only to show that $T^\times$ is a contraction mapping and invoke Theorem 3.24.

Let us introduce in $D_1 \times D_2$ the distance given by

$$\rho((x_1, y_1), (x_2, y_2)) = \|x_2 - x_1\| + L\|y_2 - y_1\| \,, \tag{3.15.12}$$

where the constant L is chosen such that

$$\frac{\|P'_y\|_\circ}{1 - \|Q'_y\|_\circ} < L < \frac{1 - \|P'_x\|_\circ}{\|Q'_x\|_\circ} \,. \tag{3.15.13}$$

To verify that the map $T^\times$ is a contraction mapping we note that by virtue of (3.15.4)

$$\|P(x_2, \bar{y}_2) - P(x_1, \bar{y}_1)\| \leq \|P'_x\|_\circ \|x_2 - x_1\| + \|P'_y\|_\circ \|\bar{y}_2 - \bar{y}_1\| \,,$$

and

$$\|Q(x_2, \bar{y}_2) - Q(x_1, \bar{y}_1)\| \leq \|Q'_x\|_\circ \|x_2 - x_1\| + \|Q'_y\|_\circ \|\bar{y}_2 - \bar{y}_1\|$$

or

$$\|\bar{x}_2 - \bar{x}_1\| \leq \|P'_x\|_\circ \|x_2 - x_1\| + \|P'_y\|_\circ \|\bar{y}_2 - \bar{y}_1\| \,,$$

and

$$\|y_2 - y_1\| \leq \|Q'_x\|_\circ \|x_2 - x_1\| + \|Q'_y\|_\circ \|\bar{y}_2 - \bar{y}_1\| \,,$$

whence

$$\begin{aligned} &\|\bar{x}_2 - \bar{x}_1\| + L\|y_2 - y_1\| \\ &\quad \leq (\|P'_x\|_\circ + L\|Q'_x\|_\circ)\|x_2 - x_1\| + (\|P'_y\|_\circ + L\|Q'_y\|_\circ)\|\bar{y}_2 - \bar{y}_1\| \\ &\quad \leq K(\|x_2 - x_1\| + L\|\bar{y}_2 - \bar{y}_1\|) \,, \end{aligned}$$

where

$$K = \max\{\|P'_x\|_\circ + L\|Q'_x\|_\circ, L^{-1}\|P'_y\|_\circ + \|Q'_y\|_\circ\} \,.$$

By virtue of (3.15.13), $K < 1$, hence it follows that $T^\times$ is a contraction mapping. End of the proof.

One can show that the obtained fixed point is of the saddle type. In fact, Theorem 4.2 from the next chapter can be applied here (both to the map T

and to its inverse T^{-1}), so one can show that the fixed point of the saddle map has a smooth stable and unstable manifolds in the form $y = \psi(x)$ and $x = \varphi(y)$, where the functions $\psi(x)$ and $\varphi(y)$ are defined everywhere on D_1 and D_2 respectively.

Let us now discuss the abstract version of Banach principle. It is obvious, that Theorem 3.24 remains valid if D is a closed subset of any *Banach space* X. Recall, that a linear space X is called Banach space if it is *complete*; *i.e.* any fundamental sequence $\{x_i\}_{i=1}^{\infty}$ of elements of X converges: if for any ϵ there exists m such that $\|x_{n+m} - x_m\| \leq \epsilon$ for all $n \geq 0$, then for some $x^* \in X$

$$\lim_{i\to\infty} x_i = x^* .$$

The distance between points of X is defined as

$$\operatorname{dist}(x^1, x^2) = \|x^1 - x^2\|$$

where the norm $\|\cdot\|$ is an arbitrary non-negative function $X \to R$ such that

$$\begin{aligned}
&\|x^1 + x^2\| \leq \|x^1\| + \|x^2\| \\
&\|\lambda x\| = |\lambda| \cdot \|x\| \quad \text{for any scalar } \lambda \\
&\|x\| > 0 \quad \text{at} \quad x \neq 0 .
\end{aligned}$$

The Euclidean space R^n is an example of the Banach space. Another important example is the space H of continuous functions $x(t)|_{t\in[0,\tau]}$ (where $x \in R^n$) with the norm

$$\|x(t)\|_\circ = \sup_{t\in[0,\tau]} \|x(t)\|$$

(we denote $\|\cdot\|_\circ$ the norm in H to distinguish with the norm in R^n). The space H is complete because R^n is complete. Thus, Theorem 3.24 is valid for any contracting operator which maps $H \to H$.

For example, the proof of Theorem 2.9 on the existence of the unique solution of a boundary-value problem near a saddle consists, essentially, of verifying that the right-hand side of the integral equation (2.8.4) defines a contracting operator on the closed ε-ball $D_\varepsilon : \|x(t)\|_\circ \leq \varepsilon$ in H (here $x(t) \equiv (u(t), v(t))$).

Analogously Theorem 3.10 (the existence of the solution of a boundary-value problem near a saddle fixed point) is proved by applying the Banach principle to an operator acting on an ε-ball in the Banach space of sequences $x = \{(u_0, v_0), (u_1, v_1), \dots, (u_k, v_k)\}$ with the norm

$$\|x\|_\circ = \max_{i=0,\dots,k} \|u_i, v_i\| .$$

The Hadamard theorem (Theorem 3.9) applies the Banach principle to the operator $\varphi \mapsto \tilde{\varphi}$ defined on the Banach space of continuous functions $u = \varphi(v)$, where v belongs to the δ-neighborhood of zero in R^{n-k} and $u \in R^k$, with the norm

$$\|\varphi\|_\circ = \sup_{\|v\| \le \delta} \|\varphi(v)\| . \tag{3.15.14}$$

In fact, the operator under consideration is well-defined (see Step 1 of the proof of the theorem) on a subset D of the Banach space, which consists of smooth functions φ satisfying (3.6.4), (3.6.5). It is not a closed subset (the sequence of smooth functions may converge in the norm (3.15.14) to a non-smooth function). Therefore, the Theorem 3.24 does not guarantee that the fixed point φ^* belongs to D, but φ^* lies *in the closure* of D: in the space of continuous functions satisfying the Lipschitz condition (3.6.3) (the smoothness of φ^* was proven later, by additional arguments).

Theorems 3.25 and 3.27 concerning the dependence of the fixed point x^* on parameters also remain valid when x and μ become elements of abstract Banach spaces X and M, respectively. To clarify the statement of Theorem 3.27, we recall the definitions.

For a map $f : Y \to X$ (where Y and X are Banach spaces) the derivative $f'(y)$ at the point $y \in Y$ is a (uniquely defined) linear operator $f'(y) : \Delta y \in Y \mapsto \Delta x \equiv f'(y)\Delta y \in X$ such that

$$\limsup_{\|\Delta y\| \to 0} \frac{\|f(y + \Delta y) - f(y) - f'(y)\Delta y\|}{\|\Delta y\|} = 0 .$$

The map f is *smooth* on a subset D of Y if $f'(y)$ depends continuously on y and is uniformly bounded for all $y \in D$ in the sense of the usual norm of linear operator:

$$\|A\| = \sup_{\|\Delta y\| = 1} \|A\Delta y\| .$$

With this norm the space of bounded linear operators $Y \to X$ is a Banach space itself. The derivative $f'(y)$ depends on the point $y \in Y$, therefore the second derivative may be considered, which is a linear operator $Y \to (Y \to$

X), and so on: the r-th derivative is an inductively defined linear operator $\underbrace{Y \to (Y \to (\dots (Y}_{r} \to X)\dots)$.

Obviously, the r-th derivative $f^{(r)}$ can be considered to be a symmetric polylinear operator $Y^r \to X$ such that

$$f(y + \Delta y) = f(y) + f'(y)\Delta y + \cdots + \frac{1}{r!} f^{(r)}(y)(\Delta y)^r + o(\|\Delta y\|^r)\,.$$

The function f is $\mathbb{C}^r$-smooth on $D \subseteq Y$ if for each $k \le r$ the k-th derivative $f^{(k)}(y)$ depends continuously on y and is uniformly bounded as an operator $Y^k \to X$; *i.e.*

$$\sup_{y \in D, \|\Delta y_1\| = \cdots = \|\Delta y_k\| = 1} \|f^{(k)}(y)\Delta y_1 \cdot \cdots \cdot \Delta y_k\|_X$$

is finite.

For example, for any $\mathbb{C}^r$-smooth function g defined on R^n, the operator $x(t) \mapsto g(x(t))$ acting on the space H of the continuous functions $x(t)_{t\in[0,\tau]}$ is $\mathbb{C}^r$-smooth. A bounded linear operator is $\mathbb{C}^r$-smooth for any r. The superposition of smooth operators is an operator of the same smoothness. In particular, the operator $H \to H$ which maps a continuous function $x(t)_{t\in[0,\tau]}$ into

$$\bar{x}(t) = \int_0^t \psi(s) g(x(s), s) ds$$

is $\mathbb{C}^r$-smooth for any continuous function ψ and any function g which is $\mathbb{C}^r$-smooth with respect to x and depends continuously on s. The smoothness of operators of this sort will be used in chapter 5 in proving the smoothness of invariant manifolds based on Theorem 3.27.

Chapter 4

INVARIANT TORI

Invariant tori appear in nonlinear dynamics in the study of periodically forced self-oscillating systems, and of the interaction of several self-oscillating systems. We restrict ourselves here to the first case, *i.e.* to non-autonomous systems of the following form

$$\dot{x} = X(x) + \mu p(x,t)\,, \tag{4.0.1}$$

where $x \in \mathbb{R}^n$, and $p(x,t)$ is a periodic function of period 2π in t. As for

$$\dot{x} = X(x) \tag{4.0.2}$$

we assume that (4.0.2) possesses a structurally stable periodic trajectory L of period τ. The phase space of (4.0.1) is the space $\mathbb{R}^n \times \mathbb{S}^1$, where $\mathbb{S}^1$ is a circle of length 2π. In principle, (4.0.1) may be recast into the form of an autonomous system

$$\begin{aligned} \dot{x} &= X(x) + \mu p(x,\theta)\,, \\ \dot{\theta} &= 1\,, \end{aligned} \tag{4.0.3}$$

in $\mathbb{R}^{n+1}$, where θ is a cyclic variable defined in modulo 2π. A particularity of (4.0.1), and consequently, of (4.0.3) is that when $\mu = 0$ (the first equation of (4.0.3) is then decoupled from the second one) both systems possess a two-dimensional invariant torus $\mathbb{T}^2_0$: $L \times \mathbb{S}^1$. We will show that there also exists a smooth invariant torus $\mathbb{T}^2_\mu$ close to $\mathbb{T}^2_0$ for all μ sufficiently small. To do this we will use a criterion on the existence of stable tori suggested by Afraimovich and L. Shilnikov [2, 3] which is called *an annulus principle*. Moreover, the annulus principle is also applicable in the case of many cyclic variables. This allows us to apply it to a non-autonomous system forced by a quasi-periodical external force.

Our next step is to study the behavior of trajectories on the two-dimensional invariant torus $\mathbb{T}^2_\mu$. In this case, the problem may be reduced to an orientable Poincaré map of a circle. The main results of the theory of such maps were obtained in pioneering works of Poincaré and Denjoy. In Sec. 4.4 we will present principal elements of this theory because it gives a mathematically correct explanation of some problems on the synchronization of oscillations.

4.1. Non-autonomous systems

An n-dimensional non-autonomous periodic system is formally written in the form

$$\dot{x} = F(x, t)\,, \tag{4.1.1}$$

where $F(x, t+2\pi) = F(x, t)$. It is assumed that the conditions of the existence and the uniqueness of a solution holds in $\mathbb{R}^n \times \mathbb{R}^1$, or in $D \times \mathbb{R}^1$, where D is some subregion of $\mathbb{R}^n$. We assume that for any initial conditions (x_0, t_0) the solution can be continued onto the interval $[t_0, t_0 + 2\pi]$. Many problems of nonlinear dynamics related to the investigation of periodically forced oscillations lead to the study of such systems. For example, the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \omega_0^2 x = A \sin \omega t\,,$$

the Düffing equation

$$\ddot{x} + h\dot{x} + \alpha x + \beta x^3 = A \sin \omega t\,,$$

etc.

Generally speaking we can enlarge system (4.1.1) up to an autonomous system by introducing a new cyclic variable θ such that $\dot{\theta} = 1$. But to do this it is necessary that both variables x and θ have an equal status, *i.e.* the function $F(x, \theta)$ must be $\mathbb{C}^r$-smooth ($r \geq 1$) with respect to all of its arguments. The feature of non-autonomous systems is that F is assumed to be only continuous with respect to t.

In principle, the study of (4.1.1) is reduced to the study of a diffeomorphism whose smoothness is equal to the smoothness of F with respect to x; of course, all derivatives of F with respect to x are assumed to be continuous functions of t. The construction is as follows: by virtue of the periodicity of F with respect to t, the trajectories of the points (x, t) and $(x, t + 2\pi m)$ are identical, where $m \in \mathbb{Z}$. Hence, we obtain the associated diffeomorphism by mapping the plane

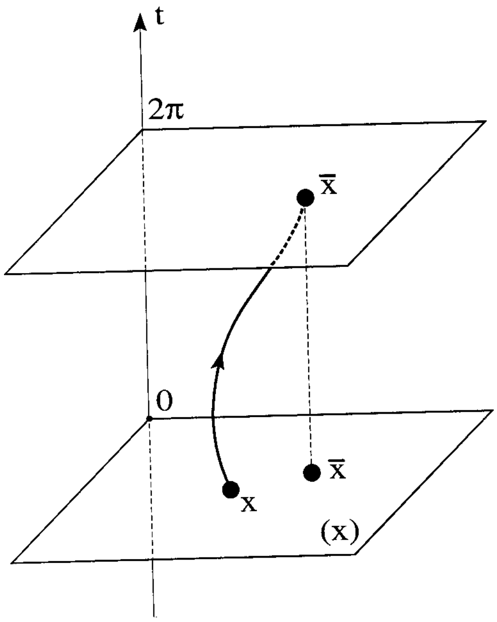


Fig. 4.1.1. Geometrical illustration of the construction of a diffeomorphism along trajectories of a 2π periodic non-autonomous system. The points of intersection of a trajectory of the non-autonomous system with a planar cross-section over every 2π period in time comprise the trajectory of the diffeomorphism.

$t = 0$ into the plane $t = 2\pi$ along the solutions of system (4.1.1) as shown in Fig. 4.1.1.

Let $\varphi(t, x)$ be a solution of system (4.1.1) which passes through the point x at $t = 0$. Then, the diffeomorphism under consideration is written in the form

$$\bar{x} = f(x)\,, \tag{4.1.2}$$

where $f(x) = \varphi(2\pi, x)$.

The possibility of such reduction is one of the features of non-autonomous systems.[1] Note that the existence of such a global cross-section in the phase space of autonomous systems is not true in general.

[1] We remark that the study of systems with a piece-wise continuous right-hand side $F(x, t)$ having a finite number of discontinuity points on period can also be reduced to such a diffeomorphism.

It is evident then that in the phase space $\mathbb{R}^n \times \mathbb{S}^1$ (or $D \times \mathbb{S}^1$) a periodic trajectory passing k times through the cross-section $t = 0$, corresponds to a k-periodic orbit $(x_0, \dots, x_{k-1})$ of the diffeomorphism.

Let us recall the definition of a periodic point of a diffeomorphism. A point x_0 is said to be a periodic point of period k if x_0 is a fixed point for the map $\bar{x} = f^k(x)$ and is not such a fixed point for $\bar{x} = f^p(x)$, for $p < k$. The points x_p along with x_0 are also periodic points, where $x_p = f^p(x_0)$, $p = 1, \dots, k-1$. It is clear that $x_{p+1} = f(x_p)$, and $x_0 = f(x_{k-1})$. Each point x_p corresponds to a solution $\varphi_p(t)\,(p = 0, \dots, k-1)$ of period $2\pi k$. Any two such solutions are identical up to a shift in the phase divisible by 2π:

$$\varphi_p(t) = \varphi_0(t + 2\pi p)\,,$$

In order to establish the existence of a fixed point the following criterion is useful. Let D be a closed, bounded region homeomorphic to a standard ball $\{x \colon \|x\| \leq 1\}$. Then, D is said to be a ball as well.

Theorem 4.1. (Brauer's criterion) *Let T be a continuous mapping of a ball D into itself, i.e. $TD \subset D$. Then, T has, at least, one fixed point.*

Brauer's criterion is usually applied in the following situation. Let all integral curves of a system, which is defined in the region $D \times \mathbb{R}^1$ enter this region on the boundary $\overline{D \times \mathbb{R}^1}$. Then the associated diffeomorphism satisfies Theorem 4.1 and, consequently, the system itself has, at least, one periodic trajectory.

Let us now return to the problem on periodically forced systems. In this case, the study of (4.0.1) is reduced to that of a family of diffeomorphisms in the form

$$\bar{x} = f(x, \mu)\,, \tag{4.1.3}$$

where f is represented as

$$f(x, \mu) = f_0(x) + \mu f_1(x, \mu)\,. \tag{4.1.4}$$

Observe that at $\mu = 0$ the diffeomorphism (4.1.3) is the shift map over 2π along the trajectories of the autonomous system (4.0.2), or, equivalently, is a mapping from $\theta = 0$ to $\theta = 2\pi$ defined along the solutions of the system

$$\begin{aligned} \dot{x} &= X(x)\,, \\ \dot{\theta} &= 1\,. \end{aligned} \tag{4.1.5}$$

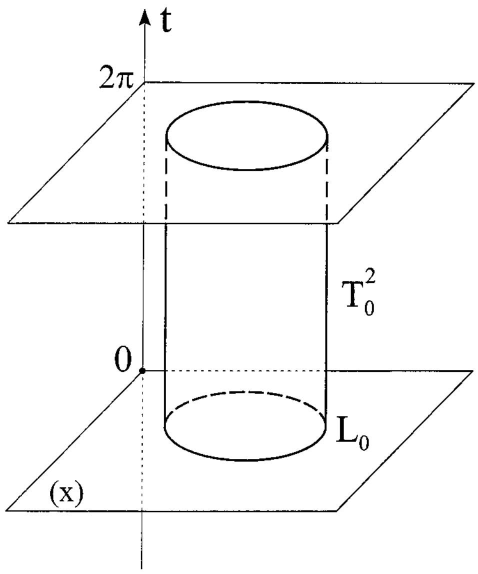


Fig. 4.1.2. An invariant torus $\mathbb{T}_0^2$ of the extended system (4.1.5) at $\mu = 0$ is represented as a direct product $L_0 \times \mathbb{S}^1$.

Under the above assumption, the system (4.0.2) has a periodic solution L of period τ, the equation of which is $x = \varphi(t)$. Hence, system (4.0.3) will have "a straight-edged" invariant torus $\mathbb{T}_0^2$ with a base defined by $\{L\colon x = \varphi(\theta_1),\ 0 \leq \theta_1 \leq \tau\}$, as shown in Fig. 4.1.2. Therefore, diffeomorphism (4.1.3) has an invariant smooth closed curve L_0 at $\mu = 0$. We will show below that if L_0 is a stable solution of (4.0.2), then for all μ sufficiently small, system (4.0.3) will possess a smooth invariant torus $\mathbb{T}_\mu^2$ close to $\mathbb{T}_0^2$, see Fig. 4.1.3. This follows from the fact that for all sufficiently small μ, diffeomorphism (4.1.3) will have a smooth invariant closed curve L_μ.

Consider now the system

$$\dot{x} = X(x, t)\,, \tag{4.1.6}$$

where we assume that $X(x, t)$ is a *quasi-periodic function* of t. This means that

$$X(x, t) = \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_{m+1}=-\infty}^{+\infty} a_{k_1 \cdots k_{m+1}}(x) e^{i(k_1\Omega_1 + \cdots + k_{m+1}\Omega_{m+1})t}\,, \tag{4.1.7}$$

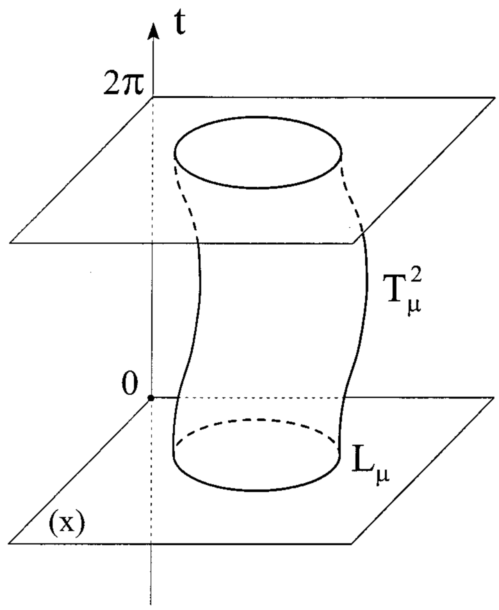


Fig. 4.1.3. A smooth invariant torus $\mathbb{T}^2_\mu$ of the perturbed system.

where $k = (k_1, \ldots, k_{m+1})$ is a vector composed of integers, and $\Omega = (\Omega_1, \ldots, \Omega_{m+1})$ is a vector of real numbers. We also assume that $\Omega_1, \ldots, \Omega_{m+1}$ comprise a basis of frequencies, *i.e.*

$$(k, \Omega) = k_1\Omega_1 + \cdots + k_{m+1}\Omega_{m+1} \neq 0 \tag{4.1.8}$$

for any $k \neq 0$. Observe that under the condition that $X(x,t) \in \mathbb{C}^r$ $(\mu \geq 1)$ it can be represented in the form:

$$X(x,t) = X(x, \theta_1, \ldots, \theta_{m+1}), \tag{4.1.9}$$

where the function $X(x, \theta_1, \ldots, \theta_{m+1}) \in \mathbb{C}^r$ is periodic of period 2π with respect to each argument $\theta_j = \Omega_j t$. Hence, system (4.1.6) may be recast as an autonomous system

$$\begin{aligned} \dot{x} &= X(x, \theta), \\ \dot{\theta} &= \Omega, \end{aligned} \tag{4.1.10}$$

where $\theta = (\theta_1, \ldots, \theta_{m+1})$. The phase space of (4.1.10) is $\mathbb{R}^n \times \mathbb{T}^{m+1}$. Furthermore, the study of system (4.1.10) may be reduced to that of the map

$$\begin{aligned} \bar{x} &= f(x, \theta)\,, \\ \bar{\theta} &= \theta + \omega \pmod{2\pi}\,, \end{aligned} \tag{4.1.11}$$

if we choose the cross-section $\theta_{m+1} = 0$. Here, $\theta = (\theta_1, \ldots, \theta_m)$, $\omega = (\omega_1, \ldots, \omega_m)$, where $\omega_j = 2\pi\Omega_j/\Omega_{m+1}\,(j = 1, \ldots, m)$. The phase space of the diffeomorphism (4.1.11) is $\mathbb{R}^n \times \mathbb{T}^m$. While studying (4.1.11) it is convenient to represent $\mathbb{T}^m$ as an m-dimensional cube

$$\Big\{(\theta_1, \ldots, \theta_m) \;\Big|\; 0 \le \theta_j \le 2\pi\,, \quad j = (1, \ldots, m)\Big\}$$

such that the points of the opposite edges of the cube are identified, *i.e.*

$$(\theta_1, \ldots, \theta_{j-1},\, 0,\, \theta_{j+1}, \ldots, \theta_m) \equiv (\theta_1, \ldots, \theta_{j-1},\, 2\pi,\, \theta_{j+1}, \ldots, \theta_m)\,.$$

Since the second group of equations in (4.1.11) is independent of x, the map

$$\bar{\theta} = \theta + \omega \pmod{2\pi}\,, \tag{4.1.12}$$

is defined on $\mathbb{T}^m$ and is a diffeomorphism. This map, due to the assumed conditions (4.1.8) on Ω, has neither fixed points nor invariant tori of a smaller dimension. In other words, $\mathbb{T}^m$ is a minimal set. Thus, the simplest objects which appear in the first stage of the study of the coupled map (4.1.11) are the m-dimensional invariant tori of the form $x = h(\theta)$ which correspond to quasi-periodic solutions with the basis of frequencies $(\omega_1, \ldots, \omega_m)$.

In Sec. 4.2 we will present a rather convenient criterion for the existence of an invariant torus for a sufficiently wide class of diffeomorphisms.

Remark. We have seen that a non-autonomous system with a periodic or a quasiperiodic dependence on time admits a natural extension up to an autonomous system of a higher dimension, where the increase in the dimension is equal to the number of independent frequencies. In the general case this extension, however, is not true for systems having an arbitrary time dependence. Furthermore, the straight-forward increasing of the dimension of the phase space in this case is not useful because the behavior of trajectories as $t \to +\infty$ must be studied on a non-compact phase space. Otherwise, the wandering set is empty. Therefore, the study of non-autonomous systems with a general time dependence call for a principally new approach. Such an approach for a class of two-dimensional non-autonomous systems has been developed by Lerman and L. Shilnikov [41].

4.2. Theorem on the existence of an invariant torus. The annulus principle

Let us consider a diffeomorphism T:

$$\begin{aligned} \bar{x} &= f(x,\theta)\,, \\ \bar{\theta} &= \theta + g_0(x,\theta) = g(x,\theta) \qquad (\operatorname{mod} 2\pi)\,, \end{aligned} \tag{4.2.1}$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{T}^m$, $n \geq 1$, $m \geq 1$, and the smooth functions f and g are 2π-periodic with respect to θ.

Let $\mathbb{K}$ be *an annulus* defined by

$$\mathbb{K} = \Big\{ (x,\theta) \,\Big|\, \|x\| \leq \delta,\ \theta \in \mathbb{T}^m \Big\}\,,$$

Introduce the following notation: for a vector-valued or matrix-valued function $\varphi(x,\theta)$

$$\|\varphi\|_\circ = \sup_{(x,\theta)\in\mathbb{K}} \|\varphi(x,\theta)\|\,,$$

where $\|\cdot\|$ is the standard Euclidean norm.

Assumption 4.1. *The map*

$$\bar{x} = f(x,\theta)$$

is a contraction for any fixed θ, *i.e.*

$$\left\| \frac{\partial f}{\partial x} \right\|_\circ < 1\,. \tag{4.2.2}$$

Assumption 4.2. *The map*

$$\bar{\theta} = \theta + g_0(x,\theta), \qquad (\operatorname{mod} 2\pi) \tag{4.2.3}$$

is a diffeomorphism for any fixed x. *This implies in particular that*

$$1 \leq \left\| \left(\frac{\partial g}{\partial \theta} \right)^{-1} \right\|_\circ \leq C < \infty\,. \tag{4.2.4}$$

Theorem 4.2. (Annulus principle) *Under the above assumptions, if*

$$1 - \left\| \left(\frac{\partial g}{\partial \theta} \right)^{-1} \right\|_\circ \cdot \left\| \frac{\partial f}{\partial x} \right\|_\circ > 2\sqrt{ \left\| \left(\frac{\partial g}{\partial \theta} \right)^{-1} \right\|_\circ \cdot \left\| \frac{\partial g}{\partial x} \right\|_\circ \cdot \left\| \frac{\partial f}{\partial \theta} \left(\frac{\partial g}{\partial \theta} \right)^{-1} \right\|_\circ }\,, \tag{4.2.5}$$

then the diffeomorphism (4.2.1) *possesses an* m*-dimensional invariant torus in* $\mathbb{K}$ *which contains all* ω*-limit points of all positive semi-trajectories in* $\mathbb{K}$*. The torus is defined by the graph* $x = h^*(\theta)$ *where* h^* *is a* $\mathbb{C}^1$*-smooth* 2π*-periodic function.*

Proof. Due to Assumption 4.2 we can rewrite (4.2.1) in the *cross-form*

$$\begin{aligned} \bar{x} &= F(x, \bar{\theta}), \\ \theta &= G(x, \bar{\theta}), \qquad (\mathrm{mod}\, 2\pi). \end{aligned} \tag{4.2.6}$$

Observe that

$$\begin{aligned} F(x, \bar{\theta}) &\equiv f(x, G(x, \bar{\theta})), \\ \bar{\theta} &\equiv g(x, G(x, \bar{\theta})). \end{aligned} \tag{4.2.7}$$

It follows from this formula that the following estimates hold for the derivatives of F and G:

$$\begin{aligned} \left\| \frac{\partial F}{\partial x} \right\|_\circ &\leq \left\| \frac{\partial f}{\partial x} \right\|_\circ + \left\| \frac{\partial g}{\partial x} \right\|_\circ \cdot \left\| \frac{\partial f}{\partial \theta} \left(\frac{\partial g}{\partial \theta} \right)^{-1} \right\|_\circ, \\ \left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_\circ &= \left\| \frac{\partial f}{\partial \theta} \left(\frac{\partial g}{\partial \theta} \right)^{-1} \right\|_\circ, \\ \left\| \frac{\partial G}{\partial x} \right\|_\circ &\leq \left\| \frac{\partial g}{\partial x} \right\|_\circ \cdot \left\| \left(\frac{\partial g}{\partial \theta} \right)^{-1} \right\|_\circ, \\ \left\| \frac{\partial G}{\partial \bar{\theta}} \right\|_\circ &= \left\| \left(\frac{\partial g}{\partial \theta} \right)^{-1} \right\|_\circ. \end{aligned} \tag{4.2.8}$$

One can check that the following inequality follows from these estimates, and from (4.2.5):

$$\sqrt{\left\| \frac{\partial F}{\partial x} \right\|_\circ \cdot \left\| \frac{\partial G}{\partial \bar{\theta}} \right\|_\circ} + \sqrt{\left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_\circ \cdot \left\| \frac{\partial G}{\partial x} \right\|_\circ} < 1. \tag{4.2.9}$$

In particular

$$\left\| \frac{\partial F}{\partial x} \right\|_\circ \cdot \left\| \frac{\partial G}{\partial \bar{\theta}} \right\|_\circ < 1.$$

According to Assumption 4.2, for each fixed x the map $\theta = G(x, \bar{\theta})$ is a diffeomorphism of the torus $\mathbb{T}^m$ onto itself, and hence it cannot be a contraction mapping. Therefore, the maximum of the norm of its Jacobian matrix is necessarily greater than 1:

$$\left\| \frac{\partial G}{\partial \bar{\theta}} \right\|_\circ \geq 1\,.$$

This implies, in turn, that

$$\left\| \frac{\partial F}{\partial x} \right\|_\circ < 1\,.$$

We can now see that (4.2.9) implies the following inequality

$$\left\| \frac{\partial F}{\partial x} \right\|_\circ + \sqrt{\left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_\circ \cdot \left\| \frac{\partial G}{\partial x} \right\|_\circ} < 1\,. \tag{4.2.10}$$

Denote

$$\mathcal{L} = \sqrt{\left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_\circ \left(\left\| \frac{\partial G}{\partial x} \right\|_\circ \right)^{-1}} \tag{4.2.11}$$

(in the special case where $\frac{\partial G}{\partial x} \equiv 0$ we simply choose a sufficiently large number for $\mathcal{L}$). It follows immediately from (4.2.9) that

$$\mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_\circ < 1\,, \tag{4.2.12}$$

$$\sup_{(x,\bar{\theta})} \left\{ \left\| \frac{\partial F}{\partial x} \right\| \cdot \left\| \frac{\partial G}{\partial \bar{\theta}} \right\| \right\} \leq \left(1 - \mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_\circ \right) \left(1 - \frac{1}{\mathcal{L}} \left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_\circ \right), \tag{4.2.13}$$

$$\sup_{(x,\bar{\theta})} \left\{ \left\| \frac{\partial F}{\partial x} \right\| \cdot \left\| \frac{\partial G}{\partial \bar{\theta}} \right\| \right\} < \left(1 - \mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_\circ \right)^2, \tag{4.2.14}$$

and from (4.2.10) we have

$$\left\| \frac{\partial F}{\partial x} \right\|_\circ < 1 - \mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_\circ\,. \tag{4.2.15}$$

The rest of the proof is based only upon these inequalities. Let us denote by $H(\mathcal{L})$ the space of vector-functions $x = h(\theta)$ with the graph in $\mathbb{K}$: $\|h\| \leq \eta_0$, where h satisfies a Lipschitz condition:

$$\|h(\theta + \Delta\theta) - h(\theta)\| \leq \mathcal{L}\|\Delta\theta\|\,. \tag{4.2.16}$$

Let us endow $H(\mathcal{L})$ by the usual norm

$$\operatorname{dist}(h_1, h_2) = \|h_1 - h_2\| = \sup_\theta \|h_1(\theta) - h_2(\theta)\| .$$

It is well known that $H(\mathcal{L})$ is closed in the Banach space of bounded continuous functions $h(\theta)$.

Lemma 4.10. *Provided* (4.2.12) *and* (4.2.13) *are satisfied, the map T induces the operator $\mathcal{T} : H(\mathcal{L}) \to H(\mathcal{L})$ (i.e. the image of the graph of a Lipschitz function $x = h(\theta)$ by the map T is the graph of a function $\bar{x} = \tilde{h}(\bar{\theta})$ that satisfies a Lipschitz condition with the same constant $\mathcal{L}$).*

Indeed, let $h \in H(\mathcal{L})$. We must prove, first, that the image $T\{x = h(\theta)\}$ is a surface of the kind $\bar{x} = \tilde{h}(\bar{\theta})$ for some single-valued function $\tilde{h}$. In other words, we must show that for any $\bar{\theta}$ there exists a unique $\bar{x}$ (which would give $\tilde{h}(\bar{\theta})$) such that $(\bar{x}, \bar{\theta}) = T(h(\theta), \theta)$ for some θ. This is equivalent (see 4.2.6) to the existence, for any $\bar{\theta}$, of a unique solution of the following equation on θ:

$$\theta = G(h(\theta), \bar{\theta}) . \tag{4.2.17}$$

We can consider this equality for each fixed $\bar{\theta}$ as an equation of a fixed point of the map

$$\theta \mapsto G(h(\theta), \bar{\theta}) \tag{4.2.18}$$

of a torus into itself. The existence and uniqueness of the sought fixed point will follow from the Banach principle if we can prove that this map is a contraction. This is, however, an easy consequence of conditions (4.2.12) and (4.2.16): for fixed $\bar{\theta}$, for any $\Delta\theta$, we have

$$\|\Delta x\| \equiv \|h(\theta + \Delta\theta) - h(\theta)\| \le \mathcal{L}\|\Delta\theta\|$$

and

$$G(x + \Delta x, \bar{\theta}) - G(x, \bar{\theta}) = \left(\int_0^1 \frac{\partial G}{\partial x}(x + s\Delta x, \bar{\theta}) ds \right) \Delta x ,$$

whence

$$\|G(h(\theta + \Delta\theta, \bar{\theta}) - G(h(\theta), \bar{\theta})\| \le \mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_\circ \|\Delta\theta\| .$$

It follows from (4.2.12) that the map under consideration is a contraction indeed.

Thus, for any $\bar{\theta}$ there exists a unique θ for which the equality (4.2.17) holds. Since the fixed point of a contracting map depends continuously on a parameter ($\bar{\theta}$ in our case), it follows that the value of θ also depends continuously on $\bar{\theta}$.

By substituting the value of θ into the first equality in (4.2.6) we obtain a function $\tilde{h} = \tilde{T}h$ in the form

$$\bar{x} \equiv \tilde{h}(\bar{\theta}) = F(h(\theta(\bar{\theta})), \bar{\theta}) \,. \tag{4.2.19}$$

We see that $\tilde{h}$ is continuous. Let us show next that $\tilde{h}$ satisfies a Lipschitz condition. Obviously, it is sufficient to prove that at each $\bar{\theta}$, we have

$$\limsup_{\Delta\bar{\theta}\to 0} \frac{\|\Delta\bar{x}\|}{\|\Delta\bar{\theta}\|} \leq \mathcal{L} \,. \tag{4.2.20}$$

In order to prove this we note that according to (4.2.6)

$$\begin{aligned} \Delta\bar{x} &= F_x \Delta x + F_{\bar{\theta}} \Delta\bar{\theta} \,, \\ \Delta\theta &= G_x \Delta x + G_{\bar{\theta}} \Delta\bar{\theta} \,, \end{aligned} \tag{4.2.21}$$

where we denote

$$F_x = \int_0^1 \frac{\partial F}{\partial x}(x + s\Delta x, \bar{\theta} + s\Delta\bar{\theta}) ds \,,$$

etc. We assume here that the points (x, θ) and $(x + \Delta x, \theta + \Delta\theta)$ belong to $\{x = h(\theta)\}$ (hence the points $(\bar{x}, \bar{\theta})$ and $(\bar{x}+\Delta\bar{x}, \bar{\theta}+\Delta\bar{\theta})$ belong to $\{\bar{x} = \tilde{h}(\bar{\theta})\}$). Therefore, $\|\Delta x\| \leq \mathcal{L}\|\Delta\theta\|$. Substituting this into (4.2.21) gives

$$\|\Delta\bar{x}\| \leq \mathcal{L} \left\{ \frac{1}{\mathcal{L}} \|F_{\bar{\theta}}\| + \frac{\|F_x\| \cdot \|G_{\bar{\theta}}\|}{1 - \mathcal{L}\|G_x\|} \right\} \|\Delta\bar{\theta}\| \,.$$

In the limit we have

$$\limsup_{\Delta\bar{\theta}\to 0} \frac{\|\Delta\bar{x}\|}{\|\Delta\bar{\theta}\|} \leq \mathcal{L} \left\{ \frac{1}{\mathcal{L}} \left\| \frac{\partial F}{\partial\bar{\theta}} \right\| + \frac{\left\| \dfrac{\partial F}{\partial x} \right\| \cdot \left\| \dfrac{\partial G}{\partial\bar{\theta}} \right\|}{1 - \mathcal{L} \left\| \dfrac{\partial G}{\partial x} \right\|} \right\} . \tag{4.2.22}$$

Thus, by virtue of (4.2.13), the function $\tilde{h}$ does satisfy a Lipschitz condition. This completes the proof.

We have defined the operator $\mathcal{T}\colon H(\mathcal{L}) \to H(\mathcal{L})$. Let us show now that $\mathcal{T}$ is contracting. Since $H(\mathcal{L})$ is a closed subset of a Banach space, the Banach principle will guarantee the existence of a unique fixed point h^* for the operator $\tilde{T}$ on $H(\mathcal{L})$. We would have $\tilde{h^*} = h^*$ which means, by definition of $\tilde{T}$, that the image of the surface $\{x = h^*\theta\}$ by the map T is the same surface; *i.e.* this surface is the sought invariant manifold (to finish the proof we will also need to establish the smoothness of h^*).

Let h_1 and h_2 be two elements of $H(\mathcal{L})$ and $\tilde{h}_1$, $\tilde{h}_2$ are their images by $\tilde{T}$. Fix any $\bar{\theta}$ and take the points $(\bar{x}_1, \bar{\theta})$ and $(\bar{x}_2, \bar{\theta})$ at which the surface of constant $\bar{\theta}$ intersects the surfaces $\{\bar{x} = \tilde{h}_1(\bar{\theta})\}$ and $\{\bar{x} = \tilde{h}_2(\bar{\theta})\}$, respectively. Since these surfaces are, by definition, the images of the surfaces $\{x = h_1(\theta)\}$ and $\{x = h_2(\theta)\}$ by the map T, there exist points $(x_1 = h_1(\theta_1), \theta_1)$ and $(x_2 = h_2(\theta_2), \theta_2)$ such that $T(x_1, \theta_1) = (\bar{x}_1, \bar{\theta})$ and $T(x_2, \theta_2) = (\bar{x}_2, \bar{\theta})$. By (4.2.6)

$$\begin{cases} \bar{x}_1 = F(h_1(\theta_1), \bar{\theta})\,, \\ \theta_1 = G(h_1(\theta_1), \bar{\theta})\,, \end{cases} \qquad \begin{cases} \bar{x}_2 = F(h_2(\theta_2), \bar{\theta})\,, \\ \theta_2 = G(h_2(\theta_2), \bar{\theta})\,, \end{cases}$$

which gives

$$\begin{aligned} \|\theta_1 - \theta_2\| &\le \left\| \frac{\partial G}{\partial x} \right\|_\circ \|h_1(\theta_1) - h_2(\theta_2)\| \\ \|\bar{x}_1 - \bar{x}_2\| &\le \left\| \frac{\partial F}{\partial x} \right\|_\circ \|h_1(\theta_1) - h_2(\theta_2)\|\,. \end{aligned} \tag{4.2.23}$$

Using Lipschitz condition (4.2.16) we have

$$\begin{aligned} \|h_1(\theta_1) - h_2(\theta_2)\| &\le \|h_1(\theta_1) - h_1(\theta_2)\| + \|h_1(\theta_2) - h_2(\theta_2)\| \\ &\le \mathcal{L}\,\|\theta_1 - \theta_2\| + \mathrm{dist}(h_1, h_2)\,. \end{aligned}$$

Thus, inequalities (4.2.23) can be written in the form

$$\|\theta_1 - \theta_2\| \le \left\| \frac{\partial G}{\partial x} \right\|_\circ (\mathcal{L}\,\|\theta_1 - \theta_2\| + \mathrm{dist}(h_1, h_2))\,,$$

and

$$\|\bar{x}_1 - \bar{x}_2\| \le \left\| \frac{\partial F}{\partial x} \right\|_\circ (\mathcal{L}\,\|\theta_1 - \theta_2\| + \mathrm{dist}(h_1, h_2))\,,$$

or

$$\|\theta_1 - \theta_2\| \le \left\| \frac{\partial G}{\partial x} \right\|_\circ \left(1 - \left\| \frac{\partial G}{\partial x} \right\|_\circ \mathcal{L}\right)^{-1} \cdot \mathrm{dist}(h_1, h_2)$$

and, finally,

$$\left\|\tilde{h}_1(\bar{\theta}) - \tilde{h}_2(\bar{\theta})\right\| \equiv \|\bar{x}_1 - \bar{x}_2\| \le \left(\frac{\left\|\dfrac{\partial F}{\partial x}\right\|_\circ}{1 - \left\|\dfrac{\partial G}{\partial x}\right\|_\circ \mathcal{L}}\right) \cdot \mathrm{dist}(h_1, h_2)\,.$$

Since $\bar{\theta}$ is chosen arbitrary, the above inequality means, by definition, that

$$\mathrm{dist}(\tilde{h}_1, \tilde{h}_2) \le \left(\frac{\left\|\dfrac{\partial F}{\partial x}\right\|_\circ}{1 - \left\|\dfrac{\partial G}{\partial x}\right\|_\circ \mathcal{L}}\right) \cdot \mathrm{dist}(h_1, h_2)$$

so, by virtue of (4.2.15), the map $\tilde{T}$ is indeed a contraction.

We have proven the existence and uniqueness of a Lipschitz invariant manifold $M^* : \{x = h^*(\theta)\}$. Since the fixed point of a contraction operator is the limit of the a sequence of successive approximations starting from any initial guess, it follows that $h^* = \lim \tilde{T}^k h_0$ for any Lipschitz function h_0; or, what is the same, the forward image of any Lipschitz surface $\{x = h_0(\theta)\}$ by the map T converges to the invariant manifold M^*. This implies the claim of the theorem that the forward iterations of any point of $\mathbb{K}$ have their limit set on M^*.

Let us now prove the smoothness of M^*. The invariance of the manifold $\{x = h^*(\theta)\}$ means, according to (4.2.6), that for any $\bar{\theta}$

$$h^*(\bar{\theta}) = F(h^*(\theta), \bar{\theta})\,, \tag{4.2.24}$$

where the value of θ is defined implicitly by the equation

$$\theta = G(h^*(\theta), \bar{\theta})\,. \tag{4.2.25}$$

The last equation defines the map T^{-1} on the invariant manifold. The same arguments as in Lemma 4.10 shows that θ is a well-defined single-valued continuous function of $\bar{\theta}$.

It follows from a formal differentiation of (4.2.24) and (4.2.25) that the derivative $\eta^* = \frac{dh^*}{d\theta}$ (if it exists) must satisfy the equation

$$\eta^*(\bar{\theta}) = \frac{\partial F}{\partial \bar{\theta}} + \frac{\partial F}{\partial x} \cdot \eta^*(\theta) \cdot \left(I - \frac{\partial G}{\partial x} \cdot \eta^*(\theta)\right)^{-1} \cdot \frac{\partial G}{\partial \bar{\theta}}\,, \tag{4.2.26}$$

where all derivatives on the right-hand side are computed at $(x = h^*(\theta), \bar{\theta})$ and θ is defined by (4.2.25) as a function of $\bar{\theta}$. Let us prove that a continuous function η^* which satisfies this equality exists. Consider the space $H'(\mathcal{L})$ of bounded ($\|\eta\|_\circ \leq \mathcal{L}$) continuous functions $x = \eta(\theta)$. It is a closed subset of a Banach space of continuous functions with the norm

$$\|\eta_1 - \eta_2\| = \operatorname{dist}(\eta_1, \eta_2) = \sup_\theta \|\eta_1(\theta) - \eta_2(\theta)\| .$$

Consider the map $\eta \mapsto \tilde{\eta}$ defined on $H'(\mathcal{L})$:

$$\tilde{\eta}(\bar{\theta}) = \frac{\partial F}{\partial \bar{\theta}} + \frac{\partial F}{\partial x} \cdot \eta(\theta) \cdot \left(I - \frac{\partial G}{\partial x} \cdot \eta(\theta)\right)^{-1} \cdot \frac{\partial G}{\partial \bar{\theta}} . \tag{4.2.27}$$

This formula gives a rule for calculating $\tilde{\eta}$ when the function η is given: for an arbitrary $\bar{\theta}$ find θ by formula (4.2.25) and substitute the result into the right-hand side of (4.2.27).

We will prove that the map given by (4.2.27) takes $H'(\mathcal{L})$ into itself and that it is contracting — this implies the existence and uniqueness of the solution η^* of (4.2.26). The continuity of $\tilde{\eta}$ is obvious so we only need to check that it is bounded by $\mathcal{L}$ provided η is bounded by the same constant. Since

$$\left\| \frac{\partial G}{\partial x} \cdot \eta \right\| \leq \left\| \frac{\partial G}{\partial x} \right\|_\circ \mathcal{L} < 1$$

(see (4.2.12)), we may write

$$\left(I - \frac{\partial G}{\partial x} \cdot \eta\right)^{-1} = \sum_{k=0}^{+\infty} \left(\frac{\partial G}{\partial x} \cdot \eta\right)^k$$

whence

$$\begin{aligned}\left\| \left(I - \frac{\partial G}{\partial x} \cdot \eta\right)^{-1} \right\| &\leq \sum_{k=0}^{+\infty} \left\| \frac{\partial G}{\partial x} \cdot \eta \right\|^k \leq \sum_{k=0}^{+\infty} \left(\mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_\circ\right)^k \\ &= \frac{1}{1 - \mathcal{L} \left\| \dfrac{\partial G}{\partial x} \right\|_\circ} .\end{aligned}$$

Using this estimate we obtain from (4.2.27)

$$\|\tilde{\eta}\| \leq \left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_\circ + \frac{\left\| \dfrac{\partial F}{\partial x} \right\| \cdot \mathcal{L} \cdot \left\| \dfrac{\partial G}{\partial \bar{\theta}} \right\|}{1 - \mathcal{L} \left\| \dfrac{\partial G}{\partial x} \right\|_\circ} .$$

By virtue of (4.2.13) this gives

$$\|\tilde{\eta}\| \leq \mathcal{L}$$

i.e. $\tilde{\eta} \in H'(\mathcal{L})$ provided $\eta \in H'(\mathcal{L})$.

To prove the contraction, note that for any η_1 and η_2 from $H'(\mathcal{L})$

$$\begin{aligned} \tilde{\eta_2}(\bar{\theta}) - \tilde{\eta_1}(\bar{\theta}) = \frac{\partial F}{\partial x} \cdot \left(I - \eta_2(\theta) \cdot \frac{\partial G}{\partial x}\right)^{-1} \\ \times (\eta_2(\theta) - \eta_1(\theta)) \cdot \left(I - \frac{\partial G}{\partial x} \cdot \eta_1(\theta)\right)^{-1} \cdot \frac{\partial G}{\partial \bar{\theta}} . \end{aligned} \tag{4.2.28}$$

To derive this formula we use

$$\eta \left(I - \frac{\partial G}{\partial x} \cdot \eta\right)^{-1} = \eta + \eta \cdot \frac{\partial G}{\partial x} \cdot \eta + \cdots = \left(I - \eta \cdot \frac{\partial G}{\partial x}\right)^{-1} \eta \tag{4.2.29}$$

whence

$$\begin{aligned} &\eta_2 \left(I - \frac{\partial G}{\partial x} \cdot \eta_2\right)^{-1} - \eta_1 \left(I - \frac{\partial G}{\partial x} \cdot \eta_1\right)^{-1} \\ &= \left(I - \eta_2 \cdot \frac{\partial G}{\partial x}\right)^{-1} \eta_2 - \eta_1 \left(I - \frac{\partial G}{\partial x} \cdot \eta_1\right)^{-1} . \end{aligned}$$

Then we apply the identity

$$\begin{aligned} &\left(I - \eta_2 \cdot \frac{\partial G}{\partial x}\right)^{-1} \eta_2 - \eta_1 \left(I - \frac{\partial G}{\partial x} \cdot \eta_1\right)^{-1} \\ &= \left(I - \eta_2 \cdot \frac{\partial G}{\partial x}\right)^{-1} (\eta_2 - \eta_1) \left(I - \frac{\partial G}{\partial x} \cdot \eta_1\right)^{-1} . \end{aligned}$$

This can be verified by multiplying $(I - \eta_2 \cdot \frac{\partial G}{\partial x})$ on the left and $(I - \frac{\partial G}{\partial x} \cdot \eta_1)$ on the right.

It follows from (4.2.28) that

$$\operatorname{dist}(\tilde{\eta_2}, \tilde{\eta_1}) \leq \frac{\sup \left(\left\| \frac{\partial F}{\partial x} \right\| \cdot \left\| \frac{\partial G}{\partial \bar{\theta}} \right\| \right)}{\left(1 - \mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_\circ \right)^2} \cdot \operatorname{dist}(\eta_2, \eta_1) ,$$

which implies a contraction by virtue of (4.2.14).

We have proven the existence of the formal derivative η^* defined uniquely in (4.2.26). Let us now show that $\eta^*(\theta) \equiv \frac{dh^*}{d\theta}$ indeed. This is the same as proving that the following quantity vanishes identically:

$$z(\theta) = \limsup_{\Delta\theta \to 0} \frac{\|h^*(\theta + \Delta\theta) - h^*(\theta) - \eta^*(\theta)\Delta\theta\|}{\|\Delta\theta\|}\,. \tag{4.2.30}$$

Observe that the function z is uniformly bounded (because η^* is bounded and h^* satisfies a Lipschitz condition).

Let us evaluate $z(\bar\theta)$ in terms of $z(\theta)$. First, we prove that

$$\begin{aligned} &h^*(\bar\theta + \Delta\bar\theta) - h^*(\bar\theta) \\ &\qquad = \frac{\partial F}{\partial x} \cdot (h^*(\theta + \Delta\theta) - h^*(\theta)) + \frac{\partial F}{\partial \bar\theta}\Delta\bar\theta + o(\Delta\bar\theta) + o(\Delta\theta) \end{aligned} \tag{4.2.31}$$

where $\Delta\theta$ satisfies

$$\Delta\theta = \frac{\partial G}{\partial x}(h^*(\theta + \Delta\theta) - h^*(\theta)) + \frac{\partial G}{\partial \bar\theta}\Delta\bar\theta + o(\Delta\bar\theta) + o(\Delta\theta) \tag{4.2.32}$$

by virtue of (4.2.25); observe that h^* satisfies a Lipschitz condition. In particular,

$$\limsup_{\Delta\bar\theta \to 0} \frac{\|\Delta\theta\|}{\|\Delta\bar\theta\|} \le \frac{\left\|\dfrac{\partial G}{\partial \bar\theta}\right\|}{1 - \mathcal{L}\left\|\dfrac{\partial G}{\partial x}\right\|_\circ}\,. \tag{4.2.33}$$

Let us rewrite (4.2.32) as

$$\begin{aligned} \Delta\theta = {} & \left(I - \frac{\partial G}{\partial x}\eta^*(\theta)\right)^{-1}\left[\frac{\partial G}{\partial x}(h^*(\theta + \Delta\theta) - h^*(\theta) - \eta^*(\theta)\Delta\theta) + \frac{\partial G}{\partial \bar\theta}\Delta\bar\theta\right] \\ & + o(\Delta\bar\theta) + o(\Delta\theta)\,. \end{aligned}$$

Now, using (4.2.26) we may write

$$\begin{aligned} h^*(\bar\theta + \Delta\bar\theta) - h^*(\bar\theta) - \eta^*(\bar\theta)\Delta\bar\theta = {} & \frac{\partial F}{\partial x} \cdot \left(I + \eta^*(\theta)\left(I - \frac{\partial G}{\partial x}\eta^*(\theta)\right)^{-1}\frac{\partial G}{\partial x}\right) \\ & \cdot (h^*(\theta + \Delta\theta) - h^*(\theta) - \eta^*(\theta)\Delta\theta) + o(\Delta\bar\theta)\,, \end{aligned}$$

or (see (4.2.29))

$$\begin{aligned} &h^*(\bar\theta + \Delta\bar\theta) - h^*(\bar\theta) - \eta^*(\bar\theta)\Delta\bar\theta \\ &\qquad = \frac{\partial F}{\partial x} \cdot \left(I - \eta^*(\theta)\frac{\partial G}{\partial x}\right)^{-1}(h^*(\theta + \Delta\theta) - h^*(\theta) - \eta^*(\theta)\Delta\theta) + o(\Delta\bar\theta)\,. \end{aligned}$$

Hence, by (4.2.30) and (4.2.33) it follows that:

$$z(\bar{\theta}) \leq \frac{\sup\limits_{(x,\bar{\theta})} \left\{ \left\| \frac{\partial F}{\partial x} \right\| \cdot \left\| \frac{\partial G}{\partial \bar{\theta}} \right\| \right\}}{\left(1 - \mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_{\circ} \right)^2} \, z(\theta) \,. \tag{4.2.34}$$

The coefficient in front of $z(\theta)$ in this formula is strictly less than 1. Recall that θ is uniquely defined by $\bar{\theta}$ for any point on the invariant manifold M^*, *i.e.* we may consider the infinite backward orbit of any point on M^* and it will stay on M^*. But if the value of z is non-zero at some point, it would grow unboundedly with the backward iterations in view of (4.2.34). Since this contradicts the uniform boundedness of $z(\theta)$, this function must be identically zero everywhere on M^*. This implies the smoothness of the invariant manifold and completes the proof.

A careful examination of the proof shows that we did not use the condition that θ is an angular variable in an essential way (we needed this only in the very beginning when we derive formulas (4.2.9) and (4.2.10) from the assumptions of the theorem). Our arguments work in the general context as well, and to avoid further repetition we simply state the result as follows.

Theorem 4.3. *Let X and Y be some convex closed subsets of some Banach spaces. Suppose a map T is defined* in the cross-form *on $X \times Y$:*

$$\begin{aligned} \bar{x} &= F(x, \bar{y}) \,, \\ y &= G(x, \bar{y}) \,, \end{aligned} \tag{4.2.35}$$

which means that two points (x, y) and $(\bar{x}, \bar{y})$ from $X \times Y$ are related by the map T if and only if (4.2.35) *holds. Let F and G be smooth functions satisfying the following two conditions*

$$\sqrt{\sup_{(x,\bar{y}) \in X \times Y} \left\{ \left\| \frac{\partial F}{\partial x} \right\| \cdot \left\| \frac{\partial G}{\partial \bar{\theta}} \right\| \right\}} + \sqrt{\left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_{\circ} \left\| \frac{\partial G}{\partial x} \right\|_{\circ}} < 1 \tag{4.2.36}$$

and

$$\left\| \frac{\partial F}{\partial x} \right\|_{\circ} + \sqrt{\left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_{\circ} \left\| \frac{\partial G}{\partial x} \right\|_{\circ}} < 1 \,. \tag{4.2.37}$$

Then the map T has an invariant $\mathbb{C}^1$-smooth manifold M^* which contains the ω-limit points of any forward orbit of T.

Note that, in principle, the map T is not supposed to be single-valued: it is not forbidden to have several different pairs $(\bar{x}, \bar{y})$ corresponding to the same (x, y) in (4.2.35). So a point in $X \times Y$ may have more than one orbit, and the theorem establishes that for all of them the ω-limit set is included in M^*.

Neither do we need the condition that $(\bar{x}, \bar{y})$ depends smoothly on (x, y). Nevertheless, as it follows from the proof above that *the inverse map* T^{-1} *is single-valued and smooth on* M^*: it is implicitly defined by the equation

$$y = G(h^*(y), \bar{y}), \tag{4.2.38}$$

where $x = h^*(y)$ is the equation of M^*.

The derivative η^* of h^* satisfies the relation

$$\eta^*(\bar{y}) = \frac{\partial F}{\partial \bar{y}} + \frac{\partial F}{\partial x} \cdot \eta^*(y) \cdot \left(I - \frac{\partial G}{\partial x} \cdot \eta^*(y) \right)^{-1} \cdot \frac{\partial G}{\partial \bar{y}} \tag{4.2.39}$$

(simply rewrite formulas (4.2.24) and (4.2.26)). We see that the derivative η^* is a function whose graph is an invariant manifold of the map written in the cross-form:

$$\begin{aligned} \bar{\eta} &= \mathcal{F}(\eta, \bar{y}), \\ y &= \mathcal{G}(\bar{y}), \end{aligned} \tag{4.2.40}$$

where $\mathcal{G}$ is given implicitly by (4.2.38) and $\mathcal{F}$ is given by the right-hand side of (4.2.39). We may apply Theorem 4.3 to this map (observe that $\frac{\partial \mathcal{G}}{\partial \eta} \equiv 0$) to obtain the following:
if

$$\sup \left\{ \left\| \frac{\partial \mathcal{F}}{\partial \eta} \right\| \cdot \left\| \frac{\partial \mathcal{G}}{\partial \bar{y}} \right\| \right\} < 1 \tag{4.2.41}$$

and

$$\sup \left\{ \left\| \frac{\partial \mathcal{F}}{\partial \eta} \right\| \right\} < 1, \tag{4.2.42}$$

then the invariant manifold $\eta = \eta^*(y)$ is *unique* and smooth. This implies that the derivative η^* is a smooth function of y, and hence, $h^* \in C^2$ in this case.

Let us rewrite the above conditions for $\mathbb{C}^2$-smoothness of M^* in terms of the original functions F and G. First, note that the formulae for $\mathcal{F}$ involve the first derivatives of F and G; therefore to have $\mathcal{F}$ smooth we need F and G to be at least $\mathbb{C}^2$. Concerning the derivative $\frac{\partial \mathcal{F}}{\partial \eta}$, recall that we have already made analogous estimates (in slightly different notations; see (4.2.28)); so to avoid repetition we simply give the result:

$$\left\| \frac{\partial \mathcal{F}}{\partial \eta} \right\| \leq \frac{\left\| \frac{\partial F}{\partial x} \right\| \cdot \left\| \frac{\partial G}{\partial \bar{y}} \right\|}{\left(1 - \mathcal{L} \left\| \frac{\partial G}{\partial x} \right\|_{\circ} \right)^2} .$$

Here $\mathcal{L}$ is the Lipschitz constant — the upper bound for the norm of the derivative η^*. By construction (see (4.2.11)),

$$\mathcal{L} = \sqrt{\left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_{\circ} \left(\left\| \frac{\partial G}{\partial x} \right\|_{\circ} \right)^{-1}} .$$

Thus,

$$\left\| \frac{\partial \mathcal{F}}{\partial \eta} \right\| \leq \frac{\left\| \frac{\partial F}{\partial x} \right\| \cdot \left\| \frac{\partial G}{\partial \bar{y}} \right\|}{\left(1 - \sqrt{\left\| \frac{\partial G}{\partial x} \right\|_{\circ} \left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_{\circ}} \right)^2} . \tag{4.2.43}$$

For the derivative $\frac{\partial \mathcal{G}}{\partial \bar{y}}$ we obtain the following estimate directly from (4.2.38):

$$\left\| \frac{\partial \mathcal{G}}{\partial \bar{y}} \right\| \leq \frac{\left\| \frac{\partial G}{\partial \bar{y}} \right\|}{\left(1 - \sqrt{\left\| \frac{\partial G}{\partial x} \right\|_{\circ} \left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_{\circ}} \right)^2} . \tag{4.2.44}$$

Substituting these two inequalities into (4.2.41) we obtain the following additional sufficient condition for $\mathbb{C}^2$-smoothness of M^* (condition (4.2.42) does not introduce new restrictions in comparison with the conditions of Theorem 4.3):

$$\frac{\sup \left\{ \left\| \frac{\partial F}{\partial x} \right\| \cdot \left\| \frac{\partial G}{\partial \bar{y}} \right\|^2 \right\}}{\left(1 - \sqrt{\left\| \frac{\partial G}{\partial x} \right\|_{\circ} \left\| \frac{\partial F}{\partial \bar{\theta}} \right\|_{\circ}} \right)^3} < 1 .$$

or

$$\sqrt[3]{\sup\left\{\left\|\frac{\partial F}{\partial x}\right\|\cdot\left\|\frac{\partial G}{\partial \bar{y}}\right\|^2\right\}}+\sqrt{\left\|\frac{\partial G}{\partial x}\right\|_\circ\left\|\frac{\partial F}{\partial \bar{y}}\right\|_\circ}<1\,. \tag{4.2.45}$$

One may repeat the above procedure to derive sufficient conditions for $\mathbb{C}^3$-smoothness (by plugging (4.2.43) and (4.2.44) into (4.2.45)), etc. By induction, we arrive at the following theorem.

Theorem 4.4. *Let the functions F and G in Theorem 4.3 be $\mathbb{C}^r$-smooth $(r \geq 1)$, and assume that they satisfy the additional condition*

$$\sqrt[q+1]{\sup_{(x,\bar{y})\in X\times Y}\left\{\left\|\frac{\partial F}{\partial x}\right\|\cdot\left\|\frac{\partial G}{\partial \bar{y}}\right\|^q\right\}}+\sqrt{\left\|\frac{\partial F}{\partial \bar{y}}\right\|_\circ\left\|\frac{\partial G}{\partial x}\right\|_\circ}<1 \tag{4.2.46}$$

for some integer $q \leq r$. In this case the invariant manifold M^* is at least $\mathbb{C}^q$-smooth.

We have formulated the theorem in terms of the cross-map and do not need the smoothness of the map T itself. Moreover, an examination of the proof shows that the theorem holds true even if we also allow the functions F and G (which define the cross-map) to have singularities on a finite number of surfaces $\{y = \text{const}\}$ (or on a finite number of smooth surfaces transverse to any surface $x = h(y)$ with $\|h'(y)\| \leq \mathcal{L}$) provided that

- the derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial G}{\partial x}$, $\frac{\partial F}{\partial \bar{y}}$, as well as all the derivatives $\frac{\partial^k F}{\partial \bar{y}^k}$ $(k \leq q)$, are continuous everywhere on $X \times Y$;
- on the surfaces of singularity

$$\lim\left\|\frac{\partial^{p_0+k_0}F}{\partial x^{k_0}\partial \bar{y}^{p_0}}\right\|\cdot\left\|\frac{\partial^{p_1+k_1}G}{\partial x^{k_1}\partial \bar{y}^{p_1}}\right\|\cdot\cdots\cdot\left\|\frac{\partial^{p_s+k_s}G}{\partial x^{k_s}\partial \bar{y}^{p_s}}\right\|=0$$

  for any integers $p_0 \geq 0$, $p_1 \geq 1, \ldots, p_s \geq 1$ and $k_0 \geq 1$, $k_1 \geq 0, \ldots, k_s \geq 0$ such that $k_0 + \cdots + k_s \leq s \leq q$ and $p_0 + \cdots + p_s \leq q$.

Returning to the annulus principle, Theorem 4.4 gives the following result (see estimates (4.2.7)–(4.2.8) relating the derivatives of the cross-map and of the initial map).

Theorem 4.5. *Let the map* (4.2.1) *be a* $\mathbb{C}^r$ $(r \geq 2)$*-smooth diffeomorphism satisfying assumptions 4.1, 4.2 and*

$$\sqrt[q+1]{\left(\left\|\frac{\partial f}{\partial x}\right\|_\circ + \left\|\frac{\partial g}{\partial x}\right\|_\circ \left\|\frac{\partial f}{\partial \theta}\left(\frac{\partial g}{\partial \theta}\right)^{-1}\right\|_\circ\right) \cdot \left\|\left(\frac{\partial g}{\partial \theta}\right)^{-1}\right\|_\circ^q} + \sqrt{\left\|\left(\frac{\partial g}{\partial \theta}\right)^{-1}\right\|_\circ \left\|\frac{\partial g}{\partial x}\right\|_\circ \left\|\frac{\partial f}{\partial \theta}\left(\frac{\partial g}{\partial \theta}\right)^{-1}\right\|_\circ} < 1\,, \tag{4.2.47}$$

where $2 \leq q \leq r$. *Then the invariant torus given by Theorem* 4.2 (*condition* (4.2.47) *implies the assumption of that theorem*) *is at least* $\mathbb{C}^q$*-smooth.*

In the following section we will focus our attention on the case where the dimension m of the second equation in (4.2.1) is equal to one. Here, we only make some remarks concerning the higher-dimensional case. By assumptions, the map (4.2.3) (corresponding to the frozen x) is a diffeomorphism. We may include it into the following family

$$\bar{\theta} = \theta + \varepsilon g(x, \theta)\,. \tag{4.2.48}$$

We obtain the original map when $\varepsilon = 1$, and the identity map when $\varepsilon = 0$. This means that (4.2.3) is homotopic to the identity. However, amongst all diffeomorphisms of the tori there are some which are non-homotopic to the identity.

Let us regard the torus $\mathbb{T}^m$ as a unit cube

$$\left\{\theta \;\middle|\; 0 \leq \theta_j \leq 1\,, \quad (j = 1, \ldots, m)\right\}$$

with the identified points

$$(\theta_1, \ldots, \theta_{j-1}, 0, \theta_{j+1}, \ldots, \theta_m) \equiv (\theta_1, \ldots, \theta_{j-1}, 1, \theta_{j+1}, \ldots, \theta_m)\,.$$

An example of a diffeomorphism of a torus which is non-homotopic to the identity, is given by

$$\bar{\theta} = A\theta \pmod 1\,, \tag{4.2.49}$$

where A is an integer matrix (other than the identity matrix) with $\det |A| = \pm 1$. An example of such a diffeomorphism is the map

$$\bar{\theta} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \theta \pmod 1\,,$$

which is illustrated in Fig. 4.2.1.

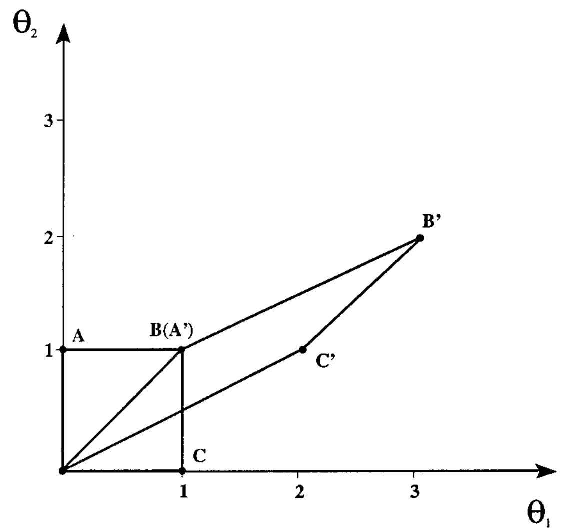


Fig. 4.2.1. An example of the action of a diffeomorphism of a torus which is non-homotopic to identity.

The map (4.2.49) is called *an algebraic automorphism of a torus.* In this case, for a diffeomorphism in the form

$$\begin{aligned} \bar{x} &= f(x,\theta)\,, \\ \bar{\theta} &= A\theta + g_0(x,\theta) = g(x,\theta) \pmod 1\,, \end{aligned} \tag{4.2.50}$$

where f and g are periodic functions of period 1 with respect to θ, the annulus principle is valid if in the ring $\mathbb{K}$ the following conditions are satisfied:

(1) The map
$$\bar{x} = f(x,\theta)$$
is contracting for $\|x\| \le r_0$, and

(2) The map
$$\bar{\theta} = A\theta + g_0(x,\theta) = g(x,\theta) \pmod 1\,,$$
is a diffeomorphism of a torus, and

(3) The map (4.2.50) satisfies conditions (4.2.5) or, for more smoothness, condition (4.2.47).

The proof of this statement is an exact copy of the proof of Theorem 4.2 or 4.5.

The map (4.2.50) when restricted on the torus $\mathbb{T}^m: x = h(\theta)$ can be written in the form

$$\bar{\theta} = A\theta + g_0(h(\theta), \theta) \pmod 1\,. \tag{4.2.51}$$

We postpone the bifurcational application of the general annulus principle to the second part of this book. We remark here that for the case $m = 1$ and $A = -1$, (4.2.51) is a non-orientable circle map of the form

$$\bar{\theta} = -\theta + g_0(\theta) \pmod 1\,, \tag{4.2.52}$$

which is the Poincaré map for certain flows on a Klein bottle.

4.3. Theorem on persistence of an invariant torus

Consider the family of systems

$$\begin{aligned} \dot{x} &= X(x) + p(x, \theta, \mu)\,, \\ \dot{\theta} &= 1\,, \end{aligned} \tag{4.3.1}$$

where X and p are $\mathbb{C}^r$ ($r \geq 1$)-smooth functions and p is 2π-periodic in θ; we will, therefore, identify θ and $\theta + 2\pi$.

We assume that p vanishes at $\mu = 0$ and that the corresponding autonomous system

$$\dot{x} = X(x)\,, \tag{4.3.2}$$

has a stable periodic trajectory L of period τ.

Theorem 4.6. *Under the above assumptions system* (4.3.1) *has a* $\mathbb{C}^r$*-smooth two-dimensional invariant torus for all sufficiently small* μ.

Proof. Let us introduce the normal coordinates (y, θ_0) (see (3.11.20)) instead of the x-coordinate in a small neighborhood of L. In the new variables the family takes the form

$$\begin{aligned} \dot{y} &= \Lambda y + F_0(\theta_0, y) + F_1(\theta_0, y, \theta, \mu)\,, \\ \dot{\theta}_0 &= \Omega_0 + b_0(\theta_0, y) + b_1(\theta_0, y, \theta, \mu)\,, \\ \dot{\theta} &= 1\,. \end{aligned} \tag{4.3.3}$$

where

$$F_0(\theta_0, 0) = 0, \qquad F'_{0y}(\theta_0, 0) = 0, \qquad b_0(\theta_0, 0) = 0\,, \tag{4.3.4}$$

and $\Omega_0 = 2\pi/\tau$.

The functions in the right-hand side of (4.3.3) are periodic of period 2π with respect to θ, and either of period 2π with respect to θ_0 or (see (3.11.22)) they are antiperiodic:[2]

$$\begin{aligned} F(\theta_0 + 2\pi, \sigma y, \theta, \mu) &= \sigma F(\theta, y, \theta, \mu) \\ b(\theta_0 + 2\pi, \sigma y, \theta, \mu) &= b(\theta, y, \theta, \mu) \end{aligned} \tag{4.3.5}$$

where σ is an involution changing sign of some of the variables y; in the periodic case, σ is the identity map. By construction (see Sec.3.11), the points (y, θ_0) and $(\sigma y, \theta_0 + 2\pi)$ correspond to the same point x.

Let us consider the diffeomorphism of the cross-section $\theta = 0$ defined by the time 2π shift by the trajectories of the system (we identify $\theta = 0$ and $\theta = 2\pi$). By continuous dependence on the parameter, this map is $\mathbb{C}^r$-close, at small μ, to the time 2π map of the autonomous system (4.3.2) (this corresponds to $F_1 = 0$ and $b_1 = 0$ in (4.3.3)). Thus, the map has the form

$$\begin{aligned} \bar{y} &= f(y, \theta_0, \mu) = e^{2\pi\Lambda} y + f_0(y, \theta_0) + f_1(y, \theta_0, \mu)\,, \\ \bar{\theta}_0 &= g(y, \theta, \mu) = \theta_0 + \omega_0 + g_0(y, \theta, \mu)\,, \end{aligned} \tag{4.3.6}$$

where the right-hand sides are either periodic or antiperiodic in θ_0, and $\omega_0 = 2\pi\Omega_0$. Moreover, (see (4.3.4))

$$\begin{aligned} f_0(0, \theta_0) = 0, \qquad f'_{0y}(0, \theta_0) = 0, \qquad f_1(y, \theta_0, 0) = 0\,, \\ g_0(0, \theta_0) = 0, \qquad g_1(y, \theta_0, 0) = 0\,. \end{aligned} \tag{4.3.7}$$

Let us now verify the conditions of the annulus principle (see the previous section). For a moment, we will not consider θ_0 as an angular variable but we assume that $\theta_0 \in (-\infty, +\infty)$; obviously, the conclusion of Theorem 4.2 on the existence of an invariant curve $y = h(\theta_0)$ will not change.

First, we must find δ such that the strip $\|y\| \le \delta$ is mapped into itself. Note that by (4.3.6), $\|f'_{0y}\|$ is small within such a strip for any sufficiently small δ. Thus, at $\mu = 0$, we have from (4.3.6), (4.3.7) that

$$\|\bar{y}\| \le (\|e^{2\pi\Lambda}\| + \varepsilon)\|y\|$$

[2] We denote $F = F_0 + F_1$ and $b = b_0 + b_1$.

where ε may be taken arbitrarily small provided that δ is small. By assumption, the periodic trajectory $L : \{y = 0\}$ is stable; *i.e.* all the eigenvalues of the matrix Λ lie strictly to the left of the imaginary axis (see Sec. 3.12). Hence, $\|e^{2\pi\Lambda}\| < 1$ and we have that at $\mu = 0$ the strip $K : \|y\| \leq \delta$ is mapped into itself at any sufficiently small δ. By continuity, the same holds true at all sufficiently small μ.

Now, we must check the fulfillment of inequalities (4.2.2), (4.2.4), (4.2.5) and (4.2.47) (at $q = r$) in K. Since these are strict inequalities, and we consider the case of small μ and δ, it is sufficient to check these conditions only at $y = 0$, $\mu = 0$. We have

$$\left\|\frac{\partial f}{\partial y}\right\|_{y=0,\mu=0} = \|e^{2\pi\Lambda}\| < 1, \ \left(\frac{\partial g}{\partial \theta_0}\right)^{-1}_{y=0,\mu=0} = 1, \ \left.\frac{\partial f}{\partial \theta_0}\right|_{y=0,\mu=0} = 0$$

and the fulfillment of the conditions of Theorems 4.2 and 4.5 follows immediately.

Thus, we established the existence, for all sufficiently small μ, a unique attractive invariant $\mathbb{C}^r$-smooth curve $y = h(\theta_0, \mu)$. By now $\theta_0 \in (-\infty, +\infty)$. Since the right-hand side of (4.3.6) is (anti)periodic, it follows that $y = \sigma h(\theta_0 + 2\pi, \mu)$ is also an invariant curve of this map. By uniqueness, we get

$$\sigma h(\theta_0 + 2\pi, \mu) = h(\theta_0, \mu)\,. \tag{4.3.8}$$

Recall, that by construction, the points (y, θ_0) and $(\sigma y, \theta_0 + 2\pi)$ must be identified because they correspond to the same point in the original x-coordinates. Thus, relation (4.3.8) shows that the invariant curve $y = h(\theta_0, \mu)$ is homeomorphic to a circle.

We have found a stable invariant circle for the time 2π map of the cross-section $\theta = 0 \pmod{2\pi}$ of system (4.3.1). The union of the trajectories starting on this circle is a two-dimensional stable invariant torus. End of the proof.

Remark. It is easy to check that our proof is applied, without changes, to the case where the function $p(x, \theta, \mu)$ in (4.3.1) depends only continuously on θ. In this case, the invariant torus is $\mathbb{C}^r$-smooth in the intersection with any cross-section $\theta = \text{const}$.

In the same way one may consider the general case where the autonomous system (4.3.2) has an arbitrary structurally stable periodic orbit, with m multipliers inside the unit circle and n multipliers outside the unit circle. The system (4.3.1) near L is written in the form (4.3.3) in normal coordinates, where the matrix Λ now has m eigenvalues strictly to the left of the imaginary axis and n eigenvalues strictly to the right of it. Let $y = (u, v)$ where $u \in \mathbb{R}^m$ be the projection onto the stable eigenspace of Λ and $v \in \mathbb{R}^n$ be the projection onto the unstable eigenspace. At $\mu = 0$ the system is written as

$$\begin{aligned} \dot{u} = \Lambda^s u + o(u, v), \qquad \dot{v} = \Lambda^u v + o(u, v)\,, \\ \dot{\theta}_0 = \Omega_0 + O(u, v), \qquad \dot{\theta} = 1 \end{aligned}$$

where the spectrum of Λ^s lies strictly to the left of the imaginary axis and the spectrum of Λ^u lies strictly to the right of it. The time 2π map $\{\theta = 0\} \to \{\theta = 2\pi\}$ is written, at $\mu = 0$, as

$$\begin{aligned} \bar{u} &= e^{2\pi\Lambda^s} u + o(u, v)\,, \\ \bar{v} &= e^{2\pi\Lambda^u} v + o(u, v)\,, \\ \bar{\theta}_0 &= \theta_0 + \omega_0 + O(u, v)\,. \end{aligned}$$

At small u and v, it is easy to see that the conditions of Theorem 4.4 are fulfilled for this map (one should consider $x = u$ and $y = (v, \theta_0)$ in (4.2.46)) and for the inverse to this map (in this case one should put $x = v$ and $y = (u, \theta_0)$ in (4.2.46)). By continuity, this holds true for all small μ. Thus, we established the existence of two smooth invariant manifolds at all small μ: a manifold $M_\mu^u : u = h^u(v, \theta_0, \mu)$ which attracts all forward iterations of the map, and a repelling manifold $M_\mu^s : v = h^s(u, \theta_0, \mu)$ which attracts all backward iterations. The trajectories which stay in a small neighborhood of L for all forward and backward iterations of the map, belong to the invariant circle $L_\mu = M_\mu^u \cap M_\mu^s$. By construction, the ω-limit set of any point in M_μ^s and the α-limit set of any point in M_μ^u belongs to L_μ. Returning from the map on the cross-section to the original system we arrive at the following result.

Theorem 4.7. *If the periodic orbit L of the autonomous system* (4.3.2) *is saddle with m multipliers inside the unite circle and n multipliers outside the unit circle, then for all sufficiently small μ system* (4.3.1) *has a $\mathbb{C}^r$-smooth saddle two-dimensional invariant torus with* $(m + 2)$*-dimensional stable and* $(n + 2)$*-dimensional unstable invariant manifolds.*

The existence of invariant manifolds in Theorem 4.4 is established by the Banach principle of contraction mappings. Therefore, it follows that the invariant circle L_μ depends continuously on μ.[3] At $\mu = 0$ it is given by the equation $y = 0$; hence the diffeomorphism on L_μ has the form (see (4.3.6)):

$$\bar{\theta}_0 = \theta_0 + \omega_0 + g^*(\theta_0, \mu) \qquad \mod 2\pi\,,$$

where $g^*(\theta_0, \mu) \equiv g_0(h(\theta_0, \mu), \theta_0, \mu)$ vanishes at $\mu = 0$.

We see that the study of non-autonomous perturbation in a neighborhood of a structurally stable periodic trajectory is reduced to the study of a diffeomorphism of a circle. We will review the theory of such diffeomorphisms in the next section.

Consider now the family of systems

$$\begin{aligned} \dot{x} &= X(x) + p(x, \theta, \mu)\,, \\ \dot{\theta} &= \Omega\,, \end{aligned} \tag{4.3.9}$$

where $\theta = (\theta_1, \ldots, \theta_k)$ is k-dimensional, and p is a $\mathbb{C}^r$-smooth 2π-periodic function with respect to each θ_j; the vector $\Omega = (\Omega_1, \ldots, \Omega_k)$ is comprised of linearly independent frequencies.

Assume that the autonomous system at $\mu = 0$

$$\dot{x} = X(x) \tag{4.3.10}$$

has a structurally stable equilibrium state O. Near the point O, the Poincaré map of the cross-section $\theta_k = 0 \bmod 2\pi$ into itself is written as

$$\begin{aligned} \bar{x} &= e^{2\pi A}x + o(x) + \ldots\,, \\ \bar{\theta}_j &= \theta_j + \omega_i \pmod{2\pi} \qquad\qquad (j = 1, \ldots, k-1) \end{aligned}$$

where $\omega_j = 2\pi\frac{\Omega_j}{\Omega_k}$; the matrix A is the linearization matrix of (4.3.10) at O; the ellipsis denotes the terms which vanish at $\mu = 0$.

Applying the annulus principle we can prove that for all sufficiently small μ the system (4.3.9) has a k-dimensional invariant torus $\mathbb{T}^k$ close to $x = 0$. Obviously, the stability of the torus is determined by the stability of the equilibrium state with respect to the autonomous system (4.3.10).

The torus has the form $x = h(\theta, \mu)$ (where $h = 0$ at $\mu = 0$). Hence, the motion on the torus is described by the second equation in (4.3.9) alone and is represented as a quasi-periodic motion with the frequency basis Ω.

[3]When the right-hand sides are smooth with respect to μ, the invariant curve depends smoothly on μ as well; for a proof it is sufficient to include μ among y-coordinates of Theorem 4.4.

Let us now assume that the system (4.3.10) has a structurally stable periodic orbit L of period $\frac{2\pi}{\Omega_0}$. In this case, at $\mu = 0$ the system (4.3.9) possesses a $(k+1)$-dimensional invariant torus $\mathbb{T}_0^{k+1} = L \times \mathbb{T}^k$. This torus is a minimal set if the frequencies $\Omega_0, \Omega_1, \ldots, \Omega_k$ form a basis. Otherwise, the torus is foliated into a family of k-dimensional tori.

In a similar manner to Theorem 4.6, we can construct the Poincaré map of the cross-section $\theta_k = 0 \pmod{2\pi}$ into itself along the trajectories near the torus $\mathbb{T}_0^{k+1}$. Applying the annulus principle we can prove that this map possesses a $\mathbb{C}^r$-smooth invariant torus $\mathbb{T}_\mu^k$ for all sufficiently small μ. Hence, the system (4.3.9) possesses a $(k+1)$-dimensional $\mathbb{C}^r$-smooth invariant torus $\mathbb{T}_\mu^{k+1}$.

The map on $\mathbb{T}_\mu^k$ has the form

$$\begin{aligned}
\bar{\theta}_0 \quad &= \theta_0 + \omega_0 + g^*(\theta_0, \ldots, \theta_{k-1}, \mu)\,, \\
\bar{\theta}_1 \quad &= \theta_1 + \omega_1\,, \\
\vdots \quad &\vdots \quad \vdots \\
\bar{\theta}_{k-1} &= \theta_{k-1} + \omega_{k-1}\,,
\end{aligned} \tag{4.3.11}$$

where each equation is taken in modulo 2π, and $\omega_j = 2\pi\frac{\Omega_j}{\Omega_k}$, $(j = 0, \ldots, k-1)$.

Suppose there is an additional small parameter α such that the first equation in (4.3.11) is represented in the form

$$\bar{\theta}_0 = \theta_0 + \omega_0 + g_0^*(\theta_0, \mu) + g_1^*(\theta_0, \theta_1, \ldots, \theta_{k-1}, \mu, \alpha), \quad \mod 2\pi\,, \tag{4.3.12}$$

where $g_1^* = 0$ at $\alpha = 0$. By assumption, at $\alpha = 0$, (4.3.12) is a diffeomorphism of a circle

$$\bar{\theta} = \theta_0 + \omega_0 + g_0^*(\theta, \mu) \qquad (\mathrm{mod}\, 2\pi)\,. \tag{4.3.13}$$

We assume that there exists an interval $\mu \in [\mu_1, \mu_2]$ where (4.3.13) has only structurally stable periodic points (see Sec. 4.4). Then applying the annulus principle we can also prove that for all α sufficiently small each stable periodic trajectory of (4.3.13) corresponds to a stable $(k-1)$ dimensional torus of the diffeomorphism (4.3.11).

Let us now rewrite (4.3.9) as a non-autonomous quasi-periodic system

$$\dot{x} = X(x) + p(x, \Omega_1 t, \ldots, \Omega_k t, \mu)\,,$$

with the frequency basis $(\Omega_1, \ldots, \Omega_k)$. As we saw, a stable equilibrium state of system (4.3.10) corresponds here to a stable quasi-periodic solution with the same basis of frequencies, for all small μ. A stable periodic orbit L of (4.3.10) corresponds to a stable quasi-periodic "tube" in $\mathbb{R}^n \times \mathbb{R}^k$ which is homeomorphic[4] to an infinite $(k+1)$-dimensional cylinder.

If the function p is represented as

$$p(x, \Omega_1 t, \dots, \Omega_k t, \mu) = p_0(x, \Omega_k t, \mu) + p_1(x, \Omega_1 t, \dots, \Omega_k t, \mu, \alpha)$$

where p_1 vanishes at $\alpha = 0$ and our assumption on the map (4.3.13) holds, then on this tube there exists a stable quasi-periodic solution with the same frequency basis $(\Omega_1, \dots, \Omega_k)$. In general, however, we cannot exclude the case where the structure of trajectories on the tube is much less trivial.

4.4. Basics of the theory of circle diffeomorphisms. Synchronization problems

An orientable circle diffeomorphism is written in the form

$$\bar{\theta} = \theta + g(\theta) \pmod{2\pi}, \tag{4.4.1}$$

where $g(\theta)$ is a periodic function of θ with period 2π. Equation (4.4.1) may be rewritten in the form

$$\bar{\theta} = \theta + \tau + g_0(\theta) \pmod{2\pi}, \tag{4.4.2}$$

where $g_0(\theta)$ is also a periodic function with zero mean value.

When $g_0(\theta) \equiv 0$ the situation is rather simple. In this case

$$\bar{\theta} = \theta + \tau \pmod{2\pi}, \tag{4.4.3}$$

and, therefore, this diffeomorphism is a rotation over an angle τ. It is easy to see that if τ is commensurable to 2π, *i.e.* $\tau = 2\pi p/q$, then all points on the circle are periodic of period q. In the case where τ is not commensurable to 2π there are no periodic points, and the trajectory of any point on the circle is everywhere dense on $\mathbb{S}^1$. In the latter case, the circle is a minimal set.

In the general case the question on the dynamics of (4.4.1) is answered by the Poincaré–Denjoy theory.

We can regard (4.4.1) not as a circle map but as the map $\mathbb{R}^1 \to \mathbb{R}^1$. In the given context this map is called *a lifting*, and $\mathbb{R}^1$ is called *a covering* of $\mathbb{S}^1$. Let

[4]And also *equimorphic*. An equimorphism is a uniformly continuous homeomorphism.

$\{\theta_j\}_{j=0}^{\infty}$ be a positive semi-trajectory of an initial point θ_0. Poincaré showed that there exists

$$\omega = \lim_{j\to\infty} \frac{\theta_j}{2\pi j}\,,$$

and that this limit ω does not depend on the choice of the initial point θ_0. The value ω is called *the Poincaré rotation number.*

Theorem 4.8. (Poincaré) *If the rotation number ω is rational, then the set of non-wandering points consists of periodic points, all having the same period. If ω is irrational, then the non-wandering set contains no periodic points.*

We note that Poincaré proved this statement when (4.4.1) is a homeomorphism.

Our next question is concerned with the structure of the non-wandering set when the rotation number is irrational.

Theorem 4.9. (Poincaré) *Let* (4.4.1) *be a homeomorphism with an irrational rotation number. Then the minimal sets of* (4.4.1) *may be either $\mathbb{S}^1$, or a finite or infinite union of minimal sets whose structures are analogous to a Cantor discontinuum.*

Theorem 4.10. (Denjoy)[5] *If* (4.4.1) *is a $\mathbb{C}^r$ $(r \geq 2)$-smooth diffeomorphism and the rotation number is irrational, then* (4.4.1) *is topologically conjugate to the map*

$$\bar{\theta} = \theta + \omega \pmod{2\pi}\,. \tag{4.4.4}$$

It follows from Denjoy's theorem that in this case the entire circle is a minimal set. When (4.4.1) is only $\mathbb{C}^1$-smooth, Denjoy constructed examples where the non-wandering set is a minimal set with a structure analogous to a Cantor discontinuum. This is the reason why we have given a special attention earlier to the necessity of proving, at least, the $\mathbb{C}^2$-smoothness of the invariant curves.

Let us consider next a one-parameter family of diffeomorphisms

$$\bar{\theta} = \theta + g(\theta, \mu) \pmod{2\pi}\,, \tag{4.4.5}$$

which depends continuously on a parameter μ. It is evident that a rotation number $\omega(\mu)$ is defined for each μ.

[5]Denjoy proved this statement under condition that $g'(\theta)$ has a bounded variation. This is true when $g(\theta) \in \mathbb{C}^2$.

Theorem 4.11. *The rotation number $\omega(\mu)$ is a continuous function of the parameter μ.*

Poincaré, and later, Krylov and Bogolyubov, certainly knew of this result, the proof of which was given in an explicit form by Maier. In this connection we must note the following result obtained by Hermann [34]: if the family depends smoothly on μ and if

$$g'_\mu(\theta, \mu) > 0$$

for all $\theta \in \mathbb{S}^1$ and for μ from some interval Δ, then $\omega(\mu)$ is a strictly monotonic function of each $\mu \in \Delta$ at which $\omega(\mu)$ is irrational.

Let us denote by B the space of all diffeomorphisms of the form (4.4.5) and let us introduce the distance between any two such diffeomorphisms as follows. Let T_1 be

$$\bar{\theta} = \theta + g_1(\theta, \mu) \pmod{2\pi},$$

and let T_2 be

$$\bar{\theta} = \theta + g_2(\theta, \mu) \pmod{2\pi}.$$

Then

$$\mathrm{dist}(T_1, T_2) = \max_\theta \Big\{ |g_1(\theta) - g_2(\theta)| + |g'_1(\theta) - g'_2(\theta)| \Big\}.$$

Theorem 4.12. (Maier) *The diffeomorphisms whose periodic points are structurally stable are everywhere dense in B.*

It follows from this theorem that any neighborhood in B of a diffeomorphism with an irrational Poincaré rotation number contains a diffeomorphism with a rational rotation number. On the other hand, if the diffeomorphism (4.4.1) possesses a structurally stable periodic trajectory, it is preserved under sufficiently small smooth perturbations of the original system. Hence it follows that nearby diffeomorphisms will have equal rotation numbers. In the case of a one-parameter family of diffeomorphisms this implies that there exists a maximal interval $[\mu^-, \mu^+]$ such that for the values μ from the interval, the function $\omega(\mu)$ is a constant one that takes rational values.

Under condition (4.4.6) de Melo and Pugh [46], in addition to Hermann's result, showed that $\omega(\mu)$ is a monotonic function for values of μ to the right of μ^- as well as to the left of μ^+.

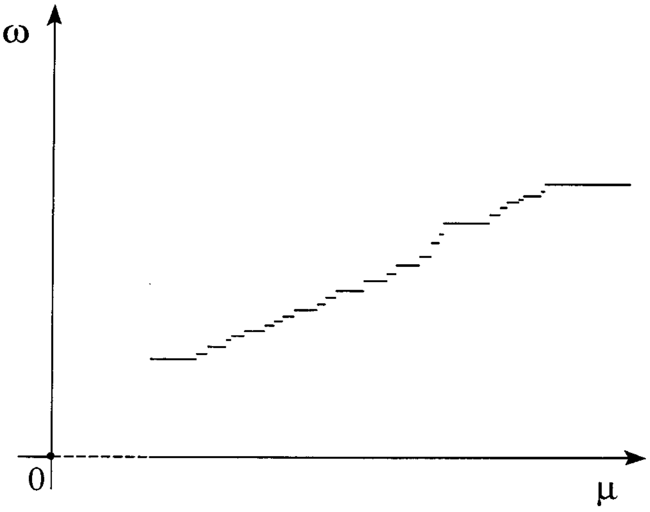


Fig. 4.4.1. A sketch of "the devil staircase".

It should be also noted that for every typical family of diffeomorphisms of a circle (without specifying the precise meaning here) for each μ within the interval of monotonicity of $\omega(\mu)$ the pre-image of a rational rotation number p/q is an interval $[\mu_{p/q}^-, \mu_{p/q}^+]$, where $\mu_{p/q}^- \neq \mu_{p/q}^+$, and the pre-image of an irrational rotation number is a point. Due to this feature the graph of the function $\omega(\mu)$ is usually referred as a "devil staircase".

The case when $\omega(\mu)$ is a monotonic function is sketched in Fig. 4.4.1. This function is constant on intervals where it takes rational values, and the pre-images of the irrational values of ω form a nowhere dense set, with possibly a non-zero Lebesgue measure.

We noticed earlier that the aim of most synchronization problems is the detection of regions in the parameter space where there are stable periodic oscillations. We can now see that there is a countable set of such synchronization regions in general. But, this does not imply that all of them are observable. This fact is well known if the problem under consideration admits a quasi-linear modeling. For example, in the case of the sine-like van der Pol generator under the action of a small external periodic force, the associated model is described by the equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + \omega_0^2 x = \mu A \sin \omega t \,,$$

where $0 < \mu \ll 1$. A careful analysis reveals that only resonances of the kind $(1 : 1)$; $(1 : 2)$; $(1 : 3)$; $(2 : 3)$ are easily observed. Using an averaging method, it can be shown that the other synchronization intervals have a size of order $e^{-1/\mu}$. The ratios of observable resonances in strongly nonlinear systems may be different. Numerous experiments in the study of the problem on the onset of turbulence have confirmed this fact.

So, only a finite number of visible synchronization regions may be detected (we do not discuss the clearance of experimental observations). The rest of the parameter regions where a two-dimensional invariant torus exists is usually interpreted, or associated with regions of modulation and beatings. Using the language of the theory of dynamical systems the modulation regimes can be translated either as a stable torus with an aperiodic trajectory on it, or with a stable periodic trajectory of a rather large period.[6] It is not superfluous to recall that systems with aperiodic trajectory behavior on a torus, in general, do not form a region in the parameter space.

Here we run into the case where the mathematical interpretation of the problem of synchronization differs in essence from that which is broadly used in nonlinear dynamics. The reason is that our traditional qualitative analysis employed the notion of irrational numbers which is a purely mathematical abstraction. To summarize we remark that the above observation is not the only example where the mathematical formalization of a problem based on the notion of irrational numbers does not agree with that suggested by common sense which lies beneath any empirical means or computer experiments.[7]

[6]Bogolubov and Mitropolsii had suggested that both situations be characterized as a multi-frequent regime.

[7]A similar situation arises when studying numerically an autonomous system under the action of a quasi-periodic external force, where the notion of a basis of independent frequencies plays a primary role.

Chapter 5

CENTER MANIFOLD. LOCAL CASE

Many physical systems can be realistically modeled by a system of ODEs. Usually, these models depend on a finite number of controlling parameters. As the parameters vary one can explain not only known behaviors exhibited by the model but can also predict new phenomena, if there are any. In most cases a comparison of the model's prediction with the real phenomenon requires both qualitative and quantitative (sufficiently close) correspondence. In the high-dimensional setting, one can encounter certain difficulties here both of mathematical and of numerical nature, although there are some special cases where well-developed methods exist.

Consider a family of dynamical systems

$$\dot{x} = X(x, \mu)\,, \tag{5.0.1}$$

where $x \in \mathbb{R}^n$, $\mu = (\mu_1, \ldots, \mu_p)$, X is a $\mathbb{C}^r$-smooth function with respect to all of its arguments and defined in some region $D \times U$, where $D \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^p$. Here, x is a vector of phase variables and μ is a vector of parameters. Let us assume that (5.0.1) has an exponentially stable equilibrium state $O_0(x = x_0)$ at $\mu = \mu^0$. This implies that the roots of the characteristic equation

$$\det |A_0 - \lambda I| = 0$$

of the associated linearized system

$$\dot{\xi} = A_0 \xi\,,$$

lie to the left of the imaginary axis, where

$$A_0 = \frac{\partial X(x_0, \mu^0)}{\partial x}\,.$$

As $\det|A_0| \neq 0$, by virtue of the implicit function theorem there exists a small $\delta > 0$ such that for $|\mu - \mu^0| < \delta$, system (5.0.1) has an equilibrium state $O_\mu(x = x(\mu))$ close to O_0. Moreover, $O\mu$ is also stable for all small $|\mu - \mu_0| < \delta_0 \leq \delta$ because the roots of the characteristic equation

$$\det|A(\mu) - \lambda I| = 0\,,$$

are continuous functions of μ, where

$$A(\mu) = \frac{\partial X(x(\mu),\mu)}{\partial x}\,.$$

Let us arbitrarily choose μ^1 which satisfies the condition $|\mu^1 - \mu^0| < \delta_0$. Repeating the above reasoning, we can find a new neighborhood $|\mu - \mu^1| < \delta_1$ where system (5.0.1) will have a stable equilibrium state O_μ, and so forth. As a result we can construct a maximal open set G in the parameter space, which is called *the stability region* of O_μ. This procedure for constructing the stability region resembles the construction of a Riemannian surface of an analytical function by means of Weierstrass's method. It may turn out that the stability region possesses a branched structure.

The boundary Γ of the stability region G corresponds to the case where some characteristic exponents of the equilibrium state O_μ, say $\lambda_1, \ldots, \lambda_m$, lie on the imaginary axis, whereas the rest of the eigenvalues $\lambda_{m+1}, \ldots, \lambda_n$ will still reside in the open left-half plane. Thus, near the bifurcating equilibrium state for some fixed parameter value on the boundary Γ, the system takes the form

$$\begin{aligned} \dot{y} &= Ay + f(x,y)\,, \\ \dot{x} &= Bx + g(x,y)\,, \end{aligned} \tag{5.0.2}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^{n-m}$, $\operatorname{spectr} A = \{\lambda_{m+1}, \ldots, \lambda_n\}$ such that $\operatorname{Re}\lambda_j < 0$ $(j = m+1, \ldots, n)$, $\operatorname{spectr} B = \{\lambda_1, \ldots, \lambda_m\}$ such that $\operatorname{Re}\lambda_i = 0$ $(i = 1, \ldots, m)$, and f and g are $\mathbb{C}^r$-smooth functions which vanish at the origin along with their first derivatives.

Now, in order to describe how the trajectories behave near O_μ we cannot rely only on the analysis of the linearized system but must account for the nonlinear terms as well. Such cases were called *critical* by Lyapunov who had derived a number of stability conditions for an equilibrium state in critical cases.

The modern approach for studying critical cases is restricted not only to the problem of stability. It also includes finding out what causes an equilibrium state to lose its stability and what happens beyond the stability boundary Γ. To answer these questions, the system under consideration must depend on some parameter μ:

$$\begin{aligned} \dot{y} &= Ay + f(x, y, \mu)\,, \\ \dot{x} &= Bx + g(x, y, \mu)\,, \end{aligned} \tag{5.0.3}$$

where μ takes the values near some critical parameter value μ^* (below we will assume that $\mu^* = 0$). All of the above problems constitute the main issue of the *theory of local bifurcations*. The basic results of this theory is the *center manifold theorem* credited to Pliss [52] and Kelley [37].

Theorem 5.1. (On center manifold) *Let* f, $g \in \mathbb{C}^r$ *in* (5.0.3), *where* $1 \leq r < \infty$. *Then, there exists a neighborhood* U *of the equilibrium state* O *such that for all* μ *sufficiently small it contains a* $\mathbb{C}^r$*-smooth*[1] *invariant* center manifold W^C *which is given by*

$$y = \psi(x, \mu)\,, \tag{5.0.4}$$

where

$$\psi(0, 0) = 0\,, \quad \frac{\partial \psi}{\partial x}(0, 0) = 0\,.$$

All trajectories which stay in U *for all times belong to the center manifold.*

The existence of a center manifold allows the problems related to the critical cases to be reduced to the study of an m-dimensional system

$$\dot{x} = Bx + g(x, \psi(x, \mu), \mu)\,. \tag{5.0.5}$$

Its dimension is equal to the number of characteristic exponents on the imaginary axis at the critical moment, *regardless of the dimension of the original system* (dim $= n$) which can be unboundedly large. Since the standard theory studies mainly bifurcations corresponding to $m = 1, 2, 3$, or $m = 4$, the reduction of an arbitrarily high-dimensional system (5.0.5) to a low-dimensional system (5.1.3) represents a tremendous advantage.

[1] We remark that if f, $g \in \mathbb{C}^\infty$, then the smoothness of the center manifold W^C may be arbitrarily large in a sufficiently small neighborhood U of the equilibrium state O. However, the larger the smoothness of W^C that we desire, the smaller the neighborhood U, and in principle, even if the original family has infinite smoothness, a $\mathbb{C}^\infty$ center manifold may not exist.

We stress again that a center invariant manifold possesses only a finite smoothness, so even when the original systems were analytic, the associated reduced system would nevertheless lose the analytic structure. Therefore, subtle results on analytic low-dimensional systems cannot be immediately applied to the study of critical cases. The *non-uniqueness* of the center manifold must also be mentioned as a possible complication.

A logical scheme of the center manifold theory is discussed in Sec. 5.1 (the proofs are presented in Sec. 5.4). Our study will also involve another geometrical object — *the strong stable invariant foliation*. Its existence allows the system to be locally reduced (by a $\mathbb{C}^{r-1}$-change of variables) to the simplest and the most suitable *triangular* form

$$\begin{aligned} \dot{y} &= (A + F(x,y,\mu))\, y\,, \\ \dot{x} &= Bx + G(x,\mu)\,, \end{aligned} \tag{5.0.6}$$

where $G(x,\mu) \equiv g(x,\psi(x,\mu),\mu)$ and $F \in \mathbb{C}^{r-1}$, $F(0,0,0) = 0$. This means that the behavior of the "critical" variables x in a small neighborhood of the structurally unstable equilibrium is independent of the other variables and repeats the behavior on the center manifold. For the y-variables we have an exponential contraction (because the spectrum of A lies strictly to the left of the imaginary axis. Compare with Sec. 2.6).

An analogue of the center manifold theorem holds true in the general case, *i.e.* when we consider an equilibrium state which has some characteristic exponents to the right of the imaginary axis as well. Therefore, in this case the qualitative study of local bifurcations can also be reduced to a lower dimensional system. Note, however, that the smooth reduction of the entire system into a triangular form of type (5.0.6) is not always possible in this general case (the corresponding coordinate transformation is only $\mathbb{C}^0$).

The same scheme works in studying the behavior of the solutions on the boundary of the existence of *periodic trajectories* but with one significant restriction. For the periodic trajectories, in contrast to the case of equilibrium states, the stability or existence boundaries may be of two different types, namely:

1. A bifurcating periodic trajectory exists when the parameter is on the boundary;

2. A bifurcating periodic trajectory does not exist on the boundary.

The boundaries of the second type do not exist in the case of equilibrium states, whereas it is well known that the periodic trajectories can disappear upon approaching a bifurcation boundary: it is accomplished by collapsing into an equilibrium state, or by merging into a homoclinic loop, or via a more complicated structure — through "a blue sky catastrophe". We will not consider boundaries of the second type in Part I of this book.

Once we restrict ourself to the first case, we can construct a cross-section through the critical periodic trajectory (which now exists by assumption) and proceed with the study of the behavior of the trajectories of an associated Poincaré map close to the bifurcating fixed point. After that the center manifold theory can be applied just as in the case of equilibrium states.

The proof of the center manifold theorem which we present in this chapter is based on the study of some boundary-value problems (Secs. 5.2 and 5.3) as in Sec. 2.8 for the proof of the existence and smoothness of the stable and unstable manifolds of a saddle equilibrium state. We will develop a unified approach for both equilibrium states and periodic trajectories. Moreover, our proof will include all other local invariant manifold theorems throughout this book, and the theorems on invariant foliations as well.

We note that besides dynamical applications of the invariant manifold theorems, these results may also be used indirectly; for example in reducing a system near a saddle point to a special form. To do this we must select the strongly stable and unstable manifolds of the saddle. This topic is discussed in detail in Appendix A.

5.1. Reduction to the center manifold

Let us consider an n-dimensional system of differential equations in a small neighborhood of a structurally unstable equilibrium state O. In particular, let us consider the case where some of the characteristic exponents of O lie on the imaginary axis and the rest of the characteristic exponents have negative real parts:

$$\operatorname{Re}\lambda_1 = \cdots = \operatorname{Re}\lambda_m = 0\,, \quad \operatorname{Re}\lambda_{m+1} < 0, \ldots, \operatorname{Re}\lambda_n < 0\,.$$

The system may be written near O in the form

$$\begin{aligned} \dot{y} &= Ay + f(x,y)\,, \\ \dot{x} &= Bx + g(x,y)\,, \end{aligned} \tag{5.1.1}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^{n-m}$, $\operatorname{spectr} A = \{\lambda_{m+1}, \dots, \lambda_n\}$, $\operatorname{spectr} B = \{\lambda_1, \dots, \lambda_m\}$, and f and g are $\mathbb{C}^r$-smooth functions which vanish at the origin along with their first derivatives.

Let us include the system in a family which depends on some set of para meters $\mu = (\mu_1, \dots, \mu_p)$, namely,

$$\begin{aligned} \dot{y} &= Ay + f(x, y, \mu)\,, \\ \dot{x} &= Bx + g(x, y, \mu)\,. \end{aligned} \tag{5.1.2}$$

Theorem 5.2. *System* (5.1.2), *where f and g depend continuously on μ along with all derivatives with respect to* (x, y), *and* $f(0,0,0) = 0$, $g(0,0,0) = 0$, $(f,g)'_{(x,y)}(0,0,0) = 0$, *has, for each small μ, an m-dimensional $\mathbb{C}^r$-smooth invariant local center manifold W^C_{loc}: $y = \psi(x, \mu)$* (*here the function ψ depends continuously on μ along with all its derivatives with respect to x*) *which is tangent at O to the x-space at $\mu = 0$* ($\psi(0,0) = 0$, $\frac{\partial \psi}{\partial x}(0,0) = 0$). *For each μ the center manifold contains all trajectories that stay in a small neighborhood of O for all times.*

The proof of this theorem is given in Sec. 5.4. Note that in the case where the right-hand side of system (5.1.2) depends *smoothly* on μ, the center manifold depends smoothly on μ as well. In particular, if the functions f and g are $\mathbb{C}^r$ with respect to (x, y, μ), then the function ψ (whose graph $y = \psi(x, \mu)$ is W^C_{loc}) may be taken to be $\mathbb{C}^r$ with respect to (x, μ). This smoothness result follows from Theorem 5.2 if we add, formally, an equation

$$\dot{\mu} = 0$$

to system (5.1.2). If we now consider the pair (x, μ) as a new variable x, then the form of the augmented system is analogous to the parameterless system (5.1.1). This mean that one may apply the center manifold theorem which gives now a center manifold depending $\mathbb{C}^r$-smoothly on the new x; *i.e.* it is $\mathbb{C}^r$ with respect to (x, μ). This trick often allows one to eliminate any dependence of the system on μ. Consequently, we will omit the dependence on μ where it is not essential.

What should also be mentioned concerning the smoothness of W^C is that even if the system is $\mathbb{C}^\infty$, the center manifold is not necessarily $\mathbb{C}^\infty$. Of course, if the original system is $\mathbb{C}^\infty$, it is $\mathbb{C}^r$ for any finite r. Therefore, in this case one may apply the center manifold theorem with any given r which implies that:

> *If the original system is $\mathbb{C}^\infty$, then for any finite r there exists a neighborhood U_r of the origin where W^C_{loc} is $\mathbb{C}^r$.*

In principle, however, these neighborhoods may shrink to zero as $r \to +\infty$. To see this, note that the equilibrium state O may persist when a parameter μ varies but the characteristic exponents of O which lie on the imaginary axis at $\mu = 0$ may move at $\mu \neq 0$, say, to the left. These exponents would correspond to the leading eigenvalues of the associated linearized system. Hence, for non-zero μ the center manifold would coincide with some leading manifold which has only finite smoothness in general (see Chap. 2).

At $\mu = 0$ the following sufficient condition of $\mathbb{C}^\infty$-smoothness can be given.

> *If every trajectory in the center manifold W^C of a $\mathbb{C}^\infty$-smooth system tends to the equilibrium state O at $\mu = 0$, either as $t \to +\infty$ or as $t \to -\infty$, then the center manifold is of $\mathbb{C}^\infty$ smoothness.*

To prove this result, let us choose a point $P \in W^C$ and let V be a small piece of W^C which contains P. By proposition, for any given r, one may take V sufficiently small such that the time-t shift V_t of V along the trajectories of the system lies in U_r for some finite t. Since the center manifold is invariant, it follows that V_t is still a subset of the center manifold. Hence, V_t is $\mathbb{C}^r$-smooth because it lies in U_r, by definition of the neighborhood U_r. Now note that the original V is a time $(-t)$ shift of V_t. The shift by the trajectories of a $\mathbb{C}^\infty$-system is a $\mathbb{C}^\infty$-map. Thus, we have that V is a $\mathbb{C}^r$-surface. Summarizing: we found that for any point $P \in W^C_{loc}$ and for any finite r there exists a neighborhood of P where W^C_{loc} is $\mathbb{C}^r$, which means the $\mathbb{C}^\infty$-smoothness of W^C.

As mentioned earlier, the main implication of the center manifold theorem is that to study the local bifurcations of a structurally unstable equilibrium O (*i.e.* to study the set of the trajectories which remain in a small neighborhood of O for all times and the dependence of this set on μ) one may restrict the system on the center manifold W^C

$$\dot{x} = Bx + g(x, \psi(x))\,. \tag{5.1.3}$$

There is an ambiguity here, caused by the fact that *the center manifold is not uniquely defined by the system.* Therefore our notion of the reduction of a system onto a center manifold requires some logical analysis.

Let N be the set of all trajectories which stay in a small neighborhood of O for all times (from $-\infty$ to $+\infty$).[2] Suppose there exist two different center manifolds: $W^{C1}_{loc}: y = \psi_1(x)$ and $W^{C2}_{loc}: y = \psi_2(x)$. It follows from the center manifold theorem that both must contain the set N; *i.e.*

$$N \subseteq W^{c1}_{loc} \cap W^{c2}_{loc} \,.$$

In other words, if for some small x the trajectory of the point $(x, \psi_1(x))$ does not leave a small neighborhood of O for all times, then $\psi_2(x) = \psi_1(x)$; *i.e.* the function ψ is uniquely defined for all x corresponding to the points of N. In fact, the following, more general, statement holds.

Theorem 5.3. *For any two center manifolds $y = \psi_1(x)$ and $y = \psi_2(x)$, at each x_0 such that $(x_0, y_0) \in N$ for some y_0, the function ψ_1 coincides with ψ_2 along with all of the derivatives*:

$$\left.\frac{d^k\psi_1}{dx^k}\right|_{x=x_0} = \left.\frac{d^k\psi_2}{dx^k}\right|_{x=x_0}, \quad k = 0, \ldots, r\,.$$

Applying this theorem to the point O (which belongs to N by definition), we obtain the following result:

> *All derivatives of the function ψ whose graph determined a center manifold are uniquely defined at the origin.*

This means that although the center manifold is not unique, a Taylor expansion of the reduced system is defined uniquely.

A counterpart of this result is the following *smooth conjugacy theorem*.

Theorem 5.4. (On smooth conjugacy) *For any two local center manifolds W^{C_1} and W^{C_2} there exists a $\mathbb{C}^{r-1}$ change of variables x which maps trajectories of the reduced system*

$$\dot{x} = Bx + g(x, \psi_1(x))$$

onto the trajectories of the other reduced system

$$\dot{x} = Bx + g(x, \psi_2(x))\,.$$

[2] Unlike the case of a stable equilibrium where $N = \{O\}$, the presence of a zero, or pure imaginary characteristic exponents may make the structure of the set N quite non-trivial.

This theorem establishes that there is essentially no difference between the dynamics on different center manifolds of the same system. We see that the system on a center manifold is a sufficiently well-defined object. The computation of the Taylor expansion of this reduced system may be done in different ways. The invariance of the manifold $y = \psi(x)$ means, according to (5.1.1), that

$$\frac{\partial \psi}{\partial x}\,(Bx + g(x, \psi(x))) = A\psi(x) + f(x, \psi(x))\,. \tag{5.1.4}$$

Expanding the involved functions in a formal series in powers of x one can consequently find all the coefficients of the Taylor expansion of ψ from this equation (compare with Sec. 2.7). Then, the Taylor expansion of the right-hand side of the reduced system (5.1.3) can be computed.

Another method is based on the computation of formal normal forms. Recall (see Sec. 2.9), that the normal form method produces an algorithm for constructing a polynomial coordinate transformation which eliminates all non-resonant monomials up to a given order in the Taylor expansion of the right-hand sides of any system of ODE's near an equilibrium state. In our case (system (5.1.1)) any monomial $x_1^{k_1} \cdot \cdots \cdot x_m^{k_m}$ in the function f is non-resonant because the resonant relation $\lambda_j = k_1\lambda_1 + \cdots + k_m\lambda_m$ is impossible for $j > m$: by assumption $\operatorname{Re}\lambda_1 = \cdots = \operatorname{Re}\lambda_m = 0$ but $\operatorname{Re}\lambda_j < 0$. Analogously, any monomial in the function g is non-resonant if it includes y-variables. Thus, one can find a polynomial coordinate transformation which brings system (5.1.1) to the form

$$\begin{aligned} \dot{y} &= (A + F(x, y))y + o(\|x, y\|^r)\,, \\ \dot{x} &= Bx + G(x) + o(\|x, y\|^r)\,, \end{aligned} \tag{5.1.5}$$

where F and G are some polynomials of orders $(r-1)$ and r, respectively, and $F(0,0) = 0$, $G(0) = 0$, $G'(0) = 0$; the $o(\|x, y\|^r)$ terms vanish at the origin along with the derivatives up to the order r. One can extract from (5.1.4) that when the system is brought to form (5.1.5), the center manifold is given by

$$y = 0 + o(\|x\|^{r-1})\,,$$

whence

$$\dot{x} = Bx + G(x)$$

is an r-th order approximation to the system on W^C_{loc}.

Our normal form observations are, in fact, covered by the following *reduction theorem*.

Theorem 5.5. (Reduction theorem) *There exists a $\mathbb{C}^{r-1}$-smooth transformation of coordinates* ($\mathbb{C}^1$*-close to the identity near the origin*) *which brings system* (5.1.1) *to the form*[3]

$$\begin{aligned} \dot{y} &= (A + F(x,y))y\,, \\ \dot{x} &= Bx + G(x)\,, \end{aligned} \tag{5.1.6}$$

where $F \in \mathbb{C}^{r-1}$, $G \in \mathbb{C}^r$

$$F(0,0) = 0\,, \quad G(0,0) = 0\,, \quad G'(0) = 0\,.$$

Here the surface $\{y = 0\}$ is an invariant center manifold, thus we have *straightened* W^C_{loc}, as in Chap. 2. The straightening of the center manifold is, of course, a $\mathbb{C}^r$-transformation. One order of smoothness is lost because, in fact, much more is achieved in the theorem: the local evolution of the x variables is now completely independent of y. Notice that although the coordinate transformation is $\mathbb{C}^{r-1}$-smooth, the function G is $\mathbb{C}^r$-smooth: it just coincides with the nonlinear part of the restriction (5.1.3) of the original system on a $\mathbb{C}^r$-smooth center manifold. Thus, for any trajectory of system (5.1.6) the variables x behave like those on the center manifold, and for the y variables there is an exponential contraction to $y = 0$ as $t \to +\infty$. The last statement can be verified exactly in the same way as when we proved the asymptotic exponential stability of equilibrium states in Chap. 2: since the function F is small near O and since all eigenvalues of the matrix A lie strictly to the left of the imaginary axis, it can be seen from the first equation of (5.1.6) that

$$\frac{d\|y\|}{dt} \leq -\lambda\|y\|$$

in a small neighborhood of O, from which the exponential contraction to $y = 0$ follows.

We will give the proof of the reduction theorem in Sec. 5.4. Note that Theorem 5.4 on smooth conjugacy follows directly from Theorem 5.5, namely, if system (5.1.6) has a center manifold other than $\{y = 0\}$, the reduced system is still given by the same second equation of (5.1.6); *i.e.* when the system is in the *triangular* form (5.1.6) the restrictions on any two center manifolds are trivially conjugate. Since the coordinate transformation that brings the system

[3]Note the difference between (5.1.5) and this formula: the functions F and G are no longer polynomials here.

to this particular form is $\mathbb{C}^{r-1}$, we have a $\mathbb{C}^{r-1}$-conjugacy when the system is not reduced to this form.

Let us give a geometrical interpretation of Theorem 5.5. Obviously, when the system is in the triangular form (5.1.6), the time-t shift of any surface $\{x = \text{const}\}$ lies again in a surface of the same kind, for any t (unless the trajectories leave a small neighborhood of O). This means that *the foliation* of a small neighborhood of O by surfaces of constant x is *invariant* with respect to the system (5.1.6). The coordinate change which transforms system (5.1.6) to the initial form (5.1.1) maps the surfaces $\{x = \text{constant}\}$ into surfaces of the kind

$$x = \xi + \eta(y, \xi)\,, \tag{5.1.7}$$

where ξ is the x-coordinate of the intersection of the surface with a center manifold; the $\mathbb{C}^{r-1}$-function η vanishes at the origin along with the first derivatives (note that $\eta \equiv 0$ everywhere on W^C).

Since the transformation which maps the surfaces $\{x = \text{constant}\}$ into the surfaces given by (5.1.7) is a diffeomorphism, it follows that Eq. (5.1.7) defines a foliation of a small neighborhood of the origin by surfaces corresponding to fixed ξ. This implies that for each point (x, y) there is a unique ξ such that the surface corresponding to the given ξ passes through the point (x, y). Such a surface is called *a leaf* of the foliation: for each point from a small neighborhood of O there is one and only one leaf which contains the point. Since the leaves are parametrized by the points on W^C_{loc}, the center manifold is *the base* of the foliation. Since the foliation $\{x = \text{constant}\}$ is invariant with respect to system (5.1.6), its image (*i.e.* the foliation given by (5.1.7)) is an invariant foliation of system (5.1.1): for any t, the time-t shift of any leaf lies in a single leaf of the same foliation unless the trajectories leave a small neighborhood of O. After straightening the center manifold, the reduction to the triangular form (5.1.6) is achieved by just transforming $x \mapsto \xi(x, y)$ (inverse of (5.1.7)): the variable x is replaced by the x-coordinate of the projection of the point along the leaves of the invariant foliation onto the center manifold. The invariance of the foliation simply means that the evolution of the new coordinate $x = \xi$ is independent of y. Thus, we see that Theorem 5.5 basically establishes the existence of a foliation of the kind (5.1.7), transverse to the center manifold and invariant with respect to system (5.1.1). We will call it *the strong stable foliation* and denote it by $\mathcal{F}^{ss}$. We will prove that the foliation is *uniquely defined* at all points whose trajectories stay in a small neighborhood of O for all *positive* times. Namely,

> *For any two strong stable invariant foliations $\mathcal{F}_1^{ss}$ and $\mathcal{F}_2^{ss}$, for any point P whose trajectory stays in a small neighborhood of O for all positive times, the leaf of $\mathcal{F}_1^{ss}$ which passes through P coincides with the corresponding leaf of $\mathcal{F}_2^{ss}$.*

Since the function η which defines the foliation is $\mathbb{C}^{r-1}$, the projection onto the center manifold by the leaves of the foliation is a $\mathbb{C}^{r-1}$ map. Moreover, for any surface transverse to the leaves, the projection onto another transversal is a $\mathbb{C}^{r-1}$-diffeomorphism. In other words, $\mathcal{F}^s$ *is a* $\mathbb{C}^{r-1}$*-smooth foliation*.

Note that for any fixed ξ, the function η is, in fact, a $\mathbb{C}^r$-smooth function with respect to y (the proof will be given in Sec. 5.4). In other words, each leaf of the foliation is a $\mathbb{C}^r$-smooth surface. The particular value $\xi = 0$ corresponds to the leaf which passes through the point O. Since O is an equilibrium state, it is not shifted with time; therefore the leaf of the point O is mapped within itself by the time-t shift for any t. It follows that the $\mathbb{C}^r$-smooth surface $x = \eta(y; 0)$ is *an invariant manifold* of system (5.1.1). This manifold is tangent to the y-axis at O; it is unique and is called *a strong stable* invariant manifold W^{ss}.

When the system depends on a parameter μ continuously, the foliation F^{ss} depends continuously on μ (*i.e.* the function η in (5.1.7) is continuous in μ, as will be proved). If the dependence on μ is smooth, the function η is $\mathbb{C}^{r-1}$ with respect to $(y; \xi, \mu)$. Thus, the leaves of F^{ss} depend $\mathbb{C}^{r-1}$ smoothly on μ. In particular, when μ varies from zero, there may exist an equilibrium state O_μ which depends smoothly on μ. In this case, the leaf of the foliation F^{ss} which passes through O_μ is uniquely defined. It is an invariant manifold, and if the position of the point O_μ is a $\mathbb{C}^k$ function of μ where $0 \leq k \leq r-1$, then the strong stable manifold depends $\mathbb{C}^k$-smoothly on μ. We remark that, in general, this statement is no longer valid when $k = r$.

Let us now consider the more general case where the equilibrium state has characteristic exponents to the right of the imaginary axis as well. Here, the system near O takes the form

$$\begin{aligned} \dot{y} &= Ay + f(x, y, z)\,, \\ \dot{z} &= Cz + h(x, y, z)\,, \\ \dot{x} &= Bx + g(x, y, z)\,, \end{aligned} \tag{5.1.8}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{n-m-k}$, the eigenvalues of the matrix A lie to the left of the imaginary axis, the eigenvalues of the matrix B are zero or purely imaginary, and the eigenvalues of the matrix C lie to the right of the imaginary axis; the $\mathbb{C}^r$-functions f, h and g vanish at the origin along with their first derivatives. The right-hand sides of the system may depend on some parameters μ, either continuously (in this case the smooth manifolds to be discussed below depend continuously on μ), or smoothly. In the latter case we will include μ among the "center" variables x so the manifolds and foliations, which we discuss below, will have the same smoothness with respect to μ as the smoothness with respect to x.

The center manifold theory is based here on the following theorem.

Theorem 5.6. (on center stable manifold) *In a small neighborhood of O there exists an $(m+k)$-dimensional invariant center stable manifold $W^{sC}_{loc}: z = \psi^{sC}(x, y)$ of class $\mathbb{C}^r$, which contains O and which is tangent to the subspace $\{z = 0\}$ at O. The manifold W^{sC}_{loc} contains all trajectories which stay in a small neighborhood of O for all positive times. Though the center stable manifold is not defined uniquely, for any two manifolds W^{sC}_1 and W^{sC}_2 the functions ψ^{sC}_1 and ψ^{sC}_2 have the same Taylor expansion at O (and at each point whose trajectory stays in a small neighborhood of O for all $t \geq 0$).*

The proof will be given in Sec. 5.4. Note that if the system is $\mathbb{C}^\infty$-smooth, the center stable manifold has, in general, only finite smoothness: for any finite r there exists a neighborhood U_r of O where W^{sC} is $\mathbb{C}^r$-smooth. Just like the reasoning above, we can conclude that

> *If the system is $\mathbb{C}^\infty$, and if every trajectory of W^{sC}_{loc} tends to O as $t \to +\infty$, then W^{sC}_{loc} is $\mathbb{C}^\infty$-smooth.*

The reversion of time $t \to -t$ changes matrices A, B and C to $-A$, $-B$ and $-C$, respectively. Thus, the part of the spectrum of characteristic exponents that corresponds to the z-variables is now to the left of the imaginary axis, and the part of the spectrum that corresponds to the y-variables is now to the right. We may apply the theorem on the center stable manifold to the system obtained from (5.1.8) by a reversion of time and obtain the following theorem on a *center unstable manifold*:

Theorem 5.7. (on center unstable manifold) *In a small neighborhood of O there exists an $(n-k)$-dimensional $\mathbb{C}^r$-smooth invariant manifold $W^{uC}_{loc}: y = \psi^{uC}(x, z)$ which contains O and which is tangent to the subspace $\{y = 0\}$ at O. The center unstable manifold contains all the trajectories which stay in a small* *neighborhood of O for all negative times. For any two manifolds W_1^{uC} and W_2^{uC} the functions ψ_1^{uC} and ψ_2^{uC} have the same Taylor expansion at O (and at each point whose trajectory stays in a small neighborhood of O for all $t \leq 0$). In the case where the system is $\mathbb{C}^\infty$-smooth, the center unstable manifold has, in general, only finite smoothness, but if every trajectory of W_{loc}^{uC} tends to O as $t \to -\infty$, then W_{loc}^{uC} is $\mathbb{C}^\infty$-smooth.*

The intersection of the center stable and center unstable manifolds is a $\mathbb{C}^r$-smooth m-dimensional invariant *center* manifold $W_{loc}^C = W_{loc}^{sC} \cap W_{loc}^{uC}$ defined by an equation of the form $(y, z) = \psi^C(x)$. By construction, the center manifold contains the set N of all trajectories which stay in a small neighborhood of O for all times $t \in (-\infty, +\infty)$. Moreover, the function ψ^C is uniquely defined at all points of N along with all derivatives. In particular, the Taylor expansion of ψ^C at O is defined uniquely by the system.

Restricted on the center stable manifold, system (5.1.8) takes the form

$$
\begin{aligned}
\dot{y} &= Ay + f(x, y, \psi^{sC}(x, y))\,, \\
\dot{x} &= Bx + g(x, y, \psi^{sC}(x, y))\,,
\end{aligned}
\tag{5.1.9}
$$

which is similar to (5.1.1). Therefore, Theorem 5.5 is applicable, namely:

On W_{loc}^{sC} there exists a $\mathbb{C}^{r-1}$-smooth invariant foliation $\mathcal{F}^{ss}$ with $\mathbb{C}^r$-smooth leaves transverse to W^C; for each point whose trajectory tends to O as $t \to +\infty$, the corresponding leaf is uniquely defined by the system.

On the center unstable manifold, the system is reduced to a form similar to (5.1.1) by reversion in time. This gives us the existence of a *strong unstable* invariant foliation on W_{loc}^{uC}:

On W_{loc}^{uC} there exists a $\mathbb{C}^{r-1}$-smooth invariant foliation $\mathcal{F}^{uu}$ with $\mathbb{C}^r$-smooth leaves transverse to W^C; for each point whose trajectory tends to O as $t \to -\infty$ the corresponding leaf of $\mathcal{F}^{uu}$ is defined uniquely.

We remark that these foliations cannot be continued outside the center stable or, respectively, center unstable manifold without loss of smoothness. In general, invariant foliations of a small neighborhood of O which are transverse to the center stable or center unstable manifolds are not smooth (of class $\mathbb{C}^0$ only, see Shoshitashvilly [70]). This means that the projection from one center stable manifold to another, or from one center unstable manifold to another by the leaves of the corresponding invariant foliation may be a non-smooth map. Therefore, there is no *smooth* conjugacy (only $\mathbb{C}^0$) between restrictions of the same system onto different center stable manifolds (or onto different center unstable manifolds).

Nevertheless, for *center* manifolds, Theorem 5.4 on smooth conjugacy still holds for the general case (where there are characteristic exponents on both sides of the imaginary axis). To prove this, note that if there are two different center manifolds W_1^C and W_2^C, then by construction there are two pairs of center stable and center unstable manifolds:

$$W_1^C = W_1^{sC} \cap W_1^{uC} \quad \text{and} \quad W_2^C = W_2^{sC} \cap W_2^{uC} .$$

The intersection $W_0^C = W_1^{sC} \cap W_2^{uC}$ is also a center manifold (by definition). The system on W_0^C is $\mathbb{C}^{r-1}$-conjugate with the system on W_1^C by means of a projection along the leaves of the strong stable foliation of W_1^{sC}, whereas the system on W_2^C is $\mathbb{C}^{r-1}$-conjugate with the system on W_0^C by means of a projection along the leaves of the strong unstable foliation of W_2^{uC}. The superposition of these two projections (from W_2^C onto W_0^C and, then, onto W_1^C) gives us a $\mathbb{C}^{r-1}$-transformation which maps trajectories of the system on W_2^C onto trajectories of the system W_1^C (this is because the foliations along which we make projections are invariant); *i.e.* we have a $\mathbb{C}^{r-1}$-conjugacy between the systems on W_2^C and W_1^C.

Thus, in this case, when studying local bifurcations, we may also restrict the system on a center manifold. Moreover, there is no significant difference between the dynamics on different center manifolds of the same system.

The straightening of the center stable and the center unstable manifolds along with the straightening of the strong stable and strong unstable invariant foliations on these manifolds lead to the following result which is similar to Theorem 5.5.

Theorem 5.8. *By a $\mathbb{C}^{r-1}$-smooth transformation system* (5.1.8) *can be locally reduced to the form*

$$\begin{aligned} \dot{y} &= (A + F(x,y,z))y \,, \\ \dot{z} &= (C + H(x,y,z))z \,, \\ \dot{x} &= Bx + G_0(x) + G_1(x,y,z)y + G_2(x,y,z)z \,, \end{aligned} \tag{5.1.10}$$

where G_0 is a $\mathbb{C}^r$-smooth function vanishing at $x = 0$ along with its first derivative, F and H are $\mathbb{C}^{r-1}$-functions vanishing at the origin, $G_{1,2} \in \mathbb{C}^{r-1}$ and G_1 vanishes identically at $z = 0$ and G_2 vanishes identically at $y = 0$.

Here, the local center unstable manifold is given by $\{y = 0\}$, the local center stable manifold is given by $\{z = 0\}$, and the local center manifold is given by $\{y = 0, z = 0\}$. The strong stable foliation is composed of the surfaces $\{x = \text{constant},\ z = 0\}$ and the leaves of the strong unstable foliation are $\{x = \text{constant},\ y = 0\}$.

An analogous theory may be applied to the study of structurally unstable periodic trajectories. The study of the dynamics in a small neighborhood of a periodic trajectory is reduced to the study of the Poincaré map on a small cross-section; the point O of intersection of the trajectory with the cross-section is a fixed point of the Poincaré map.

Let the system be $(n+1)$-dimensional, so the cross-section is n-dimensional. Let m multipliers of the periodic trajectory lie on the unit circle, k multipliers lie strictly inside the unit circle and the other $(n-m-k)$ multipliers are strictly greater than 1 in absolute value. The Poincaré map near the fixed point O is written in the following form:

$$\begin{aligned} \bar{y} &= Ay + f(x, y, z)\,, \\ \bar{z} &= Cz + h(x, y, z)\,, \\ \bar{x} &= Bx + g(x, y, z)\,, \end{aligned} \tag{5.1.11}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{n-m-k}$, the eigenvalues of the matrix A lie inside the unit circle, the eigenvalues of the matrix B equal to 1 in absolute value and the eigenvalues of the matrix C lie outside the unit circle; f, h and g are $\mathbb{C}^r$-smooth functions which vanish at the origin along with their first derivatives. We assume that the right-hand sides of the map (along with their derivatives) may depend continuously on some parameters μ. In this case, the manifolds and foliations to be discussed below will depend continuously on μ along with all derivatives. If the map depends smoothly on μ, then one can formally consider the parameters as x-variables, adding the trivial equation $\bar{\mu} = \mu$ to system (5.1.11). In this case the invariant manifolds and foliations below have the same smoothness with respect to μ as with respect to x.

The center manifold theorem may be formulated as follows.

Theorem 5.9. (On center manifold. General case) *In a small neighborhood of O there exist $\mathbb{C}^r$-smooth invariant $(m + k)$-dimensional center stable manifold $W^{sC}_{loc}\colon z = \psi^{sC}(x, y)$ and $(n - k)$-dimensional center unstable manifold $W^{uC}_{loc}\colon y = \psi^{uC}(x, y)$, which contain O and which are tangent, respectively, to the subspaces $\{z = 0\}$ and $\{y = 0\}$ at O. The manifold W^{sC}_{loc} contains the* *set* N^+ *of all points whose forward iterations by the map* (5.1.11) *stay in a small neighborhood of* O, *and* W^{uC}_{loc} *contains the set* N^- *of all points whose backward iterations never leave a small neighborhood of* O. *The intersection of* W^{sC}_{loc} *and* W^{uC}_{loc} *is a* $\mathbb{C}^r$*-smooth invariant* m*-dimensional center manifold* W^C_{loc}: $(y,z) = \psi^C(x)$ *which is tangent at* O *to the* x*-space and which contains the set* $N = N^+ \cap N^-$ *composed of all points whose iterations* (*both forward and backward*) *never leave a small neighborhood of* O. *The Taylor expansions of the functions* ψ^{uC}, ψ^{sC} *and* ψ^C *at the origin* (*and at each point of the sets* N^-, N^+ *or* N, *respectively*) *are uniquely defined by the system. On the manifolds* W^{sC}_{loc} *and* W^{uC}_{loc} *there exist, respectively, strong stable and strong unstable* $\mathbb{C}^{r-1}$*-smooth invariant foliations* $\mathcal{F}^{ss}$ *and* $\mathcal{F}^{uu}$ *with* $\mathbb{C}^r$*-smooth* k*-dimensional* (*resp.* $(n-m-k)$*-dimensional*) *leaves transverse to* W^C_{loc}. *The leaves of* $\mathcal{F}^{ss}$ *are uniquely defined at each point of the set* N^+ *and the leaves of* $\mathcal{F}^{uu}$ *are uniquely defined at each point of* N^-. *By projection along the leaves of the strong stable and strong unstable invariant foliations, the restrictions of the same map on different center manifolds are* $\mathbb{C}^{r-1}$*-conjugate.*

The proof will be given in Sec. 5.4. We remark again that if even in the case where the system under consideration is $\mathbb{C}^\infty$, the invariant manifolds are, in general, of a finite smoothness only. Nevertheless,

> *If the system is* $\mathbb{C}^\infty$*-smooth, and if the forward iterations of every trajectory of* W^{sC}_{loc} *tend to* O, *then* $W^{sC}_{loc} \in \mathbb{C}^\infty$;
> *If the backward iterations of any trajectory of* W^{uC}_{loc} *tend to* O, *then* $W^{uC}_{loc} \in \mathbb{C}^\infty$; *and*
> *If either the forward or backward iterations of every trajectory in* W^C_{loc} *tend to* O, *then* $W^C_{loc} \in \mathbb{C}^\infty$.

The straightening of the invariant manifolds and of the invariant foliations gives us the result which is completely analogous to Theorem 5.8, namely:

Theorem 5.10. *By a* $\mathbb{C}^{r-1}$*-smooth transformation system* (5.1.11) *can be locally reduced to the form*

$$\begin{aligned}
\bar{y} &= (A + F(x,y,z))y\,, \\
\bar{z} &= (C + H(x,y,z))z\,, \\
\bar{x} &= Bx + G_0(x) + G_1(x,y,z)y + G_2(x,y,z)z\,,
\end{aligned} \tag{5.1.12}$$

where G_0 is a $\mathbb{C}^r$-function vanishing at $x = 0$ along with the first derivative, F and H are $\mathbb{C}^{r-1}$-functions vanishing at the origin; $G_{1,2} \in \mathbb{C}^{r-1}$ and G_1 vanishes identically at $z = 0$, and G_2 vanishes identically at $y = 0$.

Here, the local center unstable manifold is given by $\{y = 0\}$, the local center stable manifold is given by $\{z = 0\}$ and the local center manifold is given by $\{y = 0, z = 0\}$. The strong stable foliation is composed of the surfaces $\{x = \text{constant},\ z = 0\}$, and the leaves of the strong unstable foliation are $\{x = \text{constant},\ y = 0\}$.

In the particular case where there are no multipliers outside the unit circle, one may put $z = 0$ identically and system (5.1.12) becomes

$$\begin{aligned} \bar{y} &= (A + F(x, y))y\,, \\ \bar{x} &= Bx + G(x)\,, \end{aligned} \tag{5.1.13}$$

where $F \in \mathbb{C}^{r-1}$ vanishes at the origin, and $G \in \mathbb{C}^r$ vanishes at $x = 0$ along with its first derivative.

5.2. A boundary-value problem

In this section we begin our proof of the center manifold theorems. The method, which we will use, is based on a generalization of the boundary-value problem which we have considered in Chap. 2 (see Shashkov and Turaev [52]). Since the results will be applied to the proof of various invariant manifold theorems beyond the center manifold theory, we will try to make the setting sufficiently general.

Let us consider a system of differential equations

$$\begin{aligned} \dot{z} &= Az + f(z, v, \mu, t)\,, \\ \dot{v} &= Bv + g(z, v, \mu, t)\,, \end{aligned} \tag{5.2.1}$$

where $z \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, t is the time variable and μ is a vector of parameters. We assume that f and g are $\mathbb{C}^r$-smooth ($r \geq 1$) with respect to the variables (z, v) and that they depend continuously on (μ, t) along with all the derivatives (a particular case of interest is when f and g are $\mathbb{C}^r$-smooth with respect to all of their arguments (z, v, μ, t)). Concerning the matrices A and B, we assume that the following conditions hold

$$\begin{gathered} \operatorname{spectr} A = \{\alpha_1, \ldots, \alpha_n\}\,, \quad \operatorname{spectr} B = \{\beta_1, \ldots, \beta_m\}\,, \\ \max_{i=1,\ldots,n} \operatorname{Re} \alpha_i < \alpha < \beta < \min_{j=1,\ldots,m} \operatorname{Re} \beta_j\,; \end{gathered} \tag{5.2.2}$$

i.e. there is a strip in the complex plane (the strip $\alpha \le \mathrm{Re}\,(\cdot) \le \beta$) that separates the spectra of A and B. It follows from (5.2.2) that in the appropriate (Jordan) basis the following estimates hold:

$$\begin{aligned} \left\|e^{As}\right\| &\le e^{\alpha s}\,, \\ \left\|e^{-Bs}\right\| &\le e^{-\beta s} \end{aligned} \tag{5.2.3}$$

for $s \ge 0$ (see Lemma 2.1). We also require that

$$\left\| \frac{\partial(f,g)}{\partial(z,v)} \right\| < \xi \tag{5.2.4}$$

for some sufficiently small constant ξ (the exact value of ξ can be obtained from the proofs of the theorems below). We will also assume that all of the derivatives of f and g are bounded uniformly for all z and v. The last conditions mean that the nonlinear part essentially does not affect the behavior induced by the specific structure of the linear part of the system (the separation of the spectrum). To stress this property we will call the systems satisfying (5.2.1)–(5.2.4) *globally dichotomic*.

Such systems appear naturally in the study of equilibrium states and periodic trajectories. For example, if the spectrum of the characteristic exponents of an equilibrium state is separated so that n characteristic exponents lie to the left of the line $\mathrm{Re}\,(\cdot) = \alpha$ and the other m characteristic exponents lie to the right of the line $\mathrm{Re}\,(\cdot) = \beta$ in the complex plane, then near such an equilibrium the system may be written locally in the form

$$\begin{aligned} \dot{z} &= Az + f(z, v, \mu)\,, \\ \dot{v} &= Bv + g(z, v, \mu)\,, \end{aligned} \tag{5.2.5}$$

where z belongs to a small neighborhood of the origin in $\mathbb{R}^n$, the variable v belongs to a small neighborhood of the origin in $\mathbb{R}^m$, and μ is some vector of parameters. Here, the matrices A and B satisfy (5.2.2), and the functions f, g satisfy the following conditions

$$f(0,0,0) = 0\,, \quad g(0,0,0) = 0\,, \quad \left.\frac{\partial(f,g)}{\partial(z,v)}\right|_{(z,v,\mu)=0} = 0\,. \tag{5.2.6}$$

Of course, the last equality in (5.2.6) implies the fulfillment of (5.2.4) for an arbitrarily small ξ, in a sufficiently small neighborhood of O.

The difference between systems (5.2.1) and (5.2.5) is that the nonlinear part of the latter remains small only near the origin, whereas for system (5.2.1) the linear part prevails everywhere in $\mathbb{R}^{n+m}$. A very useful trick which allows one to proceed from the local system (5.2.5) to the global version (5.2.1) is as follows. Consider a new system

$$\begin{aligned} \dot{z} &= Az + \tilde{f}(z, v, \mu) \\ \dot{v} &= Bv + \tilde{g}(z, v, \mu)\,, \end{aligned} \tag{5.2.7}$$

where the $\mathbb{C}^r$-smooth functions $\tilde{f}, \tilde{g}$ are given by

$$\begin{aligned} \tilde{f}(z, v) &= f\left(z\chi\left(\frac{\|(z,v)\|}{\rho}\right),\ v\chi\left(\frac{\|(z,v)\|}{\rho}\right)\right) \\ \tilde{g}(z, v) &= g\left(z\chi\left(\frac{\|(z,v)\|}{\rho}\right),\ v\chi\left(\frac{\|(z,v)\|}{\rho}\right)\right). \end{aligned} \tag{5.2.8}$$

Here, ρ is a small positive value and the function $\chi(u) \in \mathbb{C}^\infty$ is assumed to have the following properties

$$\chi(u) = \begin{cases} 1, & \text{for } u \leq \frac{1}{2} \\ 0, & \text{for } u \geq 1 \end{cases}, \qquad \text{and } 0 \geq \frac{d\chi}{du} \geq -3, \quad 1 \geq \chi \geq 0 \tag{5.2.9}$$

(the existence of such functions is a well-known fact).

It follows from (5.2.6), (5.2.8) and (5.2.9) that, the functions $\tilde{f}$ and $\tilde{g}$ satisfy inequality (5.2.4) for all $(z, v) \in \mathbb{R}^{n+m}$ and small μ. Moreover, the constant ξ may be made arbitrarily small if ρ is sufficiently small. Thus, system (5.2.7) is globally dichotomic and coincides with system (5.2.5) at $\|(z, v)\| \leq \rho/2$. Hence, the trajectories of system (5.2.7) coincide with the trajectories of (5.2.5) until they remain in a $\rho/2$-neighborhood of the origin.

Near a periodic trajectory L, a system of differential equations takes the form (see Chap. 3)

$$\begin{aligned} \dot{z} &= Az + f(z, v, \mu, t) \\ \dot{v} &= Bv + g(z, v, \mu, t)\,, \end{aligned} \tag{5.2.10}$$

where f and g are periodic function of t with a period equal to τ or 2τ, where τ is the period of L. The eigenvalues of the matrices A and B are equal to the ratios of the logarithms of the squares of the multipliers of L to the double period of L. Therefore, the condition (5.2.2) which separates the spectra of these matrices may be regarded as the separation of the multipliers: m multipliers must be less than $e^{\alpha\tau}$ in absolute value, and the absolute values of the other n multipliers are greater than $e^{\beta\tau}$ where τ is the period of L.

Here, the trajectory L is given by $\{z = 0, v = 0\}$, and f and g vanish at $(z, v) = 0$ along with the first derivatives with respect to (z, v). Condition (5.2.4) is fulfilled for small (z, v), so that changing (f, g) to $(\tilde{f}, \tilde{g})$ by formulae (5.2.8) results in a system of the type (5.2.1) which satisfies condition (5.2.4) for all (z, v), and which coincides locally with system (5.2.9).

Let us now return to the general setting and consider the following boundary-value problem for system (5.2.1)

$$z(0) = z^0\,, \quad v(\tau) = v^1\,, \tag{5.2.11}$$

for any $\tau > 0$, z_0 and v_1. Geometrically, this may be interpreted as finding a trajectory of system (5.2.1) which starts from the surface $\{z = z_0\}$ and finishes at the surface $\{v = v_1\}$ at the moment of time $t = \tau$. As discussed above, if the solution of the boundary-value problem stays in the region $\|(z, v)\| \leq \rho/2$ for $t \in [0, \tau]$, it is simultaneously a solution of the same boundary-value problem for the local system (5.2.5), or (5.2.10).

Theorem 5.11. *The boundary-value problem* (5.2.11) *for system* (5.2.1) *has a unique solution*[4]

$$z(t) = z^*(t; z^0, v^1, \tau, \mu)\,, \quad v(t) = v^*(t; z^0, v^1, \tau, \mu) \tag{5.2.12}$$

for any (z^0, v^1, τ).

Proof. The boundary-value problem under consideration is similar to that studied in Chap. 2 (formally speaking, Sec. 2.8 deals with the case where $\alpha < 0 < \beta$), and the proof follows very closely along the lines of Theorem 2.9. The novelty here is that we will prove the convergence of the successive approximations using an unusual norm, the so-called *γ-norm*. Namely, let us consider a space H of continuous functions $(z(t), v(t))$ defined on the segment $t \in [0, \tau]$. Let us endow the space H with the following norm:

$$\|(z(t), v(t))\|_\gamma = \sup_{t \in [0,\tau]} \left\{ \|(z(t), v(t))\| e^{-\gamma t} \right\}, \tag{5.2.13}$$

[4]Note that the solution of boundary-value problem (5.2.11) for system (5.2.1) is not prohibited from leaving a small neighborhood of zero. Therefore, the theorem cannot be directly applied to the local systems (5.2.5), or (5.2.10); namely, additional estimates are necessary to guarantee that the solution of the boundary-value problem stays bounded by a small constant.

where

$$\alpha < \gamma < \beta \,. \tag{5.2.14}$$

Obviously, H is a complete metric space.

Let us consider the integral operator $T\colon H \to H$, which maps a function $(z(t), v(t))$ onto the function $(\overline{z}(t), \overline{v}(t))$ via the following formula:

$$\begin{aligned}
\overline{z}(t) &= e^{At}z^0 + \int_0^t e^{A(t-s)} f(z(s), v(s), \mu, s)\; ds\,, \\
\overline{v}(t) &= e^{-B(\tau-t)}v^1 - \int_t^\tau e^{-B(s-t)} g(z(s), v(s), \mu, s)\; ds\,.
\end{aligned} \tag{5.2.15}$$

It is easy to check that if the solution of the boundary-value problem (5.2.11) exists, then the solution is a fixed point of the integral operator T, and vice versa (compare with Theorem 2.9).

Obviously, the operator T is smooth (in the sense of Sec. 3.15). The derivative of $(\bar{z}(t), \bar{v}(t))$ with respect to $(z(t), v(t))$ is the linear operator T' : $(\Delta z(t), \Delta v(t)) \mapsto (\Delta\bar{z}(t), \Delta\bar{v}(t))$ where

$$\begin{aligned}
\Delta\bar{z}(t) &= \int_0^t e^{A(t-s)} f'_{(z,v)}(z(s), v(s)) \cdot (\Delta z(s), \Delta v(s))ds \\
\Delta\bar{v}(t) &= -\int_t^\tau e^{-B(s-t)} g'_{(z,v)}(z(s), v(s)) \cdot (\Delta z(s), \Delta v(s))ds.
\end{aligned} \tag{5.2.16}$$

According to the Banach principle (the abstract version of Theorem 3.26), in order to show that the operator T has a unique fixed point, it is sufficient to show that $\|T'\| \le K < 1$ for any $(z(s), v(s)) \in H$.

To do this, let us plug (5.2.3), (5.2.4),(5.2.13) into (5.2.16). We get

$$\begin{aligned}
\|\Delta\bar{z}(t)\| &\le \int_0^t e^{\alpha(t-s)} \xi \|(\Delta z, \Delta v)\|_\gamma e^{\gamma s} ds \\
\|\Delta\bar{v}(t)\| &\le \int_t^\tau e^{-\beta(s-t)} \xi \|(\Delta z, \Delta v)\|_\gamma e^{\gamma s} ds\,.
\end{aligned} \tag{5.2.17}$$

From (5.2.13), (5.2.14) and (5.2.17) we obtain

$$\begin{aligned}
\|(\Delta\bar{z}, \Delta\bar{v})\|_\gamma &= \sup_{t\in[0,\tau]} \{\max\left(\|\Delta\bar{z}(t)\|, \|\Delta\bar{v}(t)\|\right) e^{-\gamma t}\} \\
&\le \xi \max\left(\frac{1}{\gamma-\alpha}, \frac{1}{\beta-\gamma}\right) \times \|(\Delta z, \Delta v)\|_\gamma\,.
\end{aligned} \tag{5.2.18}$$

Choose ξ sufficiently small such that

$$\xi \max\left(\frac{1}{\gamma-\alpha}, \frac{1}{\beta-\gamma}\right) < 1\,.$$

By (5.2.18), the integral operator T is contracting in the γ-norm.

Thus, according to the Banach principle of contraction mappings, starting with an arbitrary initial guess $(z^{(0)}(t), v^{(0)}(t))$, the sequence of successive approximations

$$\left(z^{(n+1)}(t), v^{(n+1)}(t)\right) = T\left(z^n(t), v^n(t)\right)$$

converges to a uniquely defined fixed point $(z^*(t), v^*(t))$ of T, which is, at the same time, the solution of boundary-value problem (5.2.11) for system (5.2.1). This completes the proof.

Note that the solution $(z^*(t), v^*(t))$ depends also on $\{z^0, v^1, \tau, \mu\}$. Since the integral operator T given by (5.2.15) is continuous with respect to these data, the solution depends continuously on $\{z^0, v^1, \tau, \mu\}$ (as the fixed point of a contraction operator, see Theorem 3.25). Moreover, by Theorem 3.27, since the operator T is $\mathbb{C}^r$-smooth and depends $\mathbb{C}^r$-smoothly on $\{z^0, v^1\}$, it follows that the solution of the boundary value problem is a $\mathbb{C}^r$-smooth function of $\{z^0, v^1\}$, and if the right-hand sides of the system are $\mathbb{C}^r$ with respect to all variables including t and μ, then the solution depends smoothly on $\{t, \tau, \mu\}$ as well.

The derivatives of (z^*, v^*) with respect to z^0, v^1 and μ are found as fixed points of an operator obtained by formal differentiation of (5.2.15); *i.e.* they are found as the solutions of the boundary-value problems for the corresponding variational equations with the boundary conditions obtained by formal differentiation of the boundary conditions (5.2.11) (see Sec. 2.8 for more details). For example, the solution of the boundary-value problem

$$Z(0) = I\,, \quad V(\tau) = 0\,, \tag{5.2.19}$$

(I is the identity matrix) for the system of variational equations

$$\begin{aligned} \dot{Z} &= AZ + f'_z(z^*, v^*, \mu, t)Z + f'_v(z^*, v^*, \mu, t)V\,, \\ \dot{V} &= BV + g'_z(z^*, v^*, \mu, t)Z + g'_v(z^*, v^*, \mu, t)V\,, \end{aligned} \tag{5.2.20}$$

gives the derivative of (z^*, v^*) with respect to z^0:

$$Z^* = \frac{\partial z^*}{\partial z^0}\,, \quad V^* = \frac{\partial v^*}{\partial z^0}\,.$$

The solution of another boundary-value problem

$$Z(0) = 0\,, \quad V(\tau) = I \tag{5.2.21}$$

for the same system (5.2.20) gives the derivative of (z^*, v^*) with respect to v^1. If f and g are smooth with respect to the parameter μ, then the derivatives $(Z^*, V^*) = (\partial z^*/\partial\mu, \partial v^*/\partial\mu)$ are the solution of the boundary-value problem

$$Z(0) = 0\,, \quad V(\tau) = 0 \tag{5.2.22}$$

for the system of non-homogeneous variational equations

$$\begin{aligned}\dot{Z} &= AZ + f'_z(z^*, v^*, \mu, t)Z + f'_v(z^*, v^*, \mu, t)V + f'_\mu(z^*, v^*, \mu, t)\,,\\ \dot{V} &= BV + g'_z(z^*, v^*, \mu, t)Z + g'_v(z^*, v^*, \mu, t)V + g'_\mu(z^*, v^*, \mu, t)\,.\end{aligned} \tag{5.2.23}$$

One can immediately see that system (5.2.20) or (5.2.23) is globally dichotomic: since the principal part of the right-hand sides is determined by the same matrices A and B, it follows that the condition (5.2.2) on the separation of the spectra still holds here; the residual part of the right-hand side is

$$\begin{aligned}F &= f'_z(z^*, v^*, \mu, t)Z + f'_v(z^*, v^*, \mu, t)V\\ G &= g'_z(z^*, v^*, \mu, t)Z + g'_v(z^*, v^*, \mu, t)V\,,\end{aligned}$$

or

$$\begin{aligned}F &= f'_z(z^*, v^*, \mu, t)Z + f'_v(z^*, v^*, \mu, t)V + f'_\mu(z^*, v^*, \mu, t)\\ G &= g'_z(z^*, v^*, \mu, t)Z + g'_v(z^*, v^*, \mu, t)V + g'_\mu(z^*, v^*, \mu, t)\,.\end{aligned}$$

In both cases

$$\frac{\partial(F, G)}{\partial(Z, V)} = \frac{\partial(f, g)}{\partial(z, v)}\,,$$

so condition (5.2.4) on the smallness of the derivatives is also fulfilled with the same ξ. Thus, the existence (and uniqueness) of solutions of the boundary-value problems (5.2.19), (5.2.21) and (5.2.22) simply follows from Theorem 5.11.

Now, by induction, one can see that the higher-order variational equations also belong to our class of globally dichotomic systems. Therefore, for the corresponding boundary-value problems, the existence and uniqueness of the solutions (which are the higher-order derivatives of (z^*, v^*)) is also given by Theorem 5.11.

The derivatives of (z^*, v^*) with respect to time t are given directly by system (5.2.1): because (z^*, v^*) is a solution of system (5.2.1), it follows that

$$\frac{\partial z^*}{\partial t} \equiv \dot{z}^* = Az^* + f(z^*, v^*, \mu, t)$$

and

$$\frac{\partial v^*}{\partial t} \equiv \dot{v}^* = Bz^* + g(z^*, v^*, \mu, t)\,.$$

The higher-order derivatives involving time are obtained by a repeated use of these identities. The derivatives with respect to τ may be calculated using the following lemma.

Lemma 5.1. *The solution* (z^*, v^*) *of the boundary-value problem* (5.2.11) *satisfies the following identities*:

$$\frac{\partial(z^*, v^*)}{\partial v^1}\,\frac{\partial v^*}{\partial t}\bigg|_{t=\tau} + \frac{\partial(z^*, v^*)}{\partial \tau} \equiv 0\,, \tag{5.2.24}$$

$$\frac{\partial(z^*, v^*)}{\partial \tau} + \frac{\partial(z^*, v^*)}{\partial t} - \frac{\partial(z^*, v^*)}{\partial z^0}\,\frac{\partial z^*}{\partial t}\bigg|_{t=0} \equiv 0\,. \tag{5.2.25}$$

These identities allow one to express the derivatives with respect to τ in terms of the derivatives with respect to the other variables. The proof of this lemma is achieved as follows. Recall that the uniquely defined solution of the boundary-value problem under consideration is the trajectory of system (5.2.1) which intersects the surface $\{z = z^0\}$ at $t = 0$ and the surface $\{v = v^1\}$ at $t = \tau$. We denote this trajectory as $(z^*(t; z^0, v^1, \tau, \mu), v^*(t; z^0, v^1, \tau, \mu))$.

At the moment of time $t = \tau + \delta$ the trajectory intersects the surface $\{v = v^*(\tau + \delta; z^0, v^1, \tau, \mu)\}$. By definition, we can write

$$\begin{aligned} z^*(t; z^0, v^1, \tau, \mu) &\equiv z^*(t; z^0, v^*(\tau + \delta; z^0, v^1, \tau, \mu), \tau + \delta, \mu)\,, \\ v^*(t; z^0, v^1, \tau, \mu) &\equiv v^*(t; z^0, v^*(\tau + \delta; z^0, v^1, \tau, \mu), \tau + \delta, \mu)\,. \end{aligned}$$

Differentiation of this identity with respect to δ at $\delta = 0$ gives (5.2.24). The analogous identity

$$\begin{aligned} z^*(t; z^0, v^1, \tau, \mu) &\equiv z^*(t + \delta; z^*(-\delta; z^0, v^1, \tau, \mu), v^1, \tau + \delta, \mu)\,, \\ v^*(t; z^0, v^1, \tau, \mu) &\equiv v^*(t + \delta; z^*(-\delta; z^0, v^1, \tau, \mu), v^1, \tau + \delta, \mu)\,. \end{aligned}$$

implies (5.2.25).

Our next theorem gives estimates on the derivatives of the solution of the boundary-value problem. We use the following notation for the derivatives of a vector-function $\phi = (\phi_1, \ldots, \phi_q) \in \mathbb{R}^q$ with respect to a vector-argument $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$:

$$\frac{\partial^{|s|}\phi}{\partial x^s} \equiv \left(\frac{\partial^{s_1+\cdots+s_p}\phi_1}{\partial x_1^{s_1}\cdots\partial x_p^{s_p}}\,, \ \ldots\ , \frac{\partial^{s_1+\cdots+s_p}\phi_q}{\partial x_1^{s_1}\cdots\partial x_p^{s_p}} \right)$$

where the multi-index $s = (s_1, \ldots, s_p)$ consists of non-negative integers and $|s|$ denotes $s_1 + \cdots + s_p$.

Theorem 5.12. *The following estimates hold for the solution* (z^*, v^*) *of the boundary-value problem* (5.2.11) *for system* (5.2.1) (*here* C *is a positive constant independent of* (z^0, v^1, μ, τ), *but depending on the order of differentiation* $k = |k_1| + |k_2| + |k_3|$).

1. *If* $0 < \alpha < \beta$, *then*

$$\left.\begin{array}{l} \text{(a)} \left\| \dfrac{\partial^{|k_1|+|k_2|+|k_3|}(z^*, v^*)}{\partial\,(z^0,\mu)^{k_1}\,\partial\,(v^1,\tau)^{k_2}\,\partial t^{k_3}} \right\| \\[4ex] \text{(b)} \left\| \dfrac{\partial^{|k_1|+|k_2|+|k_3|}(z^*, v^*)}{\partial\,(z^0,\tau,\mu)^{k_1}\,\partial\,(v^1)^{k_2}\,\partial(\tau - t)^{k_3}} \right\| \end{array}\right\} \le \begin{cases} C & \text{if } |k_1| = |k_2| = 0 \\[2ex] C\,e^{|k_1|\alpha t} & \begin{array}{l}\text{if } \quad |k_2| = 0 \\ \text{and } |k_1|\alpha < \beta\,,\end{array} \\[2ex] C\,e^{\beta(t-\tau)+|k_1|\alpha\tau} & \begin{array}{l}\text{if } \ |k_2| \neq 0 \\ \text{or } |k_1|\alpha > \beta\,.\end{array} \end{cases} \tag{5.2.26}$$

2. *If* $\alpha < 0 < \beta$, *then*

$$\left.\begin{array}{l} \text{(a)} \left\| \dfrac{\partial^{|k_1|+|k_2|+|k_3|}(z^*, v^*)}{\partial\,(z^0)^{k_1}\,\partial\,(v^1,\tau)^{k_2}\,\partial(t,\mu)^{k_3}} \right\| \\[4ex] \text{(b)} \left\| \dfrac{\partial^{|k_1|+|k_2|+|k_3|}(z^*, v^*)}{\partial\,(z^0,\tau)^{k_1}\,\partial\,(v^1)^{k_2}\,\partial(\tau - t,\mu)^{k_3}} \right\| \end{array}\right\} \le \begin{cases} C & \text{if } |k_1| = |k_2| = 0 \\[2ex] C\,e^{\alpha t} & \begin{array}{l}\text{if } \quad |k_2| = 0 \\ \text{and } |k_1| \neq 0\,,\end{array} \\[2ex] C\,e^{\beta(t-\tau)} & \begin{array}{l}\text{if } \quad |k_1| = 0 \\ \text{and } |k_2| \neq 0\,,\end{array} \\[2ex] C\,e^{\alpha t+\beta(t-\tau)} & \begin{array}{l}\text{if } \quad |k_1| \neq 0 \\ \text{and } |k_2| \neq 0\,.\end{array} \end{cases} \tag{5.2.27}$$

3. *If* $\alpha < \beta < 0$, *then*

$$\left.\begin{array}{l}\text{(a)}\ \left\| \dfrac{\partial^{|k_1|+|k_2|+|k_3|}(z^*, v^*)}{\partial (z^0)^{k_1} \partial (v^1, \tau, \mu)^{k_2} \partial t^{k_3}} \right\| \\ \text{(b)}\ \left\| \dfrac{\partial^{|k_1|+|k_2|+|k_3|}(z^*, v^*)}{\partial (z^0, \tau)^{k_1} \partial (v^1, \mu)^{k_2} \partial (\tau - t)^{k_3}} \right\| \end{array}\right\} \leq \begin{cases} C & \text{if } |k_1| = |k_2| = 0 \\ C\, e^{|k_2|\beta(t-\tau)} & \text{if } \ |k_1| = 0 \\ & \text{and } \alpha < |k_2|\beta\,, \\ C\, e^{\alpha t - |k_2|\beta\tau} & \text{if } \ |k_1| \neq 0 \\ & \text{or } \alpha > |k_2|\beta\,. \end{cases}$$

(5.2.28)

Proof. Note that the boundary-value problem (5.2.11) is symmetric with respect to a reversion in time via the following reassignment:

$$t \to \tau - t\,, \quad \alpha \to -\beta\,, \quad \beta \to -\alpha\,, \quad z \to v\,, \quad v \to z\,, \quad z^0 \to v^1\,, \quad v^1 \to z^0\,. \tag{5.2.29}$$

Therefore, estimates 1(b), 2(b) and 3(b) follow from estimates 3(a), 2(a) and 1(a), respectively, according to the rule above and the change $k_1 \leftrightarrow k_2$.

Note also that in the cases $\beta > \alpha > 0$ and $\alpha < \beta < 0$ the parameter μ can be included among the variables z or v respectively, by adding the equation $\dot{\mu} = 0$ to system (5.2.1) and the requirement $\mu(0) = \mu$ (in the case $\beta > \alpha > 0$) or $\mu(\tau) = \mu$ (in the case $0 > \beta > \alpha$) to the boundary conditions (5.2.11). Therefore, the derivatives involving μ have to be estimated separately only in the case $\alpha < 0 < \beta$.

To obtain the estimates for the derivatives with respect to time t we note that it follows directly from (5.2.1) that

$$\frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5|} z^*}{\partial (z^0)^{k_1} \partial (v^1)^{k_2} \partial \mu^{k_3} \partial \tau^{k_4} \partial t^{k_5+1}} = \frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5|} (Az^* + f(z^*, v^*, \mu, t))}{\partial (z^0)^{k_1} \partial (v^1)^{k_2} \partial \mu^{k_3} \partial \tau^{k_4} \partial t^{k_5}}\,, \tag{5.2.30}$$

$$\frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5|} v^*}{\partial (z^0)^{k_1} \partial (v^1)^{k_2} \partial \mu^{k_3} \partial \tau^{k_4} \partial t^{k_5+1}} = \frac{\partial^{|k_1|+|k_2|+|k_3|+|k_4|+|k_5|} (Bv^* + g(z^*, v^*, \mu, t))}{\partial (z^0)^{k_1} \partial (v^1)^{k_2} \partial \mu^{k_3} \partial \tau^{k_4} \partial t^{k_5}}\,.$$

One can see from these formulae that if the estimates of the theorem hold for the derivatives with respect to (z^0, v^1, μ, τ), then an additional differentiation with respect to t does not affect these estimates (except possibly changing the value of the constant C).

The derivatives with respect to τ are expressed in terms of the other derivatives via relation (5.2.24). It can be seen then that a differentiation with respect to τ at fixed t must give essentially the same estimates as a differentiation with respect to v^1.

Thus, in the case $\alpha < \beta < 0$, or $0 < \alpha < \beta$, it is sufficient to prove estimates (5.2.26) or, respectively, (5.2.28) for the derivatives $\frac{\partial^{|k_1|+|k_2|}(z^*,v^*)}{\partial(z^0)^{k_1}\partial(v^1)^{k_2}}$, and in the case $\alpha < 0 < \beta <$ it is sufficient to prove estimates (5.2.27) for the derivatives $\frac{\partial^{|k_1|+|k_2|+|k_3|}(z^*,v^*)}{\partial(z^0)^{k_1}\partial(v^1)^{k_2}\partial\mu^{k_3}}$. In fact, the calculation of these derivatives in the case $\alpha < \beta < 0$ is not necessary because it can be reduced to the case $0 < \alpha < \beta$ by applying the time reversion by rule (5.2.29). In the two remaining cases $0 < \alpha < \beta$ and $\alpha < 0 < \beta$ the calculations are quite similar so we will present the proof only for the more difficult case $0 < \alpha < \beta$ (estimates for the first derivatives for $\alpha < 0 < \beta$ can be found in Shilnikov [67]). It remains for us to prove that

$$\frac{\partial^{|k_1|+|k_2|}(z^*,v^*)}{\partial\,(z^0)^{k_1}\,\partial\,(v^1)^{k_2}} \leq \begin{cases} C(k)\,e^{|k_1|\alpha t} & \text{if} \quad |k_2| = 0 \\ & \text{and } |k_1|\alpha < \beta\,, \\ C(k)\,e^{\beta(t-\tau)+|k_1|\alpha\tau} & \text{if } |k_2| \neq 0 \\ & \text{or } |k_1|\alpha > \beta\,, \end{cases} \tag{5.2.31}$$

where $0 < \alpha < \beta$ and $1 \leq k \equiv |k_1| + |k_2| \leq r$. We will use induction on k, starting with $k = 1$. For the first derivatives, estimates (5.2.31) take the form

$$\begin{aligned} \frac{\partial(z^*,v^*)}{\partial z^0} &\leq C\,e^{\alpha t} \\ \frac{\partial(z^*,v^*)}{\partial v^1} &\leq C\,e^{-\beta(\tau-t)}\,. \end{aligned} \tag{5.2.32}$$

Since the first estimate is symmetric to the second with respect to the time reversion (5.2.29), it is sufficient to prove only the first inequality in (5.2.32).

As mentioned above, the derivative $(Z^*,V^*) \equiv \frac{\partial(z^*,v^*)}{\partial z^0}$ can be found as the unique solution of the boundary-value problem $Z(0) = I$, $V(\tau) = 0$ associated with the system of variational equations (5.2.20). The existence of this solution is guaranteed by Theorem 5.11 (see remarks after the theorem). Moreover, it follows that the solution is a fixed point of the integral operator

$$\begin{cases} \bar{Z}(t) = e^{At} + \int_0^t e^{A(t-s)} \left[f'_z(z^*(s), v^*(s), \mu, s) Z(s) + f'_v(z^*(s), v^*(s), \mu, s) V(s) \right] ds\,, \\ \bar{V}(t) = -\int_t^\tau e^{-B(s-t)} \left[g'_z(z^*(s), v^*(s), \mu, s) Z(s) + g'_v(z^*(s), v^*(s), \mu, s) V(s) \right] ds \end{cases} \tag{5.2.33}$$

which is obtained by a formal differentiation of integral operator (5.2.15) and which is, in fact, the integral operator of the type (5.2.15) written out for the system of variational equations (5.2.20). The fixed point is the limit of the iterations $(Z^{n+1}(t), V^{n+1}(t)) = (\overline{Z^n(t)}, \overline{V^n(t)})$ computed by formula (5.2.33), starting with an arbitrary initial point $(Z^{(0)}(t), V^{(0)}(t))$. Therefore, to derive the estimate given by the first inequality in (5.2.32), it is sufficient to prove that if this estimate holds for (Z, V), then $(\bar{Z}, \bar{V})$ in system (5.2.33) satisfies the same estimate with the same constant C (in this case, obviously, all iterations would satisfy the same estimate, as well as their limits).

Choose $\tilde{\alpha} < \alpha$ so that the spectrum of the matrix A still lies to the left of the line $\mathrm{Re}\,(\cdot) = \tilde{\alpha}$ (see (5.2.2)). We can modify (5.2.3) so that

$$\|e^{As}\| \leq e^{\tilde{\alpha}s}\,, \qquad \text{for } s \geq 0\,.$$

Since $\|(f, g)'_{z,v}\|$ is bounded by some small ξ (see (5.2.4)), it follows from (5.2.33) that

$$\|\bar{Z}(t)\| \leq e^{\alpha t} + \xi \int_0^t e^{\tilde{\alpha}(t-s)} \|(Z(s), V(s))\|\, ds\,,$$

$$\|\bar{V}(t)\| \leq \xi \int_t^\tau e^{-\beta(s-t)} \|(Z(s), V(s))\|\, ds\,.$$

Now, our desired result follows immediately: an integration of the inequalities above shows that if

$$\|(Z(t), V(t))\| \leq C e^{\alpha t}\,, \tag{5.2.34}$$

then

$$\|\bar{Z}(t)\| \le \left(1 + C\frac{\xi}{\alpha - \tilde{\alpha}}\right) e^{\alpha t}, \quad \|\bar{V}(t)\| \le C\frac{\xi}{\beta - \alpha} e^{\alpha t}.$$

Thus, if ξ is sufficiently small and C is sufficiently large, then $\|(\bar{Z}(t), \bar{V}(t))\|$ satisfies (5.2.34) with the same C.

We have proved the theorem for the case $k = 1$ and may proceed to derive the estimates for higher-order derivatives. Suppose the theorem holds for all derivatives of order less than or equal to some $q \ge 1$. Let us prove estimates (5.2.31) for the derivatives of order $k = |k_1| + |k_2| = q + 1$.

Denote

$$(Z_{k_1,k_2}, V_{k_1,k_2}) = \frac{\partial^k (z, v)}{\partial \left(z^0\right)^{k_1} \partial \left(v^1\right)^{k_2}}.$$

For $k \ge 2$, the derivatives $(Z^*_{k_1,k_2}, V^*_{k_1,k_2})$ of the solution (z^*, v^*) of the boundary-value problem (5.2.11) satisfy the equation

$$\begin{cases} Z^*_{k_1,k_2}(t) = \displaystyle\int_0^t e^{A(t-s)} \frac{\partial^k f(z^*, v^*, \mu, s)}{\partial \left(z^0\right)^{k_1} \partial \left(v^1\right)^{k_2}}\, ds, \\ V^*_{k_1,k_2}(t) = -\displaystyle\int_t^\tau e^{-B(s-t)} \frac{\partial^k g(z^*, v^*, \mu, s)}{\partial \left(z^0\right)^{k_1} \partial \left(v^1\right)^{k_2}}\, ds. \end{cases} \tag{5.2.35}$$

Recall the formula

$$\frac{\partial^{|p|} \phi(\psi(x))}{\partial x^p} = \sum_{i=1}^{|p|} \frac{\partial^i \phi}{\partial \psi^i} \sum_{\substack{j_1 + \cdots + j_i = p \\ |j_1| \ge 1, \ldots, |j_i| \ge 1}} C_{j_1,\ldots,j_i} \frac{\partial^{|j_1|}\psi}{\partial x_1^j} \cdot \cdots \cdot \frac{\partial^{|j_i|}\psi}{\partial x_i^j}$$

for the derivatives of the superposition of functions. Here ϕ and ψ are some vector functions, p and $j_1, \ldots, j_i$ are multi-indices, $\frac{\partial^i \phi}{\partial \psi^i}$ denotes the vector of all i-th order derivatives of ϕ with respect to ψ; the irrelevant constant factors $C_{j_1,\ldots,j_i}$ are independent of the specific functions ϕ and ψ. Applying this formula to (5.2.35) it follows that $(Z^*_{k_1,k_2}, V^*_{k_1,k_2})$ is the fixed point of the integral operator

$$\bar{Z}_{k_1,k_2}(t) = \int_0^t e^{A(t-s)} f'_{z,v}(z^*(s), v^*(s), \mu, s)(Z_{k_1,k_2}(s), V_{k_1,k_2}(s))\, ds$$

$$+ \sum_{i=2}^{k} \int_0^t e^{A(t-s)} \left.\frac{\partial^i f}{\partial(z,v)^i}\right|_{(z^*(s),v^*(s))}$$

$$\times \sum_j C_{j_1,\dots,j_i}(Z^*_{j_{11},j_{12}}(s), V^*_{j_{11},j_{12}}(s)) \cdots (Z^*_{j_{i1},j_{i2}}(s), V^*_{j_{i1},j_{i2}}(s))\, ds$$

$$\bar{V}_{k_1,k_2}(t) = -\int_t^\tau e^{-B(s-t)} g'_{z,v}(z^*(s), v^*(s), \mu, s)(Z_{k_1,k_2}(s), V_{k_1,k_2}(s))\, ds$$

$$- \sum_{i=2}^{k} \int_t^\tau e^{-B(s-t)} \left.\frac{\partial^i g}{\partial(z,v)^i}\right|_{(z^*(s),v^*(s))}$$

$$\times \sum_j C_{j_1,\dots,j_i}(Z^*_{j_{11},j_{12}}(s), V^*_{j_{11},j_{12}}(s)) \cdots (Z^*_{j_{i1},j_{i2}}(s), V^*_{j_{i1},j_{i2}}(s))\, ds \tag{5.2.36}$$

where the inner summation is taken over all multi-indices j such that $j_{11} + \dots + j_{i1} = k_1$, $j_{12} + \dots + j_{i2} = k_2$ and $|j_{p1}| + |j_{p2}| \geq 1$ for all $p = 1, \dots, i$.

To derive the estimates (5.2.31) for $(Z^*_{k_1,k_2}, V^*_{k_1,k_2})$ we follow the same procedure as in the case of the first derivatives. It is sufficient to check that if $(Z_{k_1,k_2}(s), V_{k_1,k_2}(s))$ satisfies these estimates, then $(\bar{Z}_{k_1,k_2}(t), \bar{V}_{k_1,k_2}(t))$ satisfies them as well, with the same constant $C(q+1)$.

Note that the second integrals in formula (5.2.36) include only the derivatives of orders less than or equal to $q = k-1$: since $|j_1| + \dots + |j_i| = k$, it follows that if $|j_p| = k$ for some $p = 1, \dots, i$, then all the other j's must be zero which is not the case (the summation is taken over non-zero multi-indices). Therefore, according to the induction hypothesis, estimates (5.2.31) hold for $(Z^*_{j_{p1},j_{p2}}(s), V^*_{j_{p1},j_{p2}}(s))$ in (5.2.36). In particular, if $|k_2| = 0$ (no differentiation with respect to v^1) and $|k_1|\alpha < \beta$, then $j_{p2} \equiv 0$ and $|j_{p1}|\alpha < \beta$ for all $p = 1, \dots, i$. Thus, in this case,

$$\|(Z^*_{j_{p1},j_{p2}}(s), V^*_{j_{p1},j_{p2}}(s))\| \leq C(q) e^{|j_{p1}|\alpha s} \tag{5.2.37}$$

and

$$\prod_{p=1}^{i} \|(Z^*_{j_{p1},j_{p2}}(s), V^*_{j_{p1},j_{p2}}(s))\| \le C(q)^i e^{(|j_{11}|+\cdots+|j_{i1}|)\alpha s} = C(q)^i e^{|k_1|\alpha s}\,. \tag{5.2.38}$$

If $|k_2| \ne 0$, then at least one of j_{p2} is non-zero and the corresponding term in the product can be estimated as follows:

$$\|(Z^*_{j_{p'1},j_{p'2}}(s), V^*_{j_{p'1},j_{p'2}}(s))\| \le C(q) e^{-\beta(\tau-s)+|j_{p'1}|\alpha\tau}\,. \tag{5.2.39}$$

All other terms may be estimated as follows:

$$\|(Z^*_{j_{p1},j_{p2}}(s), V^*_{j_{p1},j_{p2}}(s))\| \le C(q) e^{|j_{p1}|\alpha\tau} \tag{5.2.40}$$

(compare with (5.2.31): we used $\alpha > 0$ and $t \le \tau$, so $e^{|j_{p1}|\alpha t} \le e^{|j_{p1}|\alpha\tau}$; we also used $\beta > 0$ so $e^{-\beta(\tau-t)} \le 1$). It follows from these estimates that

$$\begin{aligned}\prod_{p=1}^{i} \|(Z^*_{j_{p1},j_{p2}}(s), V^*_{j_{p1},j_{p2}}(s))\| &\le C(q)^i e^{-\beta(\tau-s)} e^{(|j_{11}|+\cdots+|j_{i1}|)\alpha\tau} \\ &= C(q)^i e^{-\beta(\tau-s)+|k_1|\alpha\tau}\,. \end{aligned} \tag{5.2.41}$$

Finally, if $|k_2| = 0$ but $|k_1|\alpha > \beta$, then for some multi-indices j these products may be estimated by (5.2.38), and for the others by (5.2.41). Note that if $|k_1|\alpha > \beta$, then $e^{-\beta(\tau-s)+|k_1|\alpha\tau} > e^{|k_1|\alpha s}$ at $s \le \tau$; *i.e.* in this case the estimate (5.2.41) majorizes (5.2.38). Therefore, if $|k_2| = 0$ but $|k_1|\alpha > \beta$, then all products in the second integrals of (5.2.36) satisfy (5.2.41), as if $|k_2| \ne 0$. Recall that all derivatives of f and g are uniformly bounded. Thus, it follows from the above considerations that

$$\begin{aligned}\|\bar{Z}_{k_1,k_2}(t)\| \le &\int_0^t \xi e^{\alpha(t-s)} \|Z_{k_1,k_2}(s)\|\, ds \\ &+ \begin{cases} C^*(q) \displaystyle\int_0^t e^{\alpha(t-s)} e^{|k_1|\alpha s}\, ds & \text{if } |k_2| = 0 \text{ and } |k_1|\alpha < \beta\,, \\ C^*(q) \displaystyle\int_0^t e^{\alpha(t-s)} e^{-\beta(\tau-s)+|k_1|\alpha\tau}\, ds & \text{if } |k_2| \ne 0 \text{ or } |k_1|\alpha > \beta\,, \end{cases}\end{aligned}$$

$$
\|\bar{V}_{k_1,k_2}(t)\| \le \int_t^\tau \xi e^{-\tilde{\beta}(s-t)} \|V_{k_1,k_2}(s)\|\, ds
$$

$$
+ \begin{cases} C^*(q) \displaystyle\int_t^\tau e^{-\tilde{\beta}(s-t)} e^{|k_1|\alpha s}\, ds & \text{if } |k_2| = 0 \text{ and } |k_1|\alpha < \beta\,, \\ C^*(q) \displaystyle\int_t^\tau e^{-\tilde{\beta}(s-t)} e^{-\beta(\tau-s)+|k_1|\alpha\tau}\, ds & \text{if } |k_2| \neq 0 \text{ or } |k_1|\alpha > \beta\,, \end{cases} \tag{5.2.42}
$$

where $C^*(q)$ is some constant and $\tilde{\beta} > \beta$ is chosen close to β such that the spectrum of the matrix B still lies strictly to the right of the line $\mathrm{Re}\,(\cdot) = \tilde{\beta}$. This means that the following modification of the estimate (5.2.3) for the matrix exponent holds:

$$
\left\| e^{-Bs} \right\| \le e^{-\tilde{\beta}s} \quad \text{for } \ s \ge 0\,.
$$

According to (5.2.31), if $|k_2| = 0$ and $|k_1|\alpha < \beta$, we have $\|(Z_{k_1,k_2}(s), V_{k_1,k_2}(s))\| \le C(q+1)e^{|k_1|\alpha s}$. Substituting this into (5.2.42) gives

$$
\begin{aligned}
\|\bar{Z}_{k_1,k_2}(t)\| &\le e^{\alpha t}(\xi C(q+1) + C^*(q)) \int_0^t e^{(|k_1|-1)\alpha s}\, ds \\
&\le \frac{\xi C(q+1) + C^*(q)}{(|k_1|-1)\alpha} e^{|k_1|\alpha t}\,, \\
\|\bar{V}_{k_1,k_2}(t)\| &\le e^{\tilde{\beta}t}(\xi C(q+1) + C^*(q)) \int_t^\tau e^{-(\tilde{\beta}-|k_1|\alpha)s}\, ds \\
&\le \frac{\xi C(q+1) + C^*(q)}{\tilde{\beta} - |k_1|\alpha} e^{|k_1|\alpha t}\,,
\end{aligned} \tag{5.2.43}
$$

i.e. $\|(\bar{Z}_{k_1,k_2}(t), \bar{V}_{k_1,k_2}(t))\|$ also satisfies estimates (5.2.31) with the same constant $C(q+1)$ provided

$$
(\xi C(q+1) + C^*(q)) \max\left(\frac{1}{(|k_1|-1)\alpha}, \frac{1}{\tilde{\beta} - |k_1|\alpha} \right) \le C(q+1)\,.
$$

This finishes our proof for the particular case $|k_2| = 0$, $|k_1|\alpha < \beta$. Note that in deriving (5.2.42) we have applied the obvious inequality (here $a \le b$)

$$
\int_a^b e^{\delta s}\, ds \le \frac{1}{|\delta|} \begin{cases} e^{\delta b} & \text{if } \delta > 0\,, \\ e^{\delta a} & \text{if } \delta < 0 \end{cases} \tag{5.2.44}
$$

and the condition $\tilde{\beta} - |k_1|\alpha > 0$ was used in an essential way.

If $|k_2| \neq 0$ or $|k_1|\alpha > \beta$, we have

$$\|(Z_{k_1,k_2}(s), V_{k_1,k_2}(s))\| \leq C(q+1)e^{-\beta(\tau-s)}e^{|k_1|\alpha\tau} .$$

Substituting this into (5.2.42) gives

$$\begin{aligned}
\|\bar{Z}_{k_1,k_2}(t)\| &\leq e^{\alpha t}(\xi C(q+1) + C^*(q))e^{-\beta\tau}e^{|k_1|\alpha\tau} \int_0^t e^{(\beta-\alpha)s}\, ds \\
&\leq \frac{\xi C(q+1) + C^*(q)}{(\beta-\alpha)} e^{-\beta(\tau-t)}e^{|k_1|\alpha\tau} , \\
\|\bar{V}_{k_1,k_2}(t)\| &\leq e^{\tilde{\beta}t}(\xi C(q+1) + C^*(q))e^{-\beta\tau}e^{|k_1|\alpha\tau} \int_t^\tau e^{-(\tilde{\beta}-\beta)s}\, ds \\
&\leq \frac{\xi C(q+1) + C^*(q)}{\tilde{\beta}-\beta} e^{-\beta(\tau-t)}e^{|k_1|\alpha\tau}
\end{aligned} \tag{5.2.45}$$

(we used $\beta - \alpha > 0$ and $\tilde{\beta} - \beta > 0$). It follows that if

$$(\xi C(q+1) + C^*(q)) \max\left(\frac{1}{\beta-\alpha}, \frac{1}{\tilde{\beta}-\beta}\right) \leq C(q+1) ,$$

then $\|(\bar{Z}_{k_1,k_2}(t), \bar{V}_{k_1,k_2}(t))\|$ satisfies estimates (5.2.31) with the same constant $C(q+1)$.
This completes the proof of the theorem.

5.3. Theorem on invariant foliation

For our purposes, the most important property of the globally dichotomic systems introduced in the previous section is the existence of some invariant foliation. We will prove the existence of this foliation by considering the limit case of the above boundary-value problem which corresponds to $\tau = +\infty$ (we have already used such method in Sec. 2.8). Recall that we call a system of differential equations globally dichotomic if it has the form

$$\begin{aligned}
\dot{z} &= Az + f(z, v, \mu, t) , \\
\dot{v} &= Bv + g(z, v, \mu, t) ,
\end{aligned} \tag{5.3.1}$$

where $z \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, t is the time variable and μ is a vector of parameters. The functions f and g are $\mathbb{C}^r$-smooth ($r \geq 1$), and all their derivatives are assumed to be uniformly bounded; moreover, their first derivatives are supposed to be uniformly small:

$$\left\| \frac{\partial(f,g)}{\partial(z,v)} \right\| < \xi \tag{5.3.2}$$

for some sufficiently small constant ξ. Concerning the matrices A and B we assume that the following estimates hold for all $s \geq 0$:

$$\begin{aligned} \|e^{A\,s}\| &\leq e^{\alpha s}\,, \\ \|e^{-B\,s}\| &\leq e^{-\beta s}\,. \end{aligned} \tag{5.3.3}$$

Choose a real γ (below $\gamma \in (\alpha, \beta)$).

Definition 5.1. *Take any point (z_0, v_0). Let $(z_0(t), v_0(t))$ be the trajectory which starts with (z_0, v_0) at some $t = t_0$. We denote as $W^s_\gamma(z_0, v_0, t_0)$ the set of points (z_1, v_1) such that the trajectory $(z_1(t), v_1(t))$ of (z_1, v_1) starting with the same $t = t_0$ satisfies*

$$\|(z_1(t), v_1(t)) - (z_0(t), v_0(t))\| \leq Ce^{\gamma t} \tag{5.3.4}$$

for all $t \geq t_0$. We call $W^s_\gamma(z_0, v_0, t_0)$ the conventionally stable *or* γ-stable *set of (z_0, v_0) at $t = t_0$.*[5]

Theorem 5.13. *For any (z_0, v_0, t_0), for any $\gamma \in (\alpha, \beta)$, the conventionally stable set W^s_γ is a C^q-smooth manifold (where q is a maximal integer such that $q\alpha < \beta$ and $q \leq r$) of the type*

$$v = \varphi(z; z_0, v_0, t_0, \mu)\,,$$

where the function φ does not depend on γ; it is defined at all z and depends continuously on (z_0, v_0, μ, t_0).

[5] Here we are concerned about the starting moment $t = t_0$ because we consider a non-autonomous system, so different starting moments correspond to different trajectories. Of course, in the autonomous case where f and g do not depend on time the value of t_0 does not matter.

Proof. As in the previous section, a solution $(z(t), v(t))$ satisfies the following integral relation

$$
\begin{aligned}
z(t) &= e^{A(t-t_0)} z(t_0) + \int_{t_0}^{t} e^{A(t-s)} f(z(s), v(s), \mu, s) d\,, \\
v(t) &= e^{-B(\tau-t)} v(\tau) - \int_{t}^{\tau} e^{-B(s-t)} g(z(s), v(s), \mu, s) ds
\end{aligned}
\tag{5.3.5}
$$

for any τ. Thus, if a point (z_1, v_1) belongs to the γ-stable set of a point (z_0, v_0), then

$$
\begin{aligned}
z_1(t) - z_0(t) &= e^{A(t-t_0)}(z_1(t_0) - z_0(t_0)) \\
&\qquad + \int_{t_0}^{t} e^{A(t-s)} [f(z_1(s), v_1(s), \mu, s) - f(z_0(s), v_0(s), \mu, s)] ds\,, \\
v_1(t) - v_0(t) &= -\int_{t}^{+\infty} e^{-B(s-t)} [g(z_1(s), v_1(s), \mu, s) - g(z_0(s), v_0(s), \mu, s)] ds
\end{aligned}
\tag{5.3.6}
$$

(we took into account that $e^{-B(\tau-t)} e^{\gamma\tau} \to 0$ as $\tau \to +\infty$ for any fixed t, and that $v_1(\tau) - v_0(\tau) = O(e^{\gamma\tau})$ by the definition of the γ-stable set).

Denote $\zeta(t) = z_1(t) - z_0(t)$, $\eta(t) = v_1(t) - v_0(t)$. The solution of (5.3.6) is a fixed point of the integral operator

$$
\begin{aligned}
\bar{\zeta}(t) &= e^{A(t-t_0)} \zeta^0 + \int_{t_0}^{t} e^{A(t-s)} [f(z_0(s) + \zeta(s), v_0(s) + \eta(s), \mu, s) \\
&\qquad - f(z_0(s), v_0(s), \mu, s)] ds\,, \\
\bar{\eta}(t) &= -\int_{t}^{+\infty} e^{-B(s-t)} [g(z_0(s) + \zeta(s), v_0(s) + \eta(s), \mu, s) \\
&\qquad - g(z_0(s), v_0(s), \mu, s)] ds\,,
\end{aligned}
\tag{5.3.7}
$$

where $\zeta^0 = z_1(t_0) - z_0(t_0)$. It follows from (5.3.7) that

$$
\begin{aligned}
\|\bar{\zeta}(t)\| &\leq e^{\alpha(t-t_0)} \|\zeta^0\| + \int_{t_0}^{t} e^{\alpha(t-s)} \left\| \frac{\partial(f,g)}{\partial(z,v)} \right\| \cdot \|\zeta(s), \eta(s)\| ds\,, \\
\|\bar{\eta}(t)\| &\leq \int_{t}^{+\infty} e^{-\beta(s-t)} \left\| \frac{\partial(f,g)}{\partial(z,v)} \right\| \cdot \|\zeta(s), \eta(s)\| ds\,.
\end{aligned}
\tag{5.3.8}
$$

Based on this estimate one can immediately see, that for any $\gamma \in (\alpha, \beta)$ if a function $(\zeta(s), \eta(s))$ is *bounded in the γ-norm, i.e.* it satisfies

$$
\|\zeta(s), \eta(s)\| \leq C e^{\gamma s}
\tag{5.3.9}
$$

for all $s \geq t_0$, then the operator (5.3.7) maps such a function into a function $(\bar{\zeta}(t), \bar{\eta}(t))$ which satisfies the same condition. Moreover, exactly as in Theorem 5.11 (compare with (5.2.17)), one can prove that the operator under consideration is contracting in the γ-norm on the Banach space $H_{[t_0,+\infty)}$ of functions satisfying (5.3.9).

Thus, according to the Banach contraction mapping principle, for any given $z_1(t_0)$, the system (5.3.6) has a unique solution $(z_1(t), v_1(t))$ which satisfies (5.3.4). Due to uniqueness, this solution is independent of the choice of γ from the interval (α, β).

By Theorem 3.25 the solution depends continuously on (z_0, v_0, t_0, μ) and on initial $z = z_1(t_0) = z_0(t_0) + \zeta^0$. In particular, we have that $v = v_1(t_0)$ is a continuous function of $z = z_1(t_0)$. Thus, we have proved that the conventionally stable manifold of any point z_0 is a graph of some continuous function $v = \varphi(z)$.

Let us now prove the $\mathbb{C}^q$-smoothness of the conventionally stable manifold. It is equivalent to the $\mathbb{C}^q$-smoothness of the solution $(z_1(t), v_1(t))$ of (5.3.6) with respect to the initial condition $z_1(t_0)$. By the formal differentiation of (5.3.6), we have that the first derivative

$$(Z^*(t), V^*(t)) \equiv \left(\frac{\partial z_1(t)}{\partial z_1(t_0)}, \frac{\partial v_1(t)}{\partial z_1(t_0)} \right) ,$$

when it exists, satisfies the equation

$$\begin{aligned} Z^*(t) &= e^{A(t-t_0)} + \int_{t_0}^{t} e^{A(t-s)} f'_{z,v}(z_1(s), v_1(s), \mu, s)(Z^*(s), V^*(s)) ds \,, \\ V^*(t) &= -\int_{t}^{+\infty} e^{-B(s-t)} g'_{z,v}(z_1(s), v_1(s), \mu, s)(Z^*(s), V^*(s)) ds \,. \end{aligned} \tag{5.3.10}$$

The further derivatives

$$(Z_k^*(t), V_k^*(t)) \equiv \left(\frac{\partial^k z_1(t)}{\partial z_1(t_0)^k}, \frac{\partial^k v_1(t)}{\partial z_1(t_0)^k} \right)$$

must satisfy

$$\begin{aligned} Z_k^*(t) &= \int_{t_0}^{t} e^{A(t-s)} f'_{z,v}(z_1(s), v_1(s), \mu, s)(Z_k^*(s), V_k^*(s)) ds + P_k(t) \,, \\ V_k^*(t) &= -\int_{t}^{+\infty} e^{-B(s-t)} g'_{z,v}(z_1(s), v_1(s), \mu, s)(Z_k^*(s), V_k^*(s)) ds - Q_k(t) \end{aligned} \tag{5.3.11}$$

where

$$\begin{aligned} P_k(t) &= \int_{t_0}^{t} e^{A(t-s)} \sum_{i=2}^{k} \frac{\partial^i f}{\partial(z,v)^i}\Big|_{(z_1(s),v_1(s))} \\ &\quad \times \sum_{j_1+\cdots+j_i=k} C_{j_1,\ldots,j_i}(Z^*_{j_1}(s), V^*_{j_1}(s))\cdots(Z^*_{j_i}(s), V^*_{j_i}(s))ds \\ Q_k(t) &= \int_{t}^{+\infty} e^{-B(s-t)} \sum_{i=2}^{k} \frac{\partial^i g}{\partial(z,v)^i}\Big|_{(z_1(s),v_1(s))} \\ &\quad \times \sum_{j_1+\cdots+j_i=k} C_{j_1,\ldots,j_i}(Z^*_{j_1}(s), V^*_{j_1}(s))\cdots(Z^*_{j_i}(s), V^*_{j_i}(s))ds. \end{aligned} \tag{5.3.12}$$

Thus, when (Z^*_j, V^*_j) are known for $j < k$, the k-th derivative (Z^*_k, V^*_k) is the fixed point of the operator

$$\begin{aligned} \bar{Z}(t) &= \int_{t_0}^{t} e^{A(t-s)} f'_{z,v}(z_1(s), v_1(s), \mu, s)(Z(s), V(s))ds + P_k(t) \\ \bar{V}(t) &= -\int_{t}^{+\infty} e^{-B(s-t)} g'_{z,v}(z_1(s), v_1(s), \mu, s)(Z(s), V(s))ds - Q_k(t)\,. \end{aligned} \tag{5.3.13}$$

These equations are similar to Eqs. (5.2.35), (5.2.36) for the derivatives of the solution of the boundary-value problem (5.2.11). Absolutely in the same way as it was done there (Theorem 5.12), one can show that when $(Z^*_j(s), V^*_j(s))$ in (5.3.12) satisfy at $j < k$

$$\|Z^*_j(s), V^*_j(s)\| \le Ce^{(\max(\alpha, j\alpha)+\varepsilon)s} \tag{5.3.14}$$

for a small ε, then at $k\alpha < \beta$, the integral which defines $Q_k(t)$ is convergent and

$$\|P_k(t), Q_k(t)\| \le \text{const}\, e^{k\alpha t}\,.$$

Moreover, the operator (5.3.13) maps the space of functions $(Z(t), V(t))$ bounded in the γ-norm into itself, provided $\gamma \in (\max(\alpha, k\alpha), \beta)$, and it is contracting in that norm.

Thus, once $(Z^*_j(s), V^*_j(s))$ satisfying (5.3.14) are known at $j < k$, the formal solution $(Z^*_k(s), V^*_k(s))$ of (5.3.11) exists and satisfies (5.3.14) with $j = k$. Therefore, by induction we get the existence of bounded in the γ-norm ($\gamma \in (\max(\alpha, k\alpha), \beta)$) formal derivatives $(Z^*_k(s), V^*_k(s))$ up to the order q.

To prove that the formal derivatives are the derivatives indeed, we will show that the solution $(z_1(t), v_1(t))$ of system (5.3.6) is the limit, as $\tau \to +\infty$, of a solution $(z^*_\tau(t), v^*_\tau(t))$ of the boundary-value problem discussed in the previous section, with the boundary data $(z^0 = z_1(t_0) = z_0(t_0) + \zeta^0, v^1 = v_0(\tau))$, and that for each $k = 1, \ldots, q$ the k-th derivative $(Z^*_{k\tau}(t), V^*_{k\tau}(t))$ of $(z^*_\tau(t), v^*_\tau(t))$ with respect to z^0 converges to the solution $(Z^*_k(t), V^*_k(t))$ of (5.3.12). Precisely, we will prove that on any fixed finite interval of time

$$\sup \|(z^*_\tau(t), v^*_\tau(t)) - (z_1(t), v_1(t))\| \to 0 \ \text{ as } \ \tau \to +\infty \tag{5.3.15}$$

and

$$\sup \left\| \frac{\partial^k (z^*_\tau(t), v^*_\tau(t))}{\partial (z^0)^k} - (Z^*_k(t), V^*_k(t)) \right\| \to 0 \ \text{ as } \ \tau \to +\infty \qquad k = 1, \ldots, q \tag{5.3.16}$$

from which the $\mathbb{C}^q$-smoothness of $(z_1(t), v_1(t))$ with respect to z^0 follows immediately.

To prove (5.3.15) note that the operator given by (5.3.7) is the limit of the operator

$$\begin{aligned} \bar{\zeta}(t) &= e^{A(t-t_0)}\zeta^0 + \int_{t_0}^{t} e^{A(t-s)} \Big[f(z_0(s) + \zeta(s), v_0(s) + \eta(s), \mu, s) \\ &\qquad - f(z_0(s), v_0(s), \mu, s) \Big] ds\,, \\ \bar{\eta}(t) &= \begin{cases} -\displaystyle\int_t^{\tau} e^{-B(s-t)} \Big[g(z_0(s) + \zeta(s), v_0(s) + \eta(s), \mu, s) \\ \qquad - g(z_0(s), v_0(s), \mu, s) \Big] ds \quad \text{for } t \le \tau \\ 0 \quad \text{for } t \ge \tau \end{cases} \end{aligned} \tag{5.3.17}$$

More precisely, as $\tau \to +\infty$, the operator (5.3.17) defined on the space of functions (ζ, η) which are bounded in the γ-norm for some $\gamma \in (\alpha, \beta)$ has the operator (5.3.7) as a limit in the γ'-norm for any $\gamma' \in (\gamma, \beta)$. To prove this statement it is sufficient to check that

$$\begin{aligned} \sup_{t \ge t_0} \Bigg\| e^{-\gamma' t} \int_{\max(t,\tau)}^{+\infty} & e^{-B(s-t)} \Big[g(z_0(s) + \zeta(s), v_0(s) + \eta(s), \mu, s) \\ & - g(z_0(s), v_0(s), \mu, s) \Big] ds \Bigg\| \to 0 \quad \text{as } \tau \to +\infty \end{aligned}$$

provided $\|\zeta(s), \eta(s)\| \le Ce^{\gamma s}$. This integral can be estimated as follows:

$$\begin{aligned}
&\left\| e^{-\gamma' t} \int_{\max(t,\tau)}^{+\infty} e^{-B(s-t)} \Big[g(z_0(s)+\zeta(s), v_0(s)+\eta(s), \mu, s) \right. \\
&\qquad \left. - g(z_0(s), v_0(s), \mu, s) \Big] ds \right\| \\
&\qquad \le e^{-\gamma' t} \int_{\max(t,\tau)}^{+\infty} e^{-\beta(s-t)} \|g'_{(z,v)}\| \, \|\zeta(s), \eta(s)\| ds \\
&\qquad \le C\xi e^{-\gamma' t} \int_{\max(t,\tau)}^{+\infty} e^{-\beta(s-t)} e^{\gamma s} ds = \frac{C\xi}{\beta-\gamma} e^{(\beta-\gamma')t} e^{(\gamma-\beta)\max(t,\tau)} .
\end{aligned}$$

Thus, since $\gamma < \gamma' < \beta$,

$$\begin{aligned}
&\sup_{t\ge 0} \left\| e^{-\gamma' t} \int_{\max(t,\tau)}^{+\infty} e^{-B(s-t)} \Big[g(z_0(s)+\zeta(s), v_0(s)+\eta(s), \mu, s) \right. \\
&\qquad \left. - g(z_0(s), v_0(s), \mu, s) \Big] ds \right\| \le \frac{C\xi}{\beta-\gamma} e^{(\gamma-\gamma')\tau}
\end{aligned}$$

which proves our claim because $\gamma' > \gamma$.

Since the fixed point of a contracting operator depends continuously on parameters, it follows that as $\tau \to +\infty$, the fixed point $(\zeta^*_\tau, \eta^*_\tau)$ of (5.3.17) tends to the fixed point $(\zeta^*_\infty, \eta^*_\infty)$ of (5.3.7) in the γ'-norm. For finite τ, the fixed point $(\zeta^*_\tau, \eta^*_\tau)$ of (5.3.17) represents (on the interval $t \in [t_0, \tau]$) the solution $(z^*_\tau(t), v^*_\tau(t)) = (z_0(t)+\zeta^*_\tau(t), v_0(t)+\eta^*_\tau(t))$ of the boundary-value problem with the boundary data $z^0 = z_0(t_0) + \zeta^0$, $v^1 = v_0(\tau)$.

Thus, we have that $(z^*_\tau(t), v^*_\tau(t))$ converges to the solution $(z_1(t), v_1(t)) = (z_0(t) + \zeta^*_\infty(t), v_0(t) + \eta^*_\infty(t))$ of (5.3.6) in the γ'-norm, from which (5.3.15) obviously follows.

As we noted in Sec. 5.2, the formal differentiation with respect to the boundary data is the correct way to determine the derivatives of (z^*_τ, v^*_τ). Namely, the k-th derivative $(Z^*_{k\tau}(t), V^*_{k\tau}(t))$ is found as the fixed point of the operator

$$\begin{aligned}
\bar{Z}(t) &= \int_{t_0}^{t} e^{A(t-s)} f'_{z,v}(z^*_\tau(s), v^*_\tau(s), \mu, s)(Z(s), V(s)) ds + P_{k\tau}(t) \,, \\
\bar{V}(t) &= -\int_{t}^{\tau} e^{-B(s-t)} g'_{z,v}(z^*_\tau(s), v^*_\tau(s), \mu, s)(Z(s), V(s)) ds - Q_{k\tau}(t)
\end{aligned} \tag{5.3.18}$$

where

$$
\begin{aligned}
P_{k\tau}(t) = {} & \int_{t_0}^{t} e^{A(t-s)} \sum_{i=2}^{k} \left. \frac{\partial^i f}{\partial(z,v)^i} \right|_{(z^*_\tau(s), v^*_\tau(s))} \\
& \times \sum_{j_1+\cdots+j_i=k} C_{j_1,\ldots,j_i} (Z^*_{j_1\tau}(s), V^*_{j_1\tau}(s)) \cdots (Z^*_{j_i\tau}(s), V^*_{j_i\tau}(s)) ds \\
Q_{k\tau}(t) = {} & \int_{t}^{+\infty} e^{-B(s-t)} \sum_{i=2}^{k} \left. \frac{\partial^i g}{\partial(z,v)^i} \right|_{(z^*_\tau(s), v^*_\tau(s))} \\
& \times \sum_{j_1+\cdots+j_i=k} C_{j_1,\ldots,j_i} (Z^*_{j_1\tau}(s), V^*_{j_1\tau}(s)) \cdots (Z^*_{j_i\tau}(s), V^*_{j_i\tau}(s)) ds \,.
\end{aligned}
\tag{5.3.19}
$$

The operator (5.3.18) takes functions bounded in the γ-norm (with $\gamma \in (k\alpha, \beta)$) into functions bounded in the same norm and it is contracting in that norm, uniformly for all τ (see Theorem 5.12). Thus, the solution satisfies

$$
\|Z^*_{k\tau}(t), V^*_{k\tau}(t)\| \leq C e^{(\max(\alpha, k\alpha)+\varepsilon)t} \,. \tag{5.3.20}
$$

Now, take $k_0 \leq q$ and assume that (5.3.16) holds at all $k < k_0$. Let us extend the operator (5.3.18) onto the space of functions defined at all $t \geq t_0$, by assuming that the right hand side of the second equation in (5.3.18) vanishes identically at $t \geq \tau$. As above, one can see that on the space of functions bounded in the γ-norm (with $\gamma \in (k\alpha, \beta)$) the integral operator depends continuously on τ in the γ'-norm ($\gamma' \in (\gamma, \beta)$) and its limit as $\tau \to +\infty$ is given by the operator (5.3.11).

Indeed, by (5.3.15) and by assumed validity of (5.3.16) for all $j < k_0$, we have

$$
\|P_k(t) - P_{k\tau}(t)\|_{\gamma'} \leq \sup_{t \geq t_0} e^{-\gamma' t} \int_{t_0}^{t} e^{\alpha(t-s)} \varphi(s, \tau) ds
$$

where $\varphi(s, \tau) \to 0$ (as $\tau \to +\infty$) uniformly on any fixed bounded interval of s. Thus,

$$
\lim_{\tau \to +\infty} \|P_k(t) - P_{k\tau}(t)\|_{\gamma'} \leq \lim_{\tau \to +\infty} \sup_{t \geq t(\tau)} e^{-\gamma' t} \int_{t_0}^{t} e^{\alpha(t-s)} \varphi(s, \tau) ds \tag{5.3.21}
$$

for some $t(\tau)$ which tends to infinity as $\tau \to +\infty$. Note that φ is the norm of difference between the sums entering the integrands in (5.3.12) and (5.3.19). Therefore, by (5.3.14) and (5.3.20),

$$
\varphi \leq \text{const}\, e^{\gamma s} \,.
$$

Plugging this in (5.3.21) gives

$$\lim_{\tau\to+\infty} \|P_k(t) - P_{k\tau}(t)\|_{\gamma'} \le \text{const} \lim_{\tau\to+\infty} e^{(\gamma-\gamma')t(\tau)} = 0\,.$$

In the same way we get

$$\lim_{\tau\to+\infty} \|Q_k(t) - Q_{k\tau}(t)\|_{\gamma'} \le \text{const} \lim_{\tau\to+\infty} \sup_{t\ge t(\tau)} e^{-\gamma' t} \int_{\max(t,\tau)}^{+\infty} e^{-\beta(s-t)} e^{\gamma s} ds$$

whence

$$\lim_{\tau\to+\infty} \|Q_k(t) - Q_{k\tau}(t)\|_{\gamma'} = 0\,.$$

Absolutely analogously we prove the validity of limit transition to (5.3.11) for the first summands in (5.3.18).

As the fixed point of a contraction operator depends continuously on a parameter, the limit (in the γ'-norm and therefore in the usual norm on any finite interval of t) of the solution of (5.3.18) as $\tau \to +\infty$ is the solution $(Z^*_{k_0}, V^*_{k_0})$ of (5.3.11). Thus, the validity of (5.3.16) at all $k < k_0 \le q$ implies its validity at $k = k_0$. By induction we get that (5.3.16) holds at all $k \le q$ which gives the theorem.

Remark. The manifold W^s_γ is the same for all $\gamma \in (\alpha, \beta)$. Thus, trajectories of the points of the conventionally stable manifold of (z_0, v_0, t_0) satisfy (5.3.4) for any γ in this interval and, therefore, they satisfy

$$\|(z_1(t), v_1(t)) - (z_0(t), v_0(t))\| = o(e^{\gamma t}) \tag{5.3.22}$$

Note that the manifold $W^s_\gamma(z_0, v_0, t_0, \mu)$ is not, in general, invariant with respect to system (5.3.1), with the only exception when the system is autonomous and (z_0, v_0) is an equilibrium state. In this case W^s_γ is the set of points whose forward trajectories tend to the equilibrium in the γ-norm:

$$\|(z(t), v(t)) - (z_0, v_0)\| = o(e^{\gamma t})\,.$$

Hence, it is an invariant manifold by definition.

In the general case, the collection of all conventionally stable manifolds forms *an invariant foliation* of the extended phase space $\mathbb{R}^{n+m} \times \mathbb{R}^1$ (here $\mathbb{R}^1$ stands for the time axis). Indeed, if some point (z_1, v_1) belongs to a conventionally stable manifold of some other point (z_0, v_0), then $W^s_\gamma(z_1, v_1, t_0, \mu) = W^s_\gamma(z_0, v_0, t_0, \mu)$, by definition of W^s_γ. Therefore, if two conventionally stable manifolds intersect at some point, they must coincide. Thus, the collection of these manifolds is a continuous foliation indeed. To prove that this is an *invariant* foliation it is sufficient to note that $X_t W^s_\gamma(z_0, v_0, t_0, \mu) = W^s_\gamma(z_0(t), v_0(t), t_0 + t, \mu)$ (we denote as X_t the time t shift by the trajectories of the system). If the system is autonomous, then W^s_γ does not depend on the initial moment t_0, so we have an invariant foliation of the phase space. If the system is non-autonomous and depends periodically on time with some period T, then any surface $t = t_0 = \text{const}$ is a cross-section and the time T shift along the trajectories of the system is the Poincaré map $(z, v) \mapsto (z(t_0 + T), v(t_0 + T))$. Due to periodicity, $W^s_\gamma(z_0, v_0, t_0, \mu) = W^s_\gamma(z_0, v_0, t_0 + T, \mu)$. Thus, $X_T W^s_\gamma(z_0, v_0, t_0, \mu) = W^s_\gamma(z_0(t_0 + T), v_0(t_0 + T), t_0, \mu)$, which implies that on the cross-section the collection of conventionally stable manifolds is an invariant foliation for the Poincaré map.

Thus, Theorem 5.13 establishes the existence of a continuous invariant foliation with C^q-smooth leaves of the form $v = \varphi(z; z_0, v_0, t_0)$. Let us denote

$$\Phi(z_0, v_0, t_0, \mu) = \left.\frac{\partial \varphi}{\partial z}\right|_{z=z_0}.$$

The function Φ defines the field of tangents to the leaves of the invariant foliation: $\{(v - v_0) = \Phi(z_0, v_0, t_0, \mu)(z - z_0), t = t_0\}$. This field must be invariant with respect to the linearized system. The leaves of the invariant foliation are recovered by integrating the field of tangents; *i.e.* each leaf satisfies the equation (for each fixed t_0)

$$\frac{\partial v}{\partial z} = \Phi(z, v, t_0, \mu)\,. \tag{5.3.23}$$

Therefore, being a solution of the differential equation above, the function $v = \varphi(z; z_0, v_0, t_0, \mu)$ must have at least the same smoothness with respect to the initial conditions (z_0, v_0, t_0) and parameter μ, as the smoothness of Φ.

In general, the function Φ (and φ as well) is not smooth with respect to (z_0, v_0, t_0, μ). Let us study the question of smoothness of the foliation in more detail. Let $\tilde\beta \geq 0$ be a constant such that for the trajectory $(z(t), v(t))$, the derivatives with respect to the initial conditions $(z_0, v_0) = (z(t_0), v(t_0))$ and μ satisfy the following estimates

$$\left\| \frac{\partial^k (z(t), v(t))}{\partial (z_0, v_0, \mu)^k} \right\| \leq \text{const}\, e^{k\tilde\beta t}. \tag{5.3.24}$$

It can be proved that when the spectrum of the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

lies strictly to the left of the imaginary axis, then the constant ξ in (5.3.2) which bounds the derivatives of f and g may be taken to be so small that all the derivatives $\frac{\partial^k(z(t),v(t))}{\partial(z_0,v_0,\mu)^k}$ are bounded. Hence, $\tilde{\beta}=0$ in this case.

In general, $\tilde{\beta}$ is taken such that the spectrum of B (and A as well) lies strictly to the left of the line $\mathrm{Re}(\cdot)=\tilde{\beta}$. In this case, estimates (5.3.24) hold provided ξ is taken sufficiently small. Note that for a fixed $\tilde{\beta}$ an increase in the order of the derivative estimated by (5.3.24) requires a decrease in the value of the constant ξ.

It follows from the proof of Theorem 5.13 that the function Φ which defines the tangents to the leaves of the invariant foliation is equal to $V^*(t_0)$, where V^* is the solution of the system of integral equations (5.3.10) where $(z_1(s),v_1(s))$ is now equal to $(z_0(s),v_0(s))$ (the trajectory of the point (z_0,v_0)). Since the functions $z(t)$ and $v(t)$ involved in (5.3.10) depend on the initial conditions $(z(t_0),v(t_0))=(z_0,v_0)$, the solution V^* is also a function of (z_0,v_0). As in the proof of Theorem 5.13, one can verify that the derivatives of V^* with respect to (z_0,v_0,μ) can be found as solutions of the corresponding integral equations obtained by a formal differentiation of (5.3.10). Namely, the k-th derivative

$$(Z_k^{*0},V_k^{*0})\equiv\frac{\partial^k}{\partial(z_0,v_0,\mu)^k}(Z^*,V^*)$$

is the fixed point of the operator

$$\begin{aligned}
\bar{Z}(t)&=\int_{t_0}^{t}e^{A(t-s)}f'_{z,v}(z(s),v(s),\mu,s)(Z(s),V(s))ds\\
&\quad+\sum_{i=1}^{k}\int_{t_0}^{t}e^{A(t-s)}\left(\frac{\partial^i}{\partial(z_0,v_0,\mu)^i}f'_{z,v}(z(s),v(s),\mu,s)\right)\\
&\quad\cdot(Z_{k-i}^{*0}(s),V_{k-i}^{*0}(s))ds\,,\\
\bar{V}(t)&=-\int_{t}^{+\infty}e^{-B(s-t)}g'_{z,v}(z(s),v(s),\mu,s)(Z(s),V(s))ds\\
&\quad+\sum_{i=1}^{k}\int_{t}^{+\infty}e^{-B(s-t)}\left(\frac{\partial^i}{\partial(z_0,v_0,\mu)^i}g'_{z,v}(z(s),v(s),\mu,s)\right)\\
&\quad\cdot(Z_{k-i}^{*0}(s),V_{k-i}^{*0}(s))ds\,.
\end{aligned}\tag{5.3.25}$$

To assure that the fixed point of this operator does give the k-th derivative of (Z^*,V^*) we may consider the family of operators, depending on τ, where the infinite upper limit of the integral in the second equation is replaced by τ, and then take the limit as $\tau \to +\infty$.[6]

By (5.3.24), the derivatives

$$\frac{\partial^i}{\partial(z_0, v_0, \mu)^i}(f,g)'_{z,v}(z(s), v(s), \mu, s)\,,$$

entering (5.3.25) are estimated from above as const $e^{i\tilde{\beta}s}$. Based on this estimate, one can see, as in the proof of Theorem 5.12, that the integrals in (5.3.25) converge, provided

$$\alpha + k\tilde{\beta} < \beta\,. \tag{5.3.26}$$

Moreover, for any τ, the operators in the family under consideration take the space of the functions bounded in the γ-norm (with $\gamma \in (\alpha + k\tilde{\beta}, \beta)$) into itself, and are contracting on this space uniformly with respect to τ.

Thus, we obtain that the function Φ is $\mathbb{C}^k$ where k is the maximal possible integer such that (5.3.26) holds. Of course, the order of differentiation may not be higher than $(r-1)$ because the right-hand side of (5.3.10) contains $\mathbb{C}^{r-1}$-smooth functions $(f,g)'_{z,v}$. We arrive at the following result.

Lemma 5.2. *If for some $\tilde{\beta} \geq 0$ all the eigenvalues of the matrices B and A lie strictly to the left of the line* $\mathrm{Re}(\cdot) = \tilde{\beta}$, *then the foliation by the conventionally stable manifolds is C^k-smooth* (*provided the constant ξ in* (5.3.2) *is sufficiently small*), *where k is the maximal integer such that $k < (\beta - \alpha)/\tilde{\beta}$ and $k \leq r - 1$.*

In case $r = +\infty$ and $\tilde{\beta} = 0$ (*i.e.* the eigenvalues of A and B have strictly negative real parts) this lemma shows that the foliation is $\mathbb{C}^\infty$-smooth. On the contrary, if B has eigenvalues on the imaginary axis, then $\tilde{\beta}$ must be taken positive and for any fixed value of ξ we obtain only finitely smooth foliation.

Theorem 5.13 and Lemma 5.2 are the main technical results which we will use to prove the center manifold theorem, and the other theorems on local invariant manifolds throughout this book. Note that applying these results to the system obtained from (5.3.1) by a reversion of time gives us the existence of another invariant foliation by *conventionally unstable* manifolds of the form $z = \psi(v)$.

We emphasize that the study of the boundary-value problem of the kind introduced in the previous section is not solely for the purposes of establishing

[6]Observe that for finite τ the formal derivatives of the solution are indeed the derivatives (see the corresponding arguments in Sec. 2.8).

the invariant manifold theorems, but is also used in the analysis of non-local bifurcations. Bearing in mind a future application of this sort, we stress the following observation which was, in fact, already mentioned in the proof of Theorem 5.12.

Lemma 5.3. *Let* $(z^*(t; z^0, v^1, \tau, \mu), v^*(t; z^0, v^1, \tau, \mu))$ *be the solution of the boundary-value problem* $z^*(0) = z^0$, $v^*(\tau) = v^1$ *for system* (5.3.1), *and let* z^0 *and* v^1 *depend on* τ *so that* $(z^*(0), v^*(0))$ *have some finite limit* (z_0, v_0) *as* $\tau \to +\infty$. *Then, the derivative of* $v^*(0)$ *with respect to* z^0 *tends to a value of the function* Φ*, which defines the tangent to the conventionally stable manifold, at the point* (z_0, v_0).

To prove this lemma, observe that by hypothesis the solution $(z^*(t), v^*(t))$ of the boundary-value problem tends to the trajectory of (z_0, v_0) uniformly on any fixed finite interval of t. Therefore, the statement of the lemma is nothing more but a repetition of the claim in the proof of Theorem 5.13 that at $k = 1$ the fixed point of the integral operator (5.3.18) (finite τ) does have the solution of (5.3.10) ($\tau = +\infty$) as a limit. In the same way it follows from (5.3.16) that under the assumption of Lemma 5.3, *all the derivatives of* $v^*(0)$ *with respect to* z^0 *up to the order* q *have a finite limit as* $\tau \to +\infty$ (*where* q *is the maximal integer such that* $q\alpha < \beta$ *and* $q \leq r$).

5.4. Proof of theorems on center manifolds

In this section we complete our proof of the center manifold theorem. In fact, we prove a more general result which embraces all local invariant manifold theorems of this book.

Consider a *local* system of differential equations

$$\begin{aligned} \dot{z} &= Az + f(z, v, \mu) \\ \dot{v} &= Bv + g(z, v, \mu) \end{aligned} \tag{5.4.1}$$

defined in a small neighborhood of an equilibrium state $O(0,0)$. We assume that

$$f(0,0,0) = 0\,, \quad g(0,0,0) = 0\,, \quad (f,g)'_{z,v}(0,0,0) = 0\,.$$

We assume also that the matrices A and B satisfy inequality (5.3.3), *i.e.* the characteristic exponents corresponding to the eigenvalues of the matrix A must lie to the left of the line $\mathrm{Re}\,(\cdot) = \alpha$ and the other characteristic exponents must lie to the right of the line $\mathrm{Re}\,(\cdot) = \beta$ in the complex plane. As shown in Sec. 5.2, this system may be extended into the whole phase space such that the resulting system is globally dichotomic. Theorem 5.13 then implies the existence of an invariant manifold; namely, the γ-stable set of the point O. In fact, there is a variety of invariant manifolds, depending on how we sub-divide the phase variables into "z" and "v" parts: different choices of α and β would lead to different separations of the spectrum of characteristic exponents and, therefore, to different invariant manifolds.

Theorem 5.14. *Let an equilibrium state O of system* (5.4.1) *have n characteristic exponents to the left of the line* $\mathrm{Re}\,(\cdot) = \alpha$ *in the complex plane and let the other m characteristic exponents lie to the right of the line* $\mathrm{Re}\,(\cdot) = \beta$ *for some $\beta > \alpha$. If $\alpha < 0$, then at $\mu = 0$ the system has a uniquely defined strongly stable* (*non-leading*) *invariant $\mathbb{C}^r$-manifold W^{ss} which is tangent to $\{v = 0\}$ at O and which contains all trajectories that tend to O exponentially as $t \to +\infty$ at a rate faster than $e^{\gamma t}$ for any $0 > \gamma > \alpha$. If the equilibrium state does not disappear as μ varies and if it depends continuously on μ, then W^{ss} depends on μ continuously as well. Moreover, if the system is $\mathbb{C}^r$-smooth with respect to all variables including μ, the manifold W^{ss} is $\mathbb{C}^{r-1}$ with respect to μ* (*the tangents to W^{ss} are $\mathbb{C}^{r-1}$ with respect to all variables*).

Theorem 5.15. *Under the hypotheses of the previous theorem, if $\alpha > 0$, then for all small μ the system has an extended stable invariant $\mathbb{C}^q$-manifold W^{sE}* (*here q is the largest integer such that $q\alpha < \beta$ and $q \leq r$*) *which is tangent to $\{v = 0\}$ at O at $\mu = 0$ and which contains the set N^+ of all trajectories which stay in a small neighborhood of O for all positive times. Although W^{sE} is not unique, any two of them have the same tangent at each point of N^+. Moreover, when W^{sE} is written as by $v = \varphi^{sE}(z)$, all derivatives of the function φ^{sE} are uniquely defined at all points of N^+, up to order q. The manifold W^{sE} depends continuously on μ and if the system is $\mathbb{C}^r$-smooth with respect to all variables including μ, then the manifold W^{sE} is $\mathbb{C}^q$ with respect to μ.*

In the proof of these theorems, the local manifolds W^{ss} and W^{sE} appear as the intersection of the invariant manifold W^s_γ of system (5.3.1) (obtained by extending the local system (5.4.1) onto the whole phase space $\mathbb{R}^{m+n}$) with a small neighborhood of the equilibrium state O at the origin. Let us recall that W^s_γ is the γ-stable set of O and $\gamma \in (\alpha, \beta)$. In the case $\alpha < 0$ we can choose the value of γ to be negative and the uniqueness of W^{ss} then follows directly from the definition: W^{ss} is the set of all trajectories which tend to O faster than the decrease in the exponent $e^{\gamma t}$. If $\alpha > 0$, then $\gamma > 0$ and, therefore, the manifold W^s_γ becomes a set of trajectories of system (5.3.1) which diverge from the origin sufficiently slowly. Hence, which points in a small neighborhood of O are included in W^{sE} depends on how we extend the local system (5.4.1) onto the whole phase space. This implies that W^{sE} is not uniquely defined by the local system. Nevertheless, regardless of the method of extension of the local system, all points of the set N^+, which is composed of forward trajectories which never leave a small neighborhood of O, belong, by definition, to the γ-stable set of O for any $\gamma > 0$. Therefore, every manifold W^{sE} contains N^+. The uniqueness of the tangent to W^{sE} at any point of N^+ does not follow directly from Theorem 5.13 but this can nevertheless be extracted from its proof. Indeed, we have shown that

$$\left.\frac{\partial \varphi^{sE}}{\partial z}\right|_{z=z_0} = \left.V^*\right|_{t=0} ,$$

where V^* is found as a solution of the integral equation

$$\begin{cases} Z^*(t) = e^{At} + \displaystyle\int_0^t e^{A(t-s)} \Big(f'_z(z_0(s), v_0(s), \mu) Z^*(s) \\ \qquad\qquad + f'_v(z_0(s), v_0(s), \mu, s) V^*(s) \Big)\, ds \,, \\ V^*(t) = - \displaystyle\int_t^{+\infty} e^{-B(s-t)} (g'_z(z_0(s), v_0(s), \mu) Z^*(s) \\ \qquad\qquad + g'_v(z_0(s), v_0(s), \mu) V^*(s))\, ds \,. \end{cases} \tag{5.4.2}$$

Here, $(z_0(s), v_0(s))$ is the trajectory of the point $(z_0, v_0) = \varphi^{sE}(z_0)$. It follows from the proof of Theorem 5.13 that this solution is defined uniquely along with all derivatives with respect to z_0 up to order $(q-1)$. Consequently, since for $(z_0, v_0) \in N^+$ the trajectory of this point is defined by the local system only, it follows that the derivatives of φ^{sE} at all points of N^+ are uniquely defined.

Concerning the smoothness of W^{sE} with respect to the parameters μ, we note that in the case $\alpha > 0$ we can include μ amongst the variables z upon adding the equation $\dot{\mu} = 0$ to system (5.4.1). Therefore, in this case the smoothness with respect to μ is the same as with respect to z. If $\alpha < 0$, this no longer works, and the smoothness of the non-leading manifold with respect to the parameters does not follow from Theorem 5.13. We will study this question below in a more general framework on the smoothness of an associated invariant foliation.

Theorems 5.14 and 5.15 allow us to reconstruct the following hierarchy of local invariant manifolds. Let us choose a coordinate frame near an equilibrium state O such that the linear part of the system assumes the Jordan form. We have, in general,

$$\begin{aligned}\dot{y}_i &= A_i y_i + f_i(x,y,z,\mu)\\ \dot{z}_j &= C_j z_j + h_j(x,y,z,\mu)\\ \dot{x} &= Bx + g(x,y,z,\mu)\end{aligned}\tag{5.4.3}$$

where the spectrum of the matrix A_i lies on the straight line $\mathrm{Re}\,(\cdot) = \alpha_i$ in the complex plane, the spectrum of B lies on the imaginary axis,[7] and the spectrum of a matrix C_j lies on the straight line $\mathrm{Re}\,(\cdot) = \beta_j$ (here the indices i and j assume a finite range of values); the function f, g and h are nonlinearities. Let

$$\cdots < \alpha_2 < \alpha_1 < 0 < \beta_1 < \beta_2 < \cdots .$$

According to the theorems above, the following result holds.

Theorem 5.16. *There exists a sequence of conventionally stable smooth local invariant manifolds*

$$\cdots \subset W^s_{-2} \subset W^s_{-1} \subset W^s_0 \subset W^s_1 \subset \cdots$$

of the kind[8]

$$\begin{aligned}W^s_{-i} &: (x,z,y_1,\dots,y_{i-1}) = \varphi^{ss}_i(y_i,y_{i+1},\dots)\\ W^s_{-1} &: (x,z) = \varphi^{ss}_1(y)\\ W^s_0 &: z = \varphi^{sC}(x,y,\mu)\\ W^s_j &: (z_{j+1},\dots) = \varphi^{sE}_j(x,y,z_1,\dots,z_j,\mu)\,,\end{aligned}$$

where the functions φ vanish at zero along with the first derivatives.

[7] In the structurally stable case this part of the spectrum is missing.

[8] Here W^s_j are $\mathbb{C}^q$-smooth if $q\beta_j < \beta_{j+1}$ and $q \le r$, and W^s_{-i} are $\mathbb{C}^r$-smooth (including W_0).

Here, the manifolds with negative indices are given by Theorem 5.14 and the others are from Theorem 5.15. They are embedded into each other by construction: they are the local pieces of the corresponding conventionally stable manifolds of O for some globally defined system and the latter are embedded into each other by definition — the trajectories which converge to O in the γ-norm, converge to O in the γ'-norm as well, for any $\gamma' > \gamma$.

The manifold W_0^s is the center stable manifold of Sec. 5.1 and the manifold W_{-1}^s is the strongly stable manifold in this case. If the equilibrium is structurally stable, then there is no characteristic exponents on the imaginary axis and the manifold W_0^s is the stable manifold of O; it coincides with W_{-1}^s at $\mu = 0$ and the manifold W_{-2}^s is now the non-leading manifold of Sec. 2.6 — the other manifolds W_{-i}^s are, consequently the manifolds W^{sss}, W^{ssss}, etc., defined in that section.

For the case of structurally stable saddles, the manifold W_1^s gives the extended stable manifold defined in Sec. 2.7. In the case where all characteristic exponents of O have positive real parts and where O is completely unstable, the manifold W_1^s of O is the leading unstable manifold introduced in Sec. 2.6.

By applying Theorems 5.14 and 5.15 to the system which is derived from (5.4.1) by a reversion of time, we obtain the following sequence of conventionally unstable invariant manifolds

$$\cdots \subset W_{-2}^u \subset W_{-1}^u \subset W_0^u \subset W_1^u \subset \cdots$$

where

$$\begin{aligned}
W_{-i}^u &: (y_{i+1}, \dots) = \psi_i^{uE}(x, z, y_1, \dots, y_i, \mu) \\
W_0^u &: y = \psi^{uC}(x, z, \mu) \\
W_1^u &: (x, y) = \psi_1^{uu}(z) \\
W_j^u &: (x, y, z_1, \dots, z_{j-1}) = \psi_j^{uu}(z_j, z_{j+1}, \dots)\,,
\end{aligned}$$

where all functions ψ vanish at the origin along with their first derivatives. This sequence includes all other invariant manifolds discussed in Chap. 2 and in this chapter. In particular, $W_0^u \cap W_0^s$ is the center manifold in the structurally unstable case, and $W_{-1}^u \cap W_0^s$ is the saddle leading manifold (see Chap. 2) in the structurally stable case.

For the system on the invariant manifold W_0^s, the equilibrium state does not have positive characteristic exponents. Therefore, we can use Lemma 5.2

to assert the existence of the smooth invariant foliations on W_0^s (one should extend first the system to the whole phase space, establish the existence of the globally defined smooth invariant foliations and, then return to the local system). This results in the following theorem.

Theorem 5.17. *On W_0^s there is a family of strongly stable invariant $\mathbb{C}^{r-1}$-foliations F_{-i}^{ss} with $\mathbb{C}^r$ smooth leaves l_{-i}^{ss} of the kind*

$$(x, y_1, \dots, y_{i-1}) = \eta^{(i)}_{\xi^0,\mu}(y_i, y_{i+1}, \dots)\,,$$

where ξ^0 denotes the point $(x^0, y_1^0, \dots, y_{i-1}^0)$ of intersection of a corresponding leaf with the invariant manifold $W^u_{-(i-1)} \cap W_0^s$. For any point $M \in W_0^s$, the leaves passing through M are embedded upon each other:

$$\cdots \subset l_{-2}^{ss} \subset l_{-1}^{ss}\,.$$

If $M \in N^+$ (the forward orbit of M stays in a small neighborhood of O for all positive times), then all leaves passing through M are uniquely defined by the system.

The foliation F_{-1}^{ss} is exactly the strongly stable foliation of Sec. 5.1. It was argued there that the existence of this foliation implies the Reduction Theorem 5.5.

The leaf l_{-i}^{ss} which contains the equilibrium state O is the strongly stable or non-leading invariant manifold W_{-i}^s of Theorem 5.16. Since η is a $\mathbb{C}^{r-1}$-smooth function of ξ^0 and μ, the associated manifold has only $\mathbb{C}^{r-1}$-smoothness with respect to the parameter (when the point O does not disappear as μ varies, and when it depends smoothly on μ).

It follows from the remark to Lemma 5.2 that for $\mathbb{C}^\infty$-smooth systems the non-leading manifold is $\mathbb{C}^\infty$-smooth with respect to parameters, provided the equilibrium state is structurally stable (no characteristic exponents are on the imaginary axes). Otherwise the smoothness of W_{-i}^s with respect to μ is finite only.

In the same manner where Theorem 5.13 is used to establish the existence of different kinds of invariant manifolds near an equilibrium state, we can also use this theorem to study periodic trajectories. A system of differential equations near a periodic trajectory L of period τ may be written in the form (see Chap. 3)

$$\begin{aligned} \dot{z} &= Az + f(z, v, \mu, t)\,, \\ \dot{v} &= Bv + g(z, v, \mu, t)\,, \end{aligned} \tag{5.4.4}$$

where

$$f(0,0,0,t) \equiv 0\,, \qquad f'_{z,v}(0,0,0,t) \equiv 0\,,$$
$$g(0,0,0,t) \equiv 0\,, \qquad g'_{z,v}(0,0,0,t) \equiv 0\,.$$

The functions f and g are either τ-periodic, or τ-*antiperiodic*[9] functions of t. The eigenvalues of the matrices A and B are the ratios between the logarithms of the squares of the multipliers of L and 2τ. Condition (5.3.3) implies that the m multipliers of L must be less than $e^{\alpha\tau}$ in absolute value, and that the absolute values of the other n multipliers are greater than $e^{\beta\tau}$.

System (5.4.4) may be extended to all z and v outside of a small neighborhood of the periodic trajectory L: $(z=0, v=0)$. Applying Theorem 5.13 to the extended system, we have that for each point $M(0,0,t_0) \in L$ its γ-stable set is a smooth manifold $W^s_\gamma(0,0,t_0)$. Due to (anti)periodicity, if the trajectory of a point (z_0, v_0, t_0) tends to a trajectory of M in γ-norm, then the trajectory of the point $\sigma X_\tau(z_0, v_0, t_0)$ also tends to a trajectory of M and vice versa (in the purely periodic case we assume $\sigma = id$). This means that

$$\sigma \circ X_\tau(W^s_\gamma(0,0,t_0)) \equiv W^s_\gamma(0,0,t_0)\,,$$

i.e. the manifold $W^s_\gamma(0,0,t_0)$ is invariant with respect to the map $\sigma \circ X_\tau$. This map is nothing but the Poincaré map of the cross-section $t = t_0$ (see Chap. 3 for more details). Thus, we have established the existence of an invariant manifold for the fixed point $(0,0)$ for the Poincaré map of the extended system. The set of orbits starting from points on this manifold at the cross-section $\{t = t_0\}$ gives the corresponding invariant manifold for the system itself. Similarly, Lemma 5.2 can be used to assert the existence of certain smooth invariant foliations. Now we can return to the local system, in exactly the same way as we did in the case of equilibrium states.

Thus, we obtain a hierarchy of local invariant manifolds and foliations in a small neighborhood of the periodic trajectory. The corresponding theorems are the analogue of the above theorems which deal with equilibrium states.

Theorem 5.18. *Let a periodic trajectory L of a $\mathbb{C}^r$-smooth system have n multipliers strictly inside the circle $|(\cdot)| = e^{\alpha\tau}$ in the complex plane, and let the other m multipliers lie strictly outside the circle $|(\cdot)| = e^{\beta\tau}$ for some $\beta > \alpha$.*

[9]Recall that antiperiodicity means here that $X_t(\sigma(z_0, v_0); t_0) = \sigma X_t(z_0, v_0; t_0 + \tau)$ where X_t denotes the time-t shift and σ is some involution of the (z, v)-space: $\sigma \circ \sigma = id$.

If $\alpha < 0$, then at $\mu = 0$ the system has a uniquely defined $(n+1)$-dimensional strongly stable (non-leading) invariant $\mathbb{C}^r$-manifold W^{ss} which is tangent to the eigensubspace corresponding to the first n multipliers at each point of L and which contains all trajectories which, as $t \to +\infty$, tend to L exponentially at a rate faster than $e^{\gamma t}$ for any $0 > \gamma > \alpha$. If the periodic trajectory does not disappear as μ varies and if it depends continuously on μ, then W^{ss} depends on μ continuously as well. Moreover, if the system is $\mathbb{C}^r$-smooth with respect to all variables including μ, the manifold W^{ss} is $\mathbb{C}^{r-1}$ with respect to μ (the tangents to W^{ss} are $\mathbb{C}^{r-1}$ with respect to all variables).

Theorem 5.19. *Under the hypotheses of the previous theorem, if $\alpha > 0$, then for all small μ the system has an extended stable $(n+1)$-dimensional invariant $\mathbb{C}^q$-manifold W^{sE} (here q is the largest integer such that $q\alpha < \beta$ and $q \leq r$) which is tangent to the eigensubspace corresponding to the first n multipliers at each point of L at $\mu = 0$, and which contains the set N^+ of all trajectories which stay in a small neighborhood of L for all positive times. Though W^{sE} is not unique, any two of them have the same tangent at each point of N^+. Moreover, all derivatives up to order q are uniquely defined at all points of N^+. The manifold W^{sE} depends continuously on μ, and if the system is $\mathbb{C}^r$-smooth with respect to all variables including μ, then the manifold W^{sE} is $\mathbb{C}^q$ with respect to μ.*

By choosing different partitions of the spectrum of the multipliers of L, and by making a corresponding reformulation of the above theorems for systems obtained by a reversal of time, we can find all types of local invariant manifolds of periodic trajectories which were introduced in Chap. 3 (non-leading, leading, extended stable and unstable manifolds) and in Sec. 5.1 (strongly stable, strongly unstable, center stable and center unstable manifolds).

As above, we may recast the autonomous linear part of a system near a τ-periodic trajectory L into a Jordan form. So, we have

$$\begin{aligned} \dot{y}_i &= A_i y_i + f_i(x, y, z, \mu, t) \\ \dot{z}_j &= C_j z_j + h_j(x, y, z, \mu, t) \\ \dot{x} &= Bx + g(x, y, z, \mu, t) \end{aligned} \tag{5.4.5}$$

where the spectrum of the matrix A_i lies on the straight line $\mathrm{Re}\,(\cdot) = \alpha_i$, the spectrum of the matrix B lies on the imaginary axis and the spectrum of the matrix C_j lies on a straight line $\mathrm{Re}\,(\cdot) = \beta_j$ (here the indices i and j assume a finite range of values); the nonlinearities f, g and h are (anti)periodic in time. Let

$$\cdots < \alpha_2 < \alpha_1 < 0 < \beta_1 < \beta_2 < \cdots .$$

Theorem 5.20. *There exist sequences of the conventionally stable and conventionally unstable smooth local invariant manifolds*

$$\cdots \subset W^s_{-2} \subset W^s_{-1} \subset W^s_0 \subset W^s_1 \subset \cdots$$

and

$$\cdots \subset W^u_{-2} \subset W^u_{-1} \subset W^u_0 \subset W^u_1 \subset \cdots$$

of the kind[10]

$$\begin{aligned}
W^s_{-i} &: (x, z, y_1, \ldots, y_{i-1}) = \varphi^{ss}_i(y_i, y_{i+1}, \ldots; t)\\
W^s_{-1} &: (x, z) = \varphi^{ss}_1(y; t)\\
W^s_0 &: z = \varphi^{sC}(x, y, \mu; t)\\
W^s_j &: (z_{j+1}, \ldots) = \varphi^{sE}_j(x, y, z_1, \ldots, z_j, \mu; t)
\end{aligned}$$

and

$$\begin{aligned}
W^u_{-i} &: (y_{i+1}, \ldots) = \psi^{uE}_i(x, z, y_1, \ldots, y_i, \mu, t)\\
W^u_0 &: y = \psi^{uC}(x, z, \mu, t)\\
W^u_1 &: (x, y) = \psi^{uu}_1(z, t)\\
W^u_j &: (x, y, z_1, \ldots, z_{j-1}) = \psi^{uu}_j(z_j, z_{j+1}, \ldots; t)\,,
\end{aligned}$$

where the functions φ and ψ vanish at $(x, y, z, \mu) = 0$ along with their first derivatives.

On the invariant manifolds W^s_0 and W^u_0 there exist, respectively, a family of strongly stable and a family of strongly unstable invariant $\mathbb{C}^{r-1}$-smooth foliations F^{ss}_{-i} and F^{uu}_j with $\mathbb{C}^r$-smooth leaves l^{ss}_{-i} and, respectively, l^{uu}_j of the kind

$$(x, y_1, \ldots, y_{i-1}) = \eta^{s(i)}_{\xi^0,\mu}(y_i, y_{i+1}, \ldots; t_0)$$

and

$$(x, z_1, \ldots, z_{j-1}) = \eta^{u(j)}_{\xi^0,\mu}(z_j, z_{j+1}, \ldots; t_0)$$

[10] The dependence on t is τ-periodic.

where ξ^0 denotes the point of intersection $(x^0, y_1^0, \dots, y_{i-1}^0)$ or $(x^0, z_1^0, \dots, z_{j-1}^0)$ of a corresponding leaf with the invariant manifold $W^u_{-(i-1)} \cap W^s_0$ or, respectively, $W^s_{j-1} \cap W^u_0$; the value t_0 defines a hyperplane $\{t = t_0\}$ entirely containing the corresponding leaf. For any point $M \in W^s_0$, the leaves passing through M are embedded each into the other:

$$\cdots \subset l^{ss}_{-2} \subset l^{ss}_{-1}\,,$$

and, for any point $M \in W^u_0$,

$$\cdots \subset l^{uu}_{2} \subset l^{uu}_{1}\,.$$

If $M \in W^s_0$ and $M \in N^+$ (the forward orbit of M stays in a small neighborhood of L for all positive times), or if $M \in W^u_0$ and $M \in N^-$ (the backward orbit of M stays in a small neighborhood of L for all negative times), then all strongly stable or, respectively, strongly unstable leaves passing through M are uniquely defined by the system.

Finally, we note that these theorems are easily reformulated in terms of the Poincaré map: the intersection of the invariant manifolds with the cross-section $\{t = t_0\}$ gives the invariant manifolds for the fixed point at the origin.

Chapter 6

CENTER MANIFOLD. NON-LOCAL CASE

The local center manifold theorem is a well-known standard tool for the study of bifurcations in a small neighborhood of equilibrium states and periodic trajectories. However, as mentioned in the previous chapter, the local bifurcations do not exhaust all important bifurcations. It has been known since the work of Andronov and Leontovich [40] that among the four principal types of stability boundaries of a periodic trajectory of a two-dimensional system there are two which correspond to the disappearance of a periodic trajectory via a *homoclinic loop* — the union of an equilibrium state and a trajectory which tends to the equilibrium state both as $t \to +\infty$ and as $t \to -\infty$. Though they are at least equally important and no separation between different types of two-dimensional bifurcations was made in the classical work by Andronov and Leontovich, such objects are not considered in the theory of local bifurcations. A *global bifurcation theory* which deals with homoclinic loops, and more complicated *homoclinic and heteroclinic cycles*, as well as other non-local structures of multi-dimensional systems, had emerged after the works of Shilnikov [60–62] in the mid sixties. This theory proved to be a good source of different models of complex dynamical behaviors, as well as various scenarios of transitions between different types of non-local dynamics. In this book (in the second part) we will separate that part of the global theory which deals especially with dynamical systems with simple behavior (non-chaotic). In this part of the book we touch only the general question of the existence of an analogue of a center manifold in the non-local case.

We started the study of this particular problem at the beginning of the eighties. Since then, it has attracted the attention of many researchers. The existence of *the non-local center manifold* near a homoclinic loop has now been established by Turaev [73], Homburg [36] and Sandstede [56] (the latter also embraces infinite-dimensional cases). Results on the existence of such center manifolds for heteroclinic cycles have been derived by Shashkov [57] near certain heteroclinic cycles. Here we give a detailed proof only for the simplest case (when at least one leading exponent is real). We finish this chapter by discussing necessary and sufficient conditions for the existence of the non-local center manifold near arbitrarily complicated homoclinic and heteroclinic cycles obtained recently by Turaev [75].

It is important to note a number of significant differences between the local and the non-local center manifold theories. First, the dimension of the non-local center manifold has no relation to the level of degeneracy of the associated bifurcation problem. In the local theory the dimension of the center manifold is equal to the number of characteristic exponents on the imaginary axis, which implies that a high dimension of the center manifold corresponds to a large number of degeneracies in the linearized system. In contrast, even simple (codimension one) global bifurcation problems may not necessarily give rise to a low-dimensional center manifold. Another notable distinction of global bifurcations from local bifurcations is that in the non-local case the smoothness of the center manifold is not high. In fact, its smoothness does not correlate with the smoothness of the system and, in general, the non-local center manifold is only C^1.

Therefore, when studying specific non-local bifurcation problems, one possibly cannot apply the reduction to the center manifold directly: usually, subtle questions require calculations involving derivatives of order higher than the first order. Moreover, if the dimension of the center manifold is sufficiently high, its presence gives practically no useful information. On the other hand, if its dimension is low ($\dim W^C = 1, 2, 3, 4$), then the presence of a low-dimensional invariant manifold which captures all trajectories remaining in its neighborhood can tremendously simplify our understanding of the dynamics of the system, even if the center manifold is only C^1-smooth. In this case, one can, at least, consider a low-dimensional model having some assumed smoothness in order to make conjectures, which must be validated using the original non-reduced system.

6.1. Center manifold theorem for a homoclinic loop

Consider a family of dynamical systems

$$\dot{x} = F(x, \mu) \tag{6.1.1}$$

in $\mathbb{R}^{n+m}$, $n \geq 1$, $m \geq 1$. Assume that $F(x, \mu)$ is of class C^r ($r \geq 1$) with respect to the phase variables x and the parameter μ. Assume also that the following conditions are satisfied.

(A) *Let the system have a structurally stable equilibrium state O of the saddle type. Assume that the characteristic exponents $(\lambda_n, \ldots, \lambda_1, \gamma_1, \ldots, \gamma_m)$ of O are ordered so that*

$$\operatorname{Re}\lambda_n \leq \cdots \leq \operatorname{Re}\lambda_1 < 0 < \gamma_1 < \operatorname{Re}\gamma_2 \leq \cdots \leq \operatorname{Re}\gamma_m \,,$$

where γ_1 is assumed to be real.

In this case the dimension of the stable manifold W^s is equal to n and $\dim W^u = m$. Since the *leading* exponent γ_1 is real, there exists an $(m-1)$-dimensional non-leading (strongly) unstable sub-manifold $W^{uu} \subset W^u$. Recall that the main property of the non-leading unstable manifold asserts that as $t \to -\infty$ all trajectories lying in W^{uu} must tend to O tangentially to the eigen-space of the Jacobian matrix of the linearized system which corresponds to the non-leading eigenvalues $(\gamma_2, \ldots, \gamma_n)$, whereas the trajectories in $W^u \backslash W^{uu}$ must tend to 0 tangentially to the eigen-direction corresponding to the eigen-value γ_1.

We assume also that

(B) *at $\mu = 0$, the system possesses* a *homoclinic loop*, *i.e. there exists a trajectory Γ which tends to O as $t \to \pm\infty$ (by definition, $\Gamma \subseteq W^s \cap W^u$).*

and

(C) *the homoclinic trajectory Γ does not lie in the non-leading unstable submanifold W^{uu}.*

Assumption (**C**) implies that the trajectory Γ leaves the saddle point O along the eigen-direction corresponding to the leading eigenvalue γ_1, as shown in Fig. 6.1.1.

Conditions (**A**), (**B**) and (**C**) play different roles: condition (**A**) does not involve bifurcations: it merely selects the class of systems under consideration. If (**A**) is satisfied by the system itself, then it holds also for any nearby system (*i.e.* for any system whose right-hand side is close to F along with the first derivative). Moreover, once it is satisfied at $\mu = 0$, it remains fulfilled for all small μ as well.

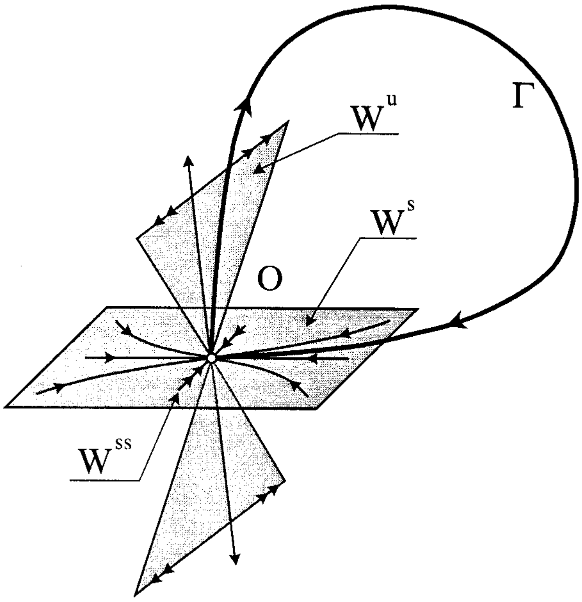


Fig. 6.1.1. Condition **(C)** implies that the trajectory Γ leaves the saddle point O along the eigen-direction corresponding to the leading eigenvalue γ_1.

As for condition (**B**), it cannot hold for all small μ; it can be shown that if a system has a homoclinic loop, then for some nearby system the loop may disappear (W^s and W^u would not have an intersection). Thus, condition (**B**) defines $\mu = 0$ as a bifurcational value for the parameter and specifies the associated bifurcation phenomenon (the bifurcation of the homoclinic loop). Generally, for any system whose right-hand side is close to $\mathcal{F}$ along with the first derivative, there would exist a value of μ near zero for which the perturbed system would also have a homoclinic loop.

Like condition (**A**), condition (**C**) does not imply any degeneracy. It just assumes that the one-parameter family under consideration is *in general position*: if it is not satisfied for a given family, it can always be achieved by a small perturbation of the right-hand side and once it is satisfied, it holds for any close family as well.

Let q be the largest integer such that $q\gamma_1 < \operatorname{Re}\gamma_2$. Recall (see Sec. 2.7) that under assumption (**A**) there exists an invariant $\mathbb{C}^{\min(q,r)}$-smooth extended stable manifold W^{sE} which is tangent at O to the eigenspace E^{sE} corresponding

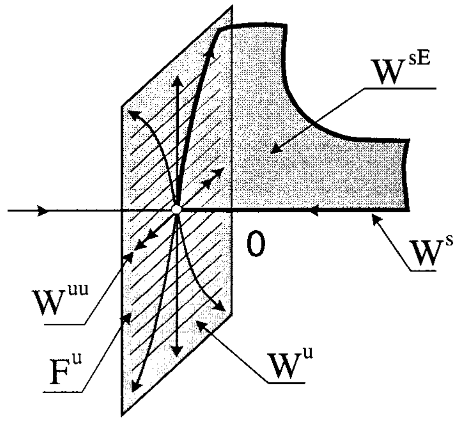


Fig. 6.1.2. The extended stable manifold W^{sE} which contains the stable manifold W^s and is tangent at the saddle point to the eigenspace corresponding to the characteristic exponents $\lambda_n, \dots, \lambda_1, \gamma_1$. The manifold W^{sE} it is not unique; any two of such manifolds have a common tangent on W^s. The strongly unstable sub-manifold W^{uu} is uniquely embedded into the smooth invariant foliation $\mathcal{F}^u$ on W^u.

to the eigenvalues $(\lambda_n, \dots, \lambda_1, \gamma_1)$. The manifold W^{sE} contains entirely the stable manifold W^s. Though it is not defined uniquely, any two such manifolds have the same tangent at any point of W^s. Another important object (see Sec. 5.4) is the smooth invariant foliation $\mathcal{F}^u$ on the unstable manifold W^u which includes the non-leading unstable manifold W^{uu} among its leaves, see Fig. 6.1.2.

The invariant extended stable manifold is defined locally, in a small neighborhood of O. However, if we take a point belonging to a piece of the trajectory Γ which belongs to W^s_{loc}, then a sufficiently small piece of W^{sE}_{loc} that contains this point may be continued by the backward trajectories of the system into a small neighborhood of any prescribed preceding point on Γ, see Fig. 6.1.3. In the same manner, the local strongly unstable foliation is extended by the forward trajectories of the system into the entire unstable manifold.

Since Γ lies simultaneously in W^u and in W^s, each point of Γ belongs to some piece of the extended stable manifold, and to some leaf of the strongly unstable foliation. Therefore, the following requirement makes sense.

(D) *The manifold W^{sE} is transverse to the leaves of the foliation $\mathcal{F}^u$ at each point of the homoclinic trajectory Γ.*

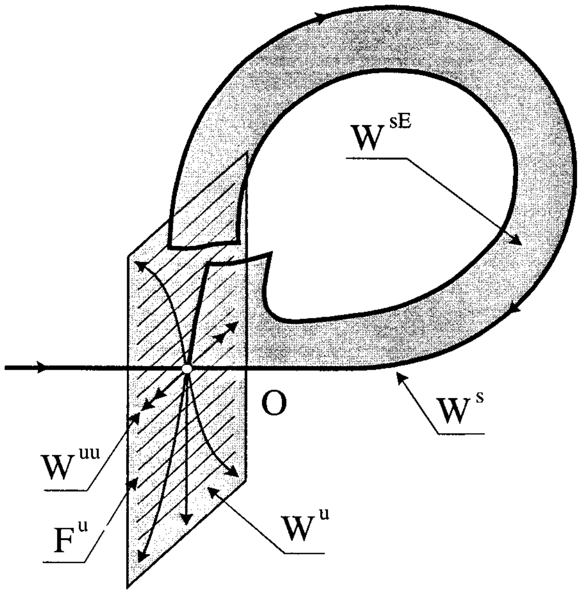


Fig. 6.1.3. Continuation of the extended stable manifold W^{sE} along the backward trajectories close to the homoclinic loop Γ.

Observe that condition (**D**) needs to be verified at only one point on the trajectory Γ because the manifold W^{sE} and the foliation $\mathcal{F}^u$ are invariant with respect to the flow defined by the system X_0. Note also that the manifold W^{sE} and the leaves of the foliation $\mathcal{F}^u$ have complementary dimensions. Therefore, our transversality condition (**D**) is well-posed. Like condition (**C**), it is a condition of general position.

Theorem 6.1. *If conditions* (**A**), (**B**), (**C**) *and* (**D**) *hold, then there exists a small neighborhood U of the homoclinic trajectory Γ such that for all sufficiently small μ the system X_μ possesses an $(n+1)$-dimensional invariant $\mathbb{C}^{\min(q,r)}$-smooth center stable manifold W^{sC} such that any trajectory which does not lie in W^{sC} leaves U as $t \to +\infty$. The manifold W^{sC} is tangent at O to the extended stable eigenspace $\mathcal{E}^{sE}$ (Fig.* 6.1.4).

The next two sections are devoted to the proof of this theorem. Note that due to the symmetry of the problem with respect to a reversion of time, it follows that there is a corresponding theorem *on the center unstable manifold*

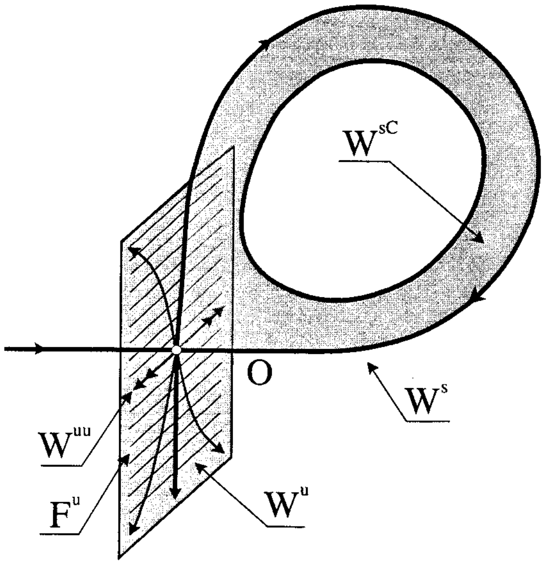


Fig. 6.1.4. The center stable manifold W^{sC}.

which may be formulated as follows. As above, suppose the system has a homoclinic loop Γ at $\mu = 0$. Let us modify conditions (**A**), (**C**) and (**D**) as follows.

(**A′**) *Let the characteristic exponents of the point O satisfy the following condition*:

$$\operatorname{Re}\lambda_n \leq \cdots \leq \operatorname{Re}\lambda_2 < \lambda_1 < 0 < \operatorname{Re}\gamma_1 \leq \cdots \leq \operatorname{Re}\gamma_m\,.$$

In this case, since the *leading stable* eigenvalue λ_1 is real, there exists an $(n-1)$-dimensional strongly stable sub-manifold $W^{ss} \subset W^s$.

(**C′**) *Assume that the homoclinic trajectory Γ does not lie in W^{ss}.*

(**D′**) *Assume that at each point of Γ the extended unstable manifold W^{uE} is transverse to the leaves of the strongly stable foliation $\mathcal{F}^s$, see Fig.* 6.1.5.

As in the above case we can continue the invariant extended unstable manifold along the forward trajectories, see Fig. 6.1.6.

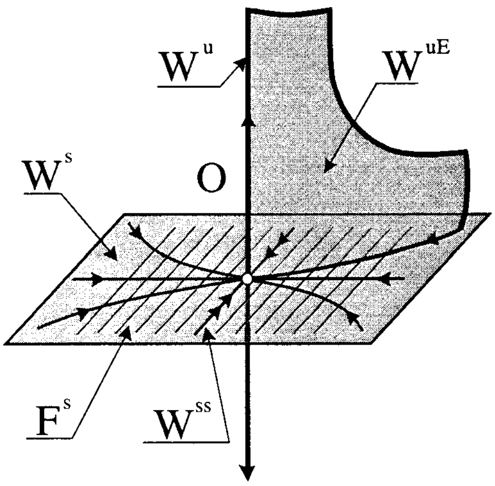


Fig. 6.1.5. The extended unstable manifold W^{uE} contains W^u and is tangent at the saddle point to the eigenspace corresponding to the characterstic exponents $\lambda_1, \gamma_1, \ldots, \gamma_m$. The manifold W^{uE} is not unique; any two of such manifolds touch each other everywhere on W^u. The strongly stable sub-manifold W^{ss} is uniquely embedded into the smooth invariant foliation F^s on W^s.

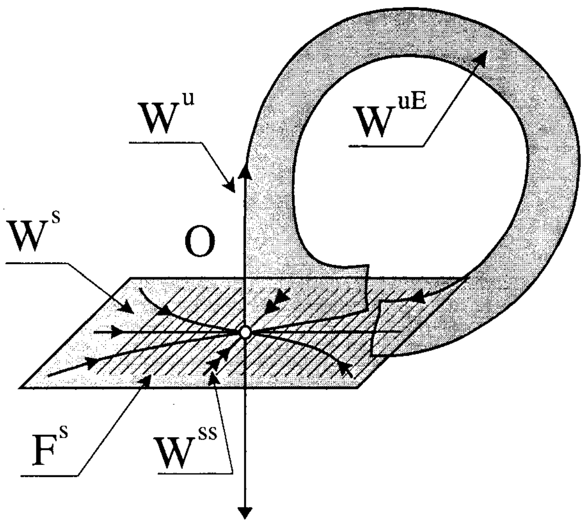


Fig. 6.1.6. Continuation of the extended unstable manifold W^{uE} along the forward trajectories close to Γ.

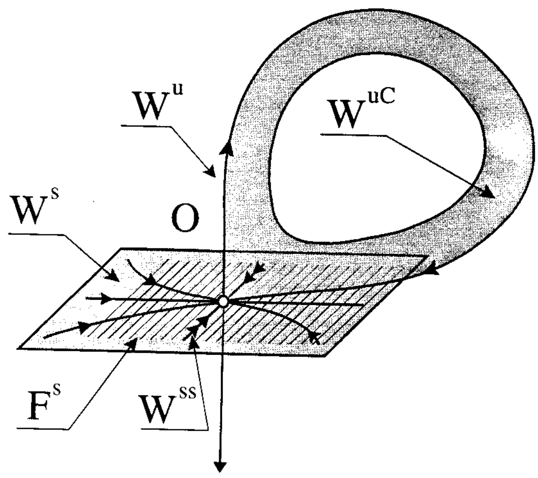


Fig. 6.1.7. The center unstable manifold W^{uC}. An inverse case to W^{sC}.

Theorem 6.2. *If the conditions* $(\mathbf{A}')$, $(\mathbf{B})$, $(\mathbf{C}')$ *and* $(\mathbf{D}')$ *hold, there exists a small neighborhood* U *of the homoclinic trajectory* Γ *such that for suffciently small* μ *the system has an* $(m+1)$*-dimensional invariant* $\mathbb{C}^{\min(p,r)}$*-smooth center unstable manifold* W^{uC} *such that any trajectory outside of* W^{uC} *leaves* U *as* $t \to -\infty$; *see Fig.* 6.1.7. (*here* p *is the largest integer such that* $p|\lambda_1| < |\mathrm{Re}\,\lambda_2|$). *The manifold* W^{uC} *is tangent at the point* O *to the eigenspace* E^{uE} *which corresponds to the characteristic exponents* $(\gamma_m, \ldots, \gamma_1, \lambda_1)$.

In the case where the conditions of both Theorems 6.1 and 6.2 hold, we have the following result

Theorem 6.3. *The intersection of* W^{uC} *and* W^{sC} *is a two-dimensional invariant* $C^{\min(p,q,r)}$*-smooth center manifold* W^c. *It contains all trajectories which stay entirely in a neighborhood* U *for all times. The manifold* W^c *is tangent at* O *to the eigenspace* E^L *corresponding to the leading characteristic exponents* (γ_1, λ_1).

This theorem reduces the problem of the bifurcations of a homoclinic loop to a saddle (1,1) to the study of a two-dimensional system on W^c (if the genericity conditions **(C)**, $(\mathbf{C}')$, **(D)**, $(\mathbf{D}')$ are satisfied). Note the importance of the condition that both leading exponents are real — generically, the dimension of the center manifold near a homoclinic loop is equal to the number of the leading characteristic exponents (both negative and positive) and when this dimension is greater than two, the bifurcations of such loop may be quite complicated in some cases.

In the case of a homoclinic loop to a saddle-(1,1), the two-dimensional dynamics is relatively simple. Nevertheless, the reduction to the center manifold requires some caution here. Our first observation is a low smoothness of W^c. Generally, it is only $\mathbb{C}^1$ and this may present an obstacle to a straightforward transcription of two-dimensional results into higher dimensions. Thus, the two-dimensional theory of bifurcations of a homoclinic loop developed by E. A. Leontovich produces a hierarchy of more and more degenerate cases (corresponding to an increasingly large number of limit cycles appearing at bifurcation). The study of these cases requires an increasingly higher smoothness of the system and, of course, the naive idea of simply repeating this hierarchy in the multidimensional situation, by referring to Theorem 6.3, would lead to erroneous results. Unlike the case of local bifurcations, Theorems 6.1–6.3 essentially contain results of a qualitative rather than analytic nature.

Our second observation is that the manifold W^c is not local (it is not homeomorphic to a disc). Since its tangent at O is the leading plane $\mathcal{E}^L$, it coincides locally with one of the saddle leading manifolds W^L_{loc}. At $\mu = 0$, this manifold must contain a piece Γ^+ of the homoclinic trajectory Γ which lies in W^s_{loc} and a piece Γ^- of Γ which lies in W^u_{loc}. From a small neighborhood of Γ^+ a small piece of W^L_{loc} can be continued by the forward trajectories along the loop Γ until it reaches Γ^-. The manifold obtained as a result of continuation must return to a neighborhood of O in such a way that it can be glued smoothly at this moment to the same local manifold W^L_{loc} — in order that a smooth invariant manifold W^C can be formed. If the orientation is preserved, the resulting glued manifold is a two-dimensional annulus. If not, the manifold W^c is a Möbius band. In fact, both cases are possible. Thus, in the multidimensional case, the bifurcation of a homoclinic loop to a saddle-(1,1) are reduced (generically) to a corresponding bifurcation either on the plane, or on a two-dimensional non-orientable manifold.

6.2. The Poincaré map near a homoclinic loop

In this and the next sections we present the proof of Theorem 6.1 which is based on a study of the Poincaré map T defined by the trajectories of the

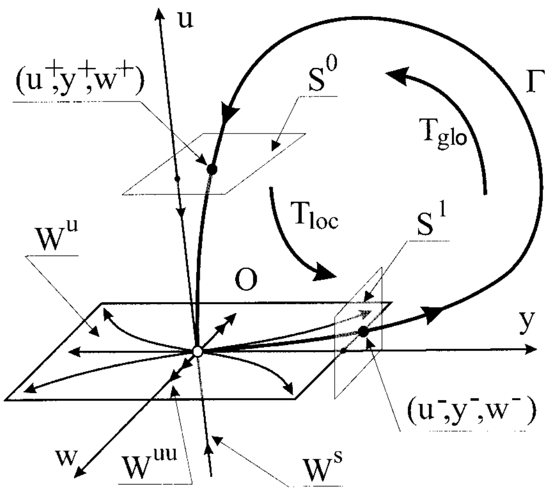


Fig. 6.2.1. The Poincaré map represented as a superposition of two maps: *the local map* T_{loc} defined along the trajectories from the cross-section S^{in} to S^{out} near the saddle point O, and *the global map* T_{glo} defined by the trajectories starting from S^{out} and ending on S^{in} along the global part of the homoclinic loop Γ.

system in a small neighborhood U of the homoclinic loop Γ. This map may be represented as a superposition of two maps: *a local map* T_{loc} defined near the saddle point O, and *a global map* T_{glo} defined by trajectories along the global part of the homoclinic trajectory Γ outside a small neighborhood of the saddle, see Fig. 6.2.1.

In a neighborhood of the saddle O let us introduce coordinates (u, y, w), $u \in \mathbb{R}^n$, $y \in \mathbb{R}^1$ and $w \in \mathbb{R}^{m-1}$ such that locally, the system assumes the form

$$\begin{aligned}\dot{u} &= Au + f(u, y, w, \mu)\,,\\ \dot{y} &= \gamma y + g(u, y, w, \mu)\,,\\ \dot{w} &= Bw + h(u, y, w, \mu)\,,\end{aligned} \tag{6.2.1}$$

where A is an $(n \times n)$-matrix and $\operatorname{spectr} A = \{\lambda_1, \ldots, \lambda_n\}$, B is an $(m-1) \times (m-1)$-matrix and $\operatorname{spectr} B = \{\gamma_2, \ldots, \gamma_m\}$, and $\gamma \equiv \gamma_1$. Let us choose some $\lambda > 0$ and $\eta > 0$ such that

$$\max\{\operatorname{Re}\lambda_1, \ldots, \operatorname{Re}\lambda_n\} < -\lambda \tag{6.2.2}$$

$$\min\{\operatorname{Re}\gamma_2, \ldots, \operatorname{Re}\gamma_m\} > \eta > \gamma\,. \tag{6.2.3}$$

The functions f, g, h are C^r-smooth and

$$(f,g,h)(0,0,0,0)=0\,, \qquad \left.\frac{\partial(f,g,h)}{\partial(u,y,w,\mu)}\right|_{(x,y,z,\mu)=0}=0\,. \tag{6.2.4}$$

In these coordinates, at $\mu = 0$, the stable manifold is tangent at O to the space $\{(y,w)=0\}$, the unstable manifold is tangent to $\{u=0\}$ and the strong unstable manifold is tangent to $\{(u,w)=0\}$.

At $\mu = 0$ the homoclinic trajectory Γ returns to a small neighborhood of O as $t \to +\infty$, lying in the local stable manifold. Therefore, for some small ξ, the trajectory intersects the surface $\|u\| = \xi$ at some point $M^+ \in W^s_{loc}$. Let us denote the coordinates of M^+ as (u^+, y^+, w^+) see Fig. 6.2.1. Choose some small $\delta > 0$ and consider a small area

$$S^{in} = \left\{\|u\| = \xi,\ \left\|\left(u-u^+, y-y^+, w-w^+\right)\right\| \le \delta\right\}. \tag{6.2.5}$$

It follows from Theorem 2.4 that $\frac{d}{dt}\|u\| < 0$ on W^s_{loc}; *i.e.* $\|u\|$ strictly decreases along the trajectories in W^s_{loc}. This implies that for a sufficiently small ξ the surface $\|u\| = \xi$ is transverse to the trajectories on W^s_{loc} and, therefore, to all close orbits. Consequently, being a part of this cross-section, the area S^{in} is transverse to the trajectories close to Γ provided that μ is sufficiently small.

Since the trajectory Γ does not lie in the non-leading unstable sub-manifold W^{uu} (condition (**C**) of Theorem 6.1), it leaves the saddle O along the leading direction which coincides with the y-axis. Without loss of generality, we can assume that Γ leaves O towards positive values of y. In this case, for sufficiently small $y^- > 0$, the homoclinic trajectory penetrates the surface $\{y = y^-\}$ at some point $M^- \in W^u_{loc}$. Denote $M^- = (u^-, y^-, w^-)$. Since at $\mu = 0$ the trajectory Γ is transverse to $\{y = y^-\}$, it follows that at all small μ the small area

$$S^{out} = \left\{y = y^-,\ \left\|\left(u-u^-, w-w^-\right)\right\| \le \delta\right\}, \tag{6.2.6}$$

is a cross-section (*i.e.* it intersects the trajectories of the system transversely). At $\mu = 0$, the trajectory of M^- (the trajectory Γ) reaches the point M^+ at some finite positive time. Therefore, due to the continuous dependence of the trajectories on initial conditions and parameters, for all small μ the trajectories which start on S^{out} near M^- must intersect S^{in} near M^+. Thus, we can define the map T_{glo} which maps a small neighborhood of M^- on S^{out} into a small neighborhood of M^+ on S^{in}.

All trajectories starting on S^{in} enter an ξ-neighborhood of the saddle point O. If a trajectory does not belong to the local stable manifold, it leaves a small neighborhood of O after some time. If a trajectory starting at some point $M^0 \in S^{in}$ leaves a small neighborhood of the saddle at a point M^1 which belongs to S^{out} we will say that M^0 and M^1 are related by the local map $T_{loc}\colon M^0 \mapsto M^1$.

Obviously, a trajectory which stays for all positive times in a small neighborhood of the homoclinic loop must intersect S^{in} and S^{out}: after leaving a neighborhood of the origin it must traverse along the global piece of Γ and then return into a neighborhood of the origin across S^{in}, and after entering this neighborhood at some point on S^{in}, it may leave only at some point on S^{out} (or it may stay in a small neighborhood of O for all times thereafter — it belongs to W^s_{loc} in this case). By definition, the consecutive points of intersection of a trajectory with the cross-sections are related by the map T_{loc}, or by T_{glo}. Thus, there is a correspondence between the trajectories under consideration and the iterations of the map $T = T_{glo} \circ T_{loc}$.

Because the flight time from S^{out} to S^{in} is bounded, the map T_{glo} is a C^r-diffeomorphism. Therefore, the necessary estimates on the map T_{glo} can be obtained simply by Taylor series expansion. We postpone the study of the global map to the end of this section and consider now the question on the structure of the local map which is much less trivial (because the time the trajectory spends in a small neighborhood of O before it reaches S^{out} may be unboundedly large, and tends to infinity as the starting point tends to W^s_{loc}). To overcome the difficulties we use the method of the boundary-value problem described in Sec. 2.8 and in Sec. 5.2.

Let us denote the coordinates on S^{in} as (u^0, y^0, w^0) ($\|u^0\| = \xi$) and the coordinates on S^{out} as (u^1, w^1). Let $\{y = \psi^s(u, \mu), w = \varphi^s(u, \mu)\}$ be the equation of W^s_{loc} and $\{u = \psi^u(y, w, \mu)\}$ be the equation of W^u_{loc}. Also let $\{w = \varphi^{sE}(u, y, \mu)\}$ be the equation of the local extended stable manifold W^{sE}_{loc}. Denote by l^{uu} that leaf of the extended unstable foliation which passes through the point M^-. Let $\{y = \psi^{uu}(w, \mu), u = \varphi^{uu}(w, \mu)\}$ be the equation of l^{uu}.

Lemma 6.1. *There exist functions u_{loc} and w_{loc} defined on $\|u^0 - u^+\| \leq \delta$, $\|w^1 - w^-\| \leq \delta$ and $0 < y^0 - \psi^s(u^0, \mu) \leq \delta'$ for some small δ', such that for two points $M^0 \in S^{in}$ and $M^1 \in S^{out}$, the relation $M^1 = T_{loc} M^0$ holds if and only if*

$$u^1 = u_{loc}(u^0, y^0, w^1, \mu)\,, \quad w_0 = w_{loc}(u^0, y^0, w^1, \mu)\,. \tag{6.2.7}$$

The functions u_{loc} and w_{loc} satisfy the following inequalities

$$\left\| \frac{\partial u_{loc}}{\partial(u^0, y^0)} \right\| \leq C\, e^{(\gamma-\lambda+\varepsilon)\tau}\,, \tag{6.2.8}$$

$$\left\| \frac{\partial u_{loc}}{\partial_\mu} \right\| \leq C\ \max\left\{1, e^{\gamma-\lambda+\varepsilon)\tau}\right\}\,, \tag{6.2.9}$$

$$\left\| \frac{\partial u_{loc}}{\partial \omega^1} \right\| \leq C\,, \tag{6.2.10}$$

$$\left\| \frac{\partial \omega_{loc}}{\partial(u^0, y^0, \mu)} \right\| \leq C\,, \tag{6.2.11}$$

$$\left\| \frac{\partial \omega_{loc}}{\partial \omega^1} \right\| \leq C\, e^{-(\eta-\gamma-\varepsilon)\tau}\,, \tag{6.2.12}$$

where C is some positive constant, λ, η and γ satisfy conditions (6.2.2) *and* (6.2.3)*, a small positive ε can be made arbitrarily small if δ is sufficiently small. Here $\tau(y^0, u^0, w^1, \mu)$ is the flight time from M^0 to M^1; it tends to infinity as $y^0 \to \psi^s(u^0, \mu)$ and*

$$\left\| \frac{\partial \gamma}{\partial(u^0, y^0, \mu)} \right\| \leq C\, e^{(\gamma+\varepsilon)\tau}\,, \qquad \left\| \frac{\partial \gamma}{\partial \omega^1} \right\| \leq C\,. \tag{6.2.13}$$

Furthermore,

$$\begin{aligned}
\lim_{y^0 \to \psi^s(u^0,\mu)} u_{loc} &= \psi^u(y^-, w^1, \mu)\,, \\
\lim_{y^0 \to \psi^s(u^0,\mu)} w_{loc} &= \psi^s(u^0, \mu)\,, \\
\lim_{y^0 \to \psi^s(u^0,\mu)} \frac{\partial u_{loc}}{\partial w^1} &= \frac{\partial \psi^u}{\partial w}(y^-, w^1, \mu)\,, \\
\lim_{y^0 \to \psi^s(u^0,\mu)} \frac{\partial u_{loc}}{\partial \mu} &= \frac{\partial \psi^u}{\partial \mu}(y^-, w^1, \mu) \quad \text{if} \quad \gamma < \lambda\,, \\
\lim_{y^0 \to \psi^s(u^0,\mu)} \frac{\partial w_{loc}}{\partial(u^0, y^0, \mu)} &= \frac{\partial \varphi^{sE}}{\partial(u, y, \mu)}(u^0, y^0, \mu)\,.
\end{aligned} \tag{6.2.14}$$

Proof. As show in Sec. 2.7, for any positive $\tau > 0$ and for any small (u^0, y^1, w^1) there is a unique trajectory $(u^*(t), y^*(t), w^*(t))$ of the system which lies in a small neighborhood of the origin and which represents a solution of the boundary-value problem:

$$u^*(0) = u_0\,, \quad y^*(\tau) = y^1\,, \quad w^*(\tau) = w^1\,.$$

Thus, the trajectory from a point M^0 reaches a point M^1 at the moment $t = \tau$ if and only if

$$\begin{aligned} u^1 &= u^*(\tau; u^0, y^1, w^1, \mu, \tau)\,, \\ y^0 &= y^*(0; u^0, y^1, w^1, \mu, \tau)\,, \\ w^0 &= w^*(0; u^0, y^1, w^1, \mu, \tau) \end{aligned} \tag{6.2.15}$$

(we took into account the fact that the solution (u^*, y^*, w^*) depends on the boundary data (u^0, y^1, w^1), on the flight time τ, and on μ; as shown in Sec. 2.8 the dependence is C^r-smooth with respect to all variables). The boundary value problem under consideration is a special case of the boundary-value problem considered in Sec. 5.2: one should consider u as the z-variable, and (y, w) as the v-variable in terms of that section. The estimates of Theorem (5.12) give in our case (one should assume $\alpha = \lambda$ and $\beta = \gamma - \varepsilon$ in (5.2.27))

$$\begin{aligned} \left\| \frac{\partial u^*}{\partial(u^0, \tau)} \right\| &\le C\, e^{-\lambda\tau}\,, \\ \left\| \frac{\partial u^*}{\partial(w^1, \mu)} \right\| &\le C \\ \left\| \frac{\partial(y^*, w^*)}{\partial(y^1, w^1, \tau)} \right\| &\le C\, e^{-(\gamma-\varepsilon)\tau}\,, \\ \left\| \frac{\partial(y^*, w^*)}{\partial(u^0, \mu)} \right\| &\le C \end{aligned} \tag{6.2.16}$$

(here, we calculate the derivatives of (y^*, w^*) at $t = 0$, by formula (5.2.27a), and the derivatives of u^* at $\tau - t = 0$, by formula (5.2.27b)).

As we argued in Sec. 2.8, the limit $\tau = +\infty$ corresponds to $M^0 \in W^s_{loc}$ and $M^1 \in W^u_{loc}$; *i.e.*

$$\begin{aligned} y^* \,|_{\tau=+\infty} &= \psi^s(u^0, \mu)\,, \\ w^* \,|_{\tau=+\infty} &= \varphi^s(u^0, \mu)\,, \\ u^* \,|_{\tau=+\infty} &= \psi^u(y^1, w^1, \mu)\,. \end{aligned} \tag{6.2.17}$$

Moreover,

$$\begin{aligned}
\left.\frac{\partial y^*}{\partial(u^0,\mu)}\right|_{\tau=+\infty} &= \frac{\partial \psi^s}{\partial(u^0,\mu)}(u^0,\mu)\,,\\
\left.\frac{\partial w^*}{\partial(u^0,\mu)}\right|_{\tau=+\infty} &= \frac{\partial \varphi^s}{\partial(u^0,\mu)}(u^0,\mu)\,,\\
\left.\frac{\partial u^*}{\partial(y^1,w^1,\mu)}\right|_{\tau=+\infty} &= \frac{\partial \psi^u}{\partial(y^1,w^1,\mu)}(y^1,w^1,\mu)\,.
\end{aligned} \tag{6.2.18}$$

At the same time one may write

$$\begin{aligned}
u^1 &= u^{**}(\tau;u^0,y^0,w^1,\mu,\tau)\,,\\
y^1 &= y^{**}(\tau;u^0,y^0,w^1,\mu,\tau)\,,\\
w^0 &= w^{**}(0;u^0,y^0,w^1,\mu,\tau)
\end{aligned} \tag{6.2.19}$$

where $(u^{**}(t),y^{**}(t),w^{**}(t))$ is the solution of the boundary-value problem

$$u^{**}(0)=u^0\,,\quad y^{**}(0)=y^0\,,\quad w^{**}(\tau)=w^1 \tag{6.2.20}$$

for a system obtained from (6.2.1) via a continuation from a small neighborhood of the origin onto the whole space R^{n+m} (note that once (6.2.15) is satisfied, the solution stays in a small neighborhood of the origin and when applying the results of Sec. 5.2 concerning the boundary-value problem (6.2.20) we should not worry about the influence of this continuation). The problem (6.2.20) is a particular case of the boundary-value problem considered in Secs. 5.2, 5.3: now one should denote the variables (u,y) as the z-variable and w as the v-variable and assume $\alpha=\gamma+\varepsilon$ and $\beta=\eta$. The estimates of Theorem 5.12 give for this case (see (5.2.26a) and (5.2.26b))

$$\begin{aligned}
\left\|\frac{\partial(y^{**},u^{**})}{\partial(u^0,y^0,\mu,\tau)}\right\| &\le C\,e^{(\gamma+\varepsilon)\tau}\,,\\
\left\|\frac{\partial(y^{**},u^{**})}{\partial w^1}\right\| &\le C\\
\left\|\frac{\partial w^{**}}{\partial(w^1,\tau)}\right\| &\le C\,e^{-\eta\tau}\,,\\
\left\|\frac{\partial w^{**}}{\partial(u^0,y^0,\mu)}\right\| &\le C
\end{aligned} \tag{6.2.21}$$

(we calculate here the derivatives of w^{**} at $t = 0$, from formula (5.2.26a), and the derivatives of (u^{**}, y^{**}) at $\tau - t = 0$, from formula (5.2.26b)).

The limit $\tau = +\infty$ was considered in Sec. 5.3. According to Lemma 5.3, the derivatives of w^{**} in the limit $\tau = +\infty$ coincide with the derivatives of the function whose graph is the conventionally stable manifold of the limiting point M^0. Since the point M^0 belongs to $W^s_{loc}(O)$ at $\tau = +\infty$, its conventionally stable manifold coincides with the conventionally stable manifold of O — in our case it is the extended stable manifold of W^{sE}. Thus,

$$\left.\frac{\partial w^{**}}{\partial(u^0, y^0, \mu)}\right|_{\tau=+\infty} = \frac{\partial \varphi^{sE}}{\partial(u^0, y^0, \mu)}(u^0, \psi^s(u^0, \mu)\,, \mu)\,. \tag{6.2.22}$$

Due to the general symmetry of the problem with respect to a reversion in time (see remarks in the proof of Theorem 5.12) the derivatives of u^{**} and y^{**} in the limit $\tau = +\infty$ coincide with the derivatives of the function whose graph is the conventionally unstable manifold of the limiting point M^1 — in our case it is the leaf l^{uu} of the strongly unstable foliation through M^1. Thus,

$$\left.\frac{\partial(y^{**}, u^{**})}{\partial w^1}\right|_{\tau=+\infty} = \frac{\partial(\psi^{uu}, \varphi^{uu})}{\partial w^1}(w^1, \mu)\,. \tag{6.2.23}$$

Fixing the value of $y^1 = y^-$ which corresponds to $M^1 \in S^{out}$ one may look at the second equation in (6.2.15) as an implicit equation for determining the flight time τ from M^0 to M^1. We will prove at once that the derivative $\frac{\partial y^*}{\partial \tau}$ does not vanish (and it is negative). Therefore, the equation

$$y^0 = y^*(0; u^0, y^-, w^1, \mu, \tau) \tag{6.2.24}$$

can be solved with respect to τ: the value $\tau = +\infty$ corresponds to $y^0 = \psi^s(u^0, \mu)$ and since $\frac{\partial y^*}{\partial \tau} < 0$, a decrease in τ to a finite value is followed by a monotonic increase in y^0. Thus, this equation uniquely defines the flight time as a function of (u^0, y^0, w^1, μ) at y^0 varying from $\psi^s(u^0, \mu)$ to $\psi^s(u^0, \mu) + \delta'$ for some sufficiently small δ'. Substituting the expression for τ into u^* and w^{**} would give the desired functions u_{loc} and w_{loc} (having fixed $y^1 = y^-$ and $\|u^0\| = \xi = \|u^+\|$).

By (6.2.24)

$$\begin{aligned} \frac{\partial \tau}{\partial y^0} &= \left(\frac{\partial y^*}{\partial \tau}\right)^{-1}, \\ \frac{\partial \tau}{\partial(u^0, w^1, \mu)} &= -\left(\frac{\partial y^*}{\partial \tau}\right)^{-1} \frac{\partial y^*}{\partial(u^0, w^1, \mu)}\,. \end{aligned} \tag{6.2.25}$$

To estimate these derivatives, let us compare the second equation of (6.2.15) with the second equation in (6.2.19). We immediately have

$$\begin{aligned} 1 &= \frac{\partial y^*}{\partial y^1}\frac{\partial y^{**}}{\partial y^0}\,, \\ 0 &= \frac{\partial y^*}{\partial(u^0,w^1,\mu)} + \frac{\partial y^*}{\partial y^1}\frac{\partial y^{**}}{\partial(u^0,w^1,\mu)}\,. \end{aligned} \tag{6.2.26}$$

Note also that according to Lemma 5.1,

$$\frac{\partial y^*}{\partial \tau} = -\frac{\partial y^*}{\partial y^1}\dot{y}\bigg|_{M^1} - \frac{\partial y^*}{\partial w^1}\,\dot{w}|_{M^1}\,.$$

It follows from this equation and from (6.2.25) and (6.2.26) that

$$\frac{\partial \tau}{\partial(u^0,y^0,w^1,\mu)} = -\frac{\partial y^{**}}{\partial(u^0,y^0,w^1,\mu)}\bigg/\left(\dot{y}|_{M^1} - \frac{\partial y^{**}}{\partial w^1}\dot{w}\bigg|_{M^1}\right). \tag{6.2.27}$$

Note that the denominator in this formula does not vanish (it is positive). Indeed, since the trajectory Γ does not belong to W^{uu}, it leaves the origin tangentially to the y-axis (see Theorem 2.5). Hence, the value of w^- is much less than y^-. In particular, this means that $\dot{y}|_{M^1} \gg \|\dot{w}\|_{M^1}$ and our claim follows from the boundedness of $\frac{\partial y^{**}}{\partial w^1}$ (note that $\dot{y}|_{M^1}$ is positive since it is equal essentially to γy^-, and y^- is positive).

Thus, the inverse $\frac{\partial \tau}{\partial y^0}$ to $\frac{\partial y^*}{\partial \tau}$ exists which proves that the flight time is indeed uniquely defined by (u^0, y^0, w^1, μ). The required negativeness of $\frac{\partial \tau}{\partial y^0}$ follows from formula (6.2.27): $\frac{\partial y^{**}}{\partial y^0}$ is equal, by definition, to 1 at $\tau = 0$ and since this derivative cannot vanish at any τ (by virtue of the first equation in (6.2.26)) it remains positive for all τ.

From the relations (6.2.27) and (6.2.21) we obtain the inequality (6.2.13). The functions u_{loc} and w_{loc} are defined as

$$\begin{aligned} u_{loc}(u^0,y^0,w^1,\mu) &\equiv u^*(\tau(u^0,y^0,w^1,\mu); u^0, y^-, w^1, \mu, \tau(u^0,y^0,w^1,\mu)) \\ w_{loc}(u^0,y^0,w^1,\mu) &\equiv w^{**}(\tau(u^0,y^0,w^1,\mu); u^0, y^0, w^1, \mu, \tau(u^0,y^0,w^1,\mu))\,. \end{aligned} \tag{6.2.28}$$

One may check now that the estimates (6.2.16), (6.2.21) and (6.2.13) imply (6.2.8)–(6.2.12) and the limit relations (6.2.18), (6.2.22) and (6.2.23) imply (6.2.14). This completes our proof of the lemma.

The higher derivatives of the functions u_{loc} and w_{loc}, can also be easily estimated using Theorem 5.12 and the identities (6.2.28) and (6.2.27) (in (6.2.27), the values of $\dot{y}$ and $\dot{w}$ at the point $M^1(u^1, y^-, w^1)$ are evaluated by formulas (6.2.1)). Omitting the obvious calculations the final result is as follow:

Lemma 6.2. *In Lemma* 6.1 *the following estimates hold*:

$$
\begin{aligned}
\left\| \frac{\partial^{|k_1|+|k_2|+|k_3|} u_{loc}}{\partial(u^0, y^0)^{k_1} \partial \mu^{k_2} \partial(w^1)^{k_3}} \right\| &\leq C\ e^{((|k_1|+|k_2|)(\gamma+\varepsilon)-\lambda)\tau} \qquad (k_1 \neq 0)\,, \\
\left\| \frac{\partial^{|k_2|+|k_3|} u_{loc}}{\partial \mu^{k_2} \partial(w^1)^{k_3}} \right\| &\leq C\ \max\left(1, e^{(|k_2|(\gamma+\varepsilon)-\lambda)\tau}\right), \\
\left\| \frac{\partial^{|k_1|+|k_2|} w_{loc}}{\partial(u^0, y^0, \mu)^{k_1} \partial(w^1)^{k_2}} \right\| &\leq \begin{cases} C & \text{if } k_2 = 0 \quad \text{and} \\ & |k_1|(\gamma+\varepsilon) < \eta\,, \\ C\ e^{-(\eta - |k_1|(\gamma+\varepsilon))\tau} & \text{if } k_2 \neq 0 \quad \text{or} \\ & |k_1|(\gamma+\varepsilon) > \eta\,. \end{cases}
\end{aligned}
\tag{6.2.29}
$$

We remark that as in Lemma 6.1, those derivatives which are bounded by a constant in these formulae have, in fact, a finite limit, equal to the derivatives of the corresponding conventionally stable manifolds, as $\tau \to +\infty$ (see remarks after Lemma 5.3).

The estimates in Lemmas 6.1 and 6.2 are more than enough for our purposes. In fact, what we need to prove the invariant manifold theorem is summarized by the following lemma.

Lemma 6.3. *Let us change the coordinates on the cross-sections S^{in} and S^{out} in the following way*:

$$
\begin{aligned}
y^0_{new} &= y^0 - \psi^s(u^0, \mu)\,, \\
w^0_{new} &= w^0 - \varphi^{sE}(u^0, y^0, \mu) \quad \text{on } S^{in} \\
u^1_{new} &= u^1 - \psi^u(y^1, w^1, \mu) \quad \text{on } S^{out}
\end{aligned}
$$

(*we straighten the intersections $W^s_{loc} \cap S^{in}$ and $W^u_{loc} \cap S^{out}$ and make the intersection $W^{sE}_{loc} \cap S^{in}$ tangent to $\{w^0 = 0\}$ at the point $M^+ = \Gamma \cap S^{in}$ at $\mu = 0$*). *Points $M^0 \in S^{in}$ and $M^1 \in S^{out}$ are related by the map T_{loc} if, and only if,*

$$
u^1 = u_{loc}(u^0, y^0, w^1, \mu)\,, \quad w_0 = w_{loc}(u^0, y^0, w^1, \mu) \tag{6.2.30}
$$

where the functions u_{loc} and w_{loc} are now defined at $y^0 \in [0, \delta']$; they satisfy the following inequalities in the new coordinates:

$$\begin{aligned}
\left\| \frac{\partial^{|k_1|+|k_2|} u_{loc}}{\partial(u^0, y^0, \mu)^{k_1} \partial(w^1)^{k_2}} \right\| &\leq C\, e^{|k_1|(\gamma+\varepsilon)\tau}\,, \quad (k_1 \neq 0)\,, \\
\left\| \frac{\partial^{|k_1|+|k_2|} w_{loc}}{\partial(u^0, y^0, \mu)^{k_1} \partial(w^1)^{k_2}} \right\| &\leq C\, e^{-(\eta-|k_1|(\gamma+\varepsilon))\tau}\,, \quad (k_2 \neq 0)\,.
\end{aligned} \tag{6.2.31}$$

All derivatives $\frac{\partial^{|k|} u_{loc}}{\partial(w^1)^k}$ and $\frac{\partial^{|k|} w_{loc}}{\partial(u^0, y^0, \mu)^k}$ *up to order* $\min(q, r)$ *are continuous and bounded where r is the smoothness of the system and q is the largest integer such that $q\gamma < \eta$. Moreover, in the new coordinates*

$$u_{loc}(u^0, 0, w^1, \mu) \equiv 0\,, \quad w_{loc}(u^0, 0, w^1, \mu) \equiv 0 \tag{6.2.32}$$

and

$$\frac{\partial^k w_{loc}}{\partial(u^0, y^0, \mu)^k}(u^0, 0, w^1, \mu) \equiv 0\,, \tag{6.2.33}$$

at $k \leq \min(q, r)$.

This lemma follows immediately from the two previous lemmas. Observe that it follows from (6.2.32) that

$$\frac{\partial u_{loc}}{\partial w^1} \equiv 0 \quad \text{at} \quad y^0 = 0\,. \tag{6.2.34}$$

Let us now consider the global map $T_{glo}\colon S^{out} \mapsto S^{in}$. Since the flight time from S^{in} to S^{out} is bounded (and it depends smoothly on the initial point) the map T_{glo} is a $\mathbb{C}^r$-diffeomorphism. Somehow, it is more convenient for us to consider the inverse map T_{glo}^{-1}. Since it is $\mathbb{C}^r$-smooth as well, one may write near the point M^+ the map $T_{glo}^{-1}\colon S^{in} \to S^{out}$ in the form

$$\begin{aligned}
\begin{pmatrix} u^1 - u^-(\mu) \\ w^1 - w^-(\mu) \end{pmatrix} &= \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{pmatrix} \begin{pmatrix} u^0 - u^+ \\ y^0 \\ w^0 \end{pmatrix} \\
&\quad + \begin{pmatrix} u_{glo}(u^0, y^0, w^0, \mu) \\ w_{glo}(u^0, y^0, w^0, \mu) \end{pmatrix}\,.
\end{aligned} \tag{6.2.35}$$

Here $(u^-(\mu), w^-(\mu))$ are the coordinates of the image $T_{glo}^{-1} M^+$ of the point M^+. At $\mu = 0$ it is the point $M^- = \Gamma \cap S^{out}$. Recall that in the coordinates of Lemma 6.3 $u^-(0) = 0$. The constants d_{11}, d_{12}, d_{13}, d_{21}, d_{22} and d_{23} are matrices of dimensions $n\times(n-1)$, $n\times 1$, $n\times(m-1)$, $(m-1)\times(n-1)$, $(m-1)\times 1$ and $(m-1)\times(m-1)$, respectively; the functions u_{glo} and w_{glo} denote the nonlinear terms.

Recall that by assumption the manifold W^{sE} intersects transversely the leaves of the strongly unstable foliation (condition (**D**)) at the points of Γ at $\mu = 0$. Therefore, the intersection of the tangent to the continuation of W^{sE} at the point M^- with the tangent to the leaf l^{uu} at the same point is zero. The tangent to W^u_{loc} at this point is spanned to the tangent to l^{uu} and to the phase velocity vector $(\dot{u}, \dot{y}, \dot{w})|_{M^-}$ which is tangent to Γ at M^-. This vector is also contained in the tangent to W^{sE} (because W^{sE} contains Γ). Thus, the intersection of the tangents to W^{sE} and W^u_{loc} at M^- is one-dimensional (it is spanned on the phase velocity vector). This implies that W^{sE} and W^u_{loc} meet transversely at M^-. Therefore, the image of $W^{sE}_{loc} \cap S^{in}$ by the map T^{-1}_{glo} must be transverse to the intersection $W^u_{loc} \cap S^{out}$. In the coordinates of Lemma 6.3 the latter is given by $\{u^1 = 0\}$ and $W^{sE}_{loc} \cap S^{in}$ is tangent to $\{w^0 = 0\}$ at $\mu = 0$. Therefore, the transversality condition (**D**) transcripts in these coordinates as a transversality of the space

$$u^1 = (d_{11}(0), d_{12}(0)) \cdot \begin{pmatrix} u^0 - u^+ \\ y^0 \end{pmatrix}$$

with respect to the space $u^1 = 0$, which means that

$$\det(d_{11}, d_{12}) \neq 0\,. \tag{6.2.36}$$

6.3. Proof of the center manifold theorem near a homoclinic loop

In order to prove Theorem 6.1, we must establish the existence of an invariant manifold for the inverse of the Poincaré map $T^{-1} = T^{-1}_{glo} \circ T^{-1}_{loc}$ on the cross-section S_{in}. We achieve this by using Theorem 4.4 (a generalization of the annulus principle).

Recall (see Lemma 6.3) that we have represented the map $T_{loc}\colon M_0 \mapsto M_1$ in the cross-form by terms of the functions u_{loc} and w_{loc} which are defined for $\{\|u^0 - u^+\| \leq \delta, \|w^1\| \leq \delta, y^0 \in [0, \delta']\}$ for some small δ and δ'. The map T^{-1}_{glo} is given by formula (6.2.35). It follows from the superposition of these two maps that two points (u^0, y^0, w^0) and $(\bar{u}^0, \bar{y}^0, \bar{w}^0)$ are related by the map $T^{-1}\colon (u^0, y^0, w^0) \mapsto (\bar{u}^0, \bar{y}^0, \bar{w}^0)$ if, and only if,

$$\begin{aligned} \bar{w}^0 &= w_{loc}(\bar{u}^0, \bar{y}^0, w^1, \mu)\,, \\ u^1 &= u_{loc}(\bar{u}^0, \bar{y}^0, w^1, \mu)\,, \\ u^1 - u^-(\mu) &= d_{11}(u^0 - u^+) + d_{12}y^0 + d_{13}w^0 + u_{glo}(u^0, y^0, w^0, \mu)\,, \\ w^1 - w^-(\mu) &= d_{21}(u^0 - u^+) + d_{22}y^0 + d_{23}w^0 + w_{glo}(u^0, y^0, w^0, \mu)\,, \end{aligned} \tag{6.3.1}$$

where (u^1, w^1) is an intermediate point where the backward trajectory of the point (u^0, y^0, w^0) intersects S^{out}.

Let us extend the domain of the functions involved here in the following way. First, let us assume $(u_{loc}, w_{loc}) \equiv 0$ for $y^0 \le 0$, and then change these functions by multiplying them by some factor which vanishes outside a small neighborhood of $(u^+, 0, 0)$:

$$\begin{aligned} u_{loc} &\to u_{loc} \cdot \chi\left(\frac{\|u^0 - u^+, y^0, w^1\|}{\rho}\right), \\ w_{loc} &\to w_{loc} \cdot \chi\left(\frac{\|u^0 - u^+, y^0, w^1\|}{\rho}\right), \end{aligned}$$

where χ is a C^r-smooth function such that

$$\chi(s) = \begin{cases} 1, & \text{if } s \le 1/2\,, \\ 0, & \text{if } s \ge 1\,, \end{cases} \qquad \text{and} \qquad \left|\frac{\partial \chi}{\partial s}\right| < 3\,. \tag{6.3.2}$$

Here ρ is a small constant. One can see that such multiplications do not change the estimates of Lemma 6.3 in an essential way, just an additional constant factor may appear. Observe that the functions u_{loc} and w_{loc} are kept the same in a small $\frac{\rho}{2}$-neighborhood of $(u^+, 0, 0)$ whereas they now vanish identically on the boundary of the domain of definition, whence we may consider them to be identically zero outside, without any loss of smoothness.

The same procedure may be applied to the map T_{glo}^{-1} — the functions u_{glo} and w_{glo} may be modified outside the $\frac{\rho}{2}$-neighborhood of the point $(u^0 = u^+, y^0 = 0, w^0 = 0)$ so that they vanish at a distance ρ of that point, and this allows one to assume that T_{glo}^{-1} is defined at all (u^0, y^0, w^0). Recall that u_{glo} and w_{glo} are nonlinear functions. Hence, if ρ is sufficiently small, then the modified map T_{glo}^{-1} is very close to its linear part everywhere.

In particular, this means (by virtue of the transversality condition (6.2.42)) that the third equation in (6.3.1) can be solved with respect to (u^0, y^0) as follows:

$$(u^0, y^0) = f(u^1, w^0, \mu)\,, \tag{6.3.3}$$

where f is some smooth function all of whose derivatives are uniformly bounded. Consequently, the fourth equation in (6.3.2) can be recast into the form

$$w_1 = g(u^1, w^0, \mu)\,, \tag{6.3.4}$$

where g is a smooth function with the uniformly bounded derivatives. Substituting the last equation into the second equation of (6.3.1) we obtain

$$u^1 = u_{loc}(\bar{u}^0, \bar{y}^0, g(u^1, w^0, \mu), \mu)\,.$$

Since $\frac{\partial u_{loc}}{\partial w^1}$ tends to zero for small $\bar{y}^0$ (see Lemma 6.3), this derivative can be made uniformly small for the modified function u_{loc} by taking ρ small enough. Therefore, the equation above can be solved with respect to u^1. This gives

$$u_1 = \tilde{u}_{loc}(\bar{u}^0, \bar{y}^0, w^0, \mu)\,, \tag{6.3.5}$$

where the function $\tilde{u}_{loc}$ essentially satisfies the same estimates (given by a Lemma 6.3) as the function u_{loc}.

The substitution of this expression into (6.3.3) and into the first equation of (6.3.1) represents the map T^{-1} in the cross-form

$$\begin{aligned} \bar{w}^0 &= F(w^0, (\bar{u}^0, \bar{y}^0))\,, \\ (u^0, y^0) &= G(w^0, (\bar{u}^0, \bar{y}^0))\,. \end{aligned} \tag{6.3.6}$$

One can see, using the estimates of Lemma 6.3, that the functions F and G satisfy the conditions of Theorem 4.4 (under the weakened smoothness conditions given below that theorem; observe that the function u_{loc} is not smooth at $y^0 = 0$). Thus, we immediately have the existence of a $C^{\min(q,r)}$-smooth manifold $w^0 = \phi^*(u^0, y^0, \mu)$ which is invariant with respect to the modified map T^{-1}. The function ϕ^* is defined at all u^0, y^0, μ. Since the modified map coincides with the original map T^{-1} in a small neighborhood of $(u^+, 0, 0)$ at $y^0 \geq 0$, it follows that the intersection of the above manifold with this domain is a smooth invariant manifold of the original map.

By construction (see the proof of the annulus principle in Sec. 4.2), forward iterations of any point by the modified map T^{-1} converge exponentially to the "large" invariant manifold we found. This implies that all points whose backward iterations are at a bounded distance on this manifold must lie in this manifold. In terms of the original Poincaré map T this means that all trajectories whose forward iterations lie in a small neighborhood of the point $(u^0, 0, 0)$ must belong to the "small" invariant manifold.

The set of trajectories which start from points of this manifold on the cross-section is an invariant manifold for the system of differential equations under consideration (one should choose the pieces of the trajectories until they remain in a small neighborhood U of the homoclinic loop). By construction, this manifold contains all trajectories which stay in U for all positive times. In particular, it contains the intersection $W^s_{loc} \cap U$. The point O must also be included in the resulting invariant manifold. Note that the smoothness of the above invariant manifold follows from the proven smoothness of its intersection with the cross-section S_{in} — everywhere except at the equilibrium state O. The smoothness of O must be verified separately, but we refrain from giving a complete proof here because it is irrelevant for our purposes. Just note that the resulting invariant manifold coincides locally with one of the extended stable manifolds $W^{sE}_{loc}(O)$ from which the smoothness at O follows.

6.4. Center manifold theorem for heteroclinic cycles

The non-local center manifold theorem which we have proved for a homoclinic loop admits a straightforward generalization onto a class of heteroclinic cycles. Namely, suppose a family of C^r-smooth dynamical systems

$$\dot{x} = X(x, \mu) \tag{6.4.1}$$

depending smoothly on some vector of parameters μ has a number of saddle equilibrium states $O_1, \ldots, O_k$ which satisfy condition (**A**) of the previous sections: for each saddle the leading positive characteristic exponent is real and simple. Let the stable manifold of each O_i be n-dimensional and the unstable manifold be m-dimensional. Suppose that *at* $\mu = 0$, *for each* $i = 1, \ldots, k$ *there exists a trajectory* Γ_i *of intersection* $W^u(O_i) \cap W^s(O_{i+1})$ (*respectively,* $W^u(O_k) \cap W^s(O_1)$ *for* $i = k$).

The trajectories Γ_i are called *heteroclinic* because Γ_i tends to O_i as $t \to -\infty$ (it lies in $W^u(O_i)$) but it tends to another equilibrium state O_{i+1} as $t \to +\infty$ (it lies in $W^s(O_{i+1})$). The union $C = O_1 \cup \Gamma_1 \cup O_2 \cup \cdots \cup O_k \cup \Gamma_k$ is called *a heteroclinic cycle* or *a heteroclinic contour.* Note that generically

k-independent governing parameters $\mu_1, \ldots, \mu_k$ are necessary for the system to have the cycle with k-heteroclinic trajectories.

Let us impose on the trajectories Γ_i the same genericity conditions given by conditions (**C**) and (**D**) of the previous sections. Namely,

> *For each heteroclinic trajectory Γ_i, suppose that it does not lie in the non-leading unstable submanifold $W^{uu}(O_i)$,* and *suppose that the extended stable manifold $W^{sE}(O_{i+1})$ is transverse to the leaves of the strongly unstable foliation $\mathcal{F}^u(O_i)$ of $W^u(O_i)$ at each point of the heteroclinic trajectory Γ_i (for each $i = 1, \ldots, k$).*

Theorem 6.4. *There exists a small neighborhood U of the heteroclinic cycle C such that for all sufficiently small μ the system possesses an $(n+1)$-dimensional invariant C^q-smooth*[1] *center stable manifold W^{sC} such that any trajectory which does not lie in W^{sC} leaves U as $t \to +\infty$. The manifold W^{sC} is tangent at O_i to the extended stable eigenspace $\mathcal{E}^{sE}(O_i)$.*

The proof is identical to that of Theorem 6.1. One may construct a local cross-section $S_{in}^{(i)}$ to each Γ_i near O_{i+1}. Then, consider the inverse $T_i \colon S_{in}^{(i)} \to S_{in}^{(i-1)}$ of the Poincaré map (the map T_i is defined by the backward trajectories of the system). The maps T_i can be modified exactly in the same way as in Sec. 6.3 and, after that, they are written in the cross-form

$$\begin{aligned} \bar{w}_{i-1} &= F_i(w_i, (\bar{u}_{i-1}, \bar{y}_{i-1}), \mu)\,, \\ (u_i, y_i) &= G_i(w_i, (\bar{u}_{i-1}, \bar{y}_{i-1}), \mu)\,, \end{aligned} \tag{6.4.2}$$

where the w-variables belong to R^{m-1}, the (u, y)-variables to R^n and the functions F_i, G_i satisfy the conditions of Theorem 4.4. One can check that denoting $x = (w_1, \ldots, w_k)$ and $y = ((u_1, y_1), \ldots, (u_k, y_k); \mu)$ the relations (6.4.2) (along with the artificial equation $\mu = \bar{\mu}$) define a cross-map

$$\begin{aligned} \bar{x} &= F(x, \bar{y}) \\ y &= G(x, \bar{y}) \end{aligned} \tag{6.4.3}$$

which satisfy the conditions of Theorem 4.4. Thus, there exists a smooth invariant manifold of the kind

$$(w_1, \ldots, w_k) = \varphi^*((u_1, y_1), \ldots, (u_k, y_k); \mu)\,.$$

[1]The integer q must satisfy $q \leq r$ and $q\gamma_1^{(i)} < \operatorname{Re} \gamma_2^{(i)}$ for all O_i, where $\gamma_1^{(i)}$ is the leading positive characteristic exponent of O_i and $\gamma_2^{(i)}$ is the next positive exponent.

We need somewhat more: each w_i here should only depend on (u_i, y_i, μ). The graphs $\mathcal{L}_i^*$ of these dependencies would define the invariant manifold on the extended cross-sections $S_{in}^{(i)}$:

$$T_i \mathcal{L}_i^* = \mathcal{L}_{i-1}^* \, .$$

To prove that the invariant manifold of the map (6.4.3) has the required structure, it is sufficient to note that the invariant manifold is obtained in Theorem 4.4 as the limit of the iterations of an arbitrary Lipschitz manifold. Thus, if we make as our initial guess the manifold, say, $(w_1 = 0, \dots, w_k = 0)$ which indeed represents the collection of independent surfaces on $S_{in}^{(1)}, \dots, S_{in}^{(k)}$, respectively, then just by the essence of the problem all iterations will have the same structure. Hence their limit will also have the same structure.

The intersection of the derived surfaces $\mathcal{L}_i^*$ with the original local pieces of the cross-sections $S_{in}^{(i)}$ define the invariant manifold for the original Poincaré map. So, the set of trajectories which start on any of these surfaces is the desired invariant manifold of the system itself.

Note that a reversion of time allows one to obtain an analogous *center unstable manifold* theorem for the case when the leading *negative* exponent is real and simple for each O_i, and a theorem on two-dimensional *center* manifold when both positive and negative leading exponents are real and simple, as was done in Sec. 6.1 for a homoclinic loop.

The heteroclinic cycles under consideration represents one of the simplest cases among a large variety of possible heteroclinic or homoclinic structures. For example, a single saddle equilibrium state may have more than one homoclinic loop at some value of μ (in a two-parameter family). We distinguish two generic cases here:

- *a figure-eight* — the homoclinic trajectories Γ_1 and Γ_2 enter the saddle O from the opposite directions, as shown in Fig. 6.4.1,

- *a homoclinic butterfly* — the homoclinic trajectories Γ_1 and Γ_2 come back to O along the same direction (positive y), so they are tangent to each other at O (as $t \to +\infty$) as shown in Fig. 6.4.2.

Note that both cases correspond to the case where condition (**C**) holds for both homoclinic trajectories: they do not belong to W^{uu} and therefore leave O along the leading direction, namely the y-axis.

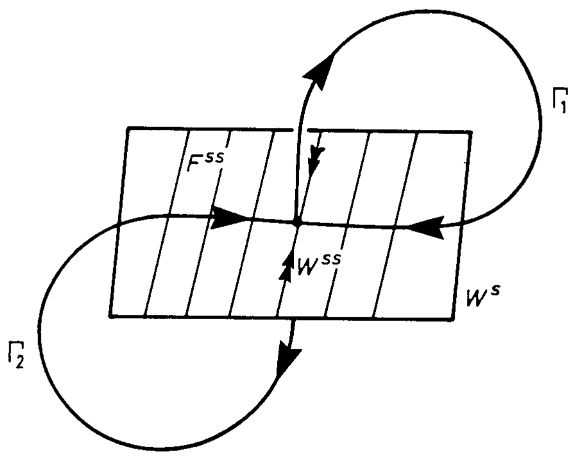


Fig. 6.4.1. The homoclinic figure-eight for which the non-coincidence conditions are fulfilled: the separatrix Γ_1 intersects only those strongly stable leaves which are not intersected by the separatrix Γ_2.

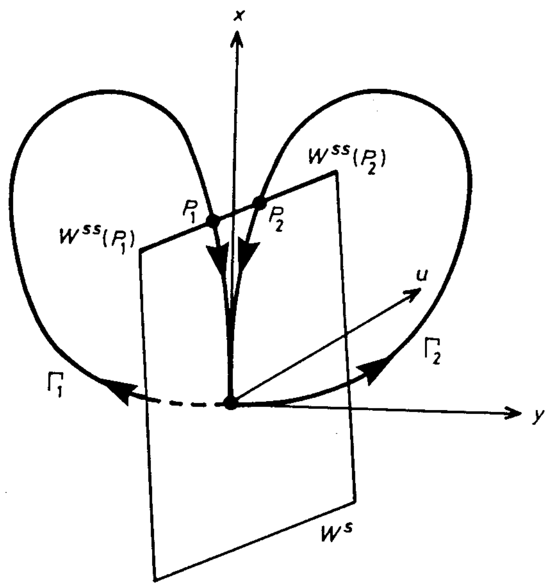


Fig. 6.4.2. The homoclinic butterfly composed by two loops Γ_1 and Γ_2 which does not satisfy the non-coincidence conditions: the strongly stable leaf of an arbitrary point $P_1 \in \Gamma_1$ lying near the equilibrium state coincides with the strongly stable leaf of some point $P_2 \in \Gamma_2$.

Suppose that the transversality condition (**D**) is fulfilled for both homoclinic trajectories (see Sec. 6.1). Again, revisiting the construction of the previous section one may prove the following result.

Theorem 6.5. *There exists a small neighborhood U of the homoclinic figure-eight such that for all sufficiently small μ the system possesses an $(n+1)$-dimensional invariant C^q-smooth* center stable manifold W^{sC} *such that any trajectory which does not lie in W^{sC} leaves U as $t \to +\infty$. The manifold W^{sC} is tangent at O to the extended stable eigenspace $\mathcal{E}^{sE}$.*

At the same time, near a homoclinic butterfly there obviously cannot be an $(n+1)$-dimensional smooth invariant manifold which is tangent to $\mathcal{E}^{sE}$ at O. Indeed, the intersection of such a manifold with $W^u(O)$ would be one-dimensional and it must contain both the homoclinic trajectories Γ_1 and Γ_2. Therefore, it follows that the smooth system on the invariant manifold had a saddle equilibrium state with a *non-smooth* unstable manifold (it must be one-dimensional and contain two trajectories tangent to each other at O). This is impossible for structurally stable saddles.

We see that the transversality condition (**D**) which in fact plays a crucial role in the proof of the non-local center manifold is not always sufficient. Nevertheless, simple enough necessary and sufficient conditions exist for the non-local center manifold theorem near an arbitrarily complicated homoclinic or heteroclinic cycle.

Let C be the union of a finite number of equilibrium states $O_1, O_2, \dots$, periodic trajectories $L_1, L_2, \dots$ and homo/heteroclinic trajectories $\Gamma_1, \Gamma_2, \dots$: each trajectory Γ_s tends to some of the trajectories O_i or L_i as $t \to +\infty$, and to the same or the another trajectory O_i or L_i as $t \to -\infty$ (thus, Γ_s lies in the intersection of the stable and unstable manifolds of the corresponding trajectories. In the case of a structurally unstable equilibrium or periodic trajectory we should consider the center stable or center unstable manifolds). We call any such set C *a heteroclinic cycle*.

Suppose the following *trichotomy condition* holds.
There exist non-negative integers $k \geq 1$, m, n ($k + m + n =$ the dimension of the phase space) such that for each equilibrium state or periodic trajectory in the heteroclinic cycle, for some positive β_i^u and β_i^s, exactly k characteristic exponents λ lie in the strip

$$-\beta_i^s < \operatorname{Re}\lambda < \beta_i^u$$

(*this is the* center part *of the spectrum*); n *characteristic exponents lie to the right of this strip*:

$$\operatorname{Re}\lambda > \beta_i^u$$

and m *characteristic exponents lie to the left of this strip*:

$$\operatorname{Re}\lambda < -\beta_i^s\,.$$

Schematically, we can write

$$\operatorname{Re}\Lambda^{ss} < -\beta_i^s < \operatorname{Re}\Lambda^c < \beta_i^u < \operatorname{Re}\Lambda^{uu}\,.$$

To be more accurate, we take into account the gap between the center part and the strongly stable and strongly unstable parts and write

$$\operatorname{Re}\Lambda^{ss} < -\beta_i^{ss} < -\beta_i^s < \operatorname{Re}\Lambda^c < \beta_i^u < \beta_i^{uu} < \operatorname{Re}\Lambda^{uu}\,, \tag{6.4.4}$$

where $\beta_i^{uu} > \beta_i^u > 0$, $\beta_i^{ss} > \beta_i^s > 0$.

The separating values β_i can be different for different equilibria and periodic trajectories in the cycle. The important requirement is that the numbers k, m, n of the characteristic exponents belonging to each part of the spectrum do *not* depend on a specific trajectory. Note that the numbers k, m, n are not uniquely determined by the system. For instance, if the cycle contains only one recurrent trajectory, namely, a saddle periodic trajectory L, one may, in principle, consider all characteristic exponents of L as critical and in this case $m = n = 0$ and k equals to the dimension of the phase space, or one may consider all characteristic exponents with negative real parts as strongly stable, the characteristic exponents with positive real parts as strongly unstable and only trivial characteristic exponent equal to zero is critical in this case (*i.e.* $k = 1$); other variants corresponding to intermediate values of k are also allowed.

Implicitly, when studying concrete multidimensional global bifurcational problems, such a separation of the spectrum of characteristic exponents was always done. Usually, the leading characteristic exponents are taken as critical, and the non-leading as strongly stable and (or) strongly unstable.

We restrict our freedom in the choice of the trichotomy decomposition by an additional requirement. Namely, we suppose that for each homo/heteroclinic trajectory Γ_s in the cycle C a pair of the *transversality conditions* is fulfilled.

These conditions are analogous to conditions (**D**) and (**D′**) from Sec. 6.1. According to Theorems 5.16, 5.17 and 5.20, a trajectory Γ_s which tends to an equilibrium state O_i, or to a periodic trajectory L_i, as $t \to +\infty$, lies in an $(m+k)$-dimensional extended stable manifold W^{sE} of O_i or L_i and through each point of Γ_s a uniquely defined m-dimensional leaf of the strongly stable foliation $\mathcal{F}^{ss}$ exists; the tangent to W^{sE} is also uniquely defined at each point of Γ_s. Analogously, the trajectory Γ_s in the heteroclinic cycle tends to some equilibrium state O_j, or to a periodic trajectory L_j, as $t \to -\infty$ and this implies that Γ_s lies in an $(n+k)$-dimensional extended unstable manifold W^{uE} of O_j, or L_j, (the tangent to W^{uE} is uniquely defined at each point of Γ_s) and through each point of Γ_s a uniquely defined n-dimensional leaf of the strongly unstable foliation $\mathcal{F}^{uu}$ exists. The manifold W^{sE} is tangent at O_i or L_i to the extended stable invariant subspace $\mathcal{E}^{sE}$ of the linearized system, corresponding to the critical and strongly stable parts of the spectrum of characteristic exponents. The manifold W^{uE} is tangent at O_j, or L_j, to the extended unstable invariant subspace $\mathcal{E}^{uE}$ corresponding to the critical and strongly unstable parts of the spectrum. The foliation $\mathcal{F}^{ss}$ includes the strongly stable manifold which is tangent at O_i, or L_i, to the strongly stable invariant subspace $\mathcal{E}^{ss}$, and the foliation $\mathcal{F}^{uu}$ includes the strongly unstable manifold which is tangent at O_j or L_j, to the strongly unstable invariant subspace $\mathcal{E}^{uu}$.

The transversality conditions are:

At each point of each trajectory $\Gamma_s \subset C$ the extended unstable manifold is transverse to a leaf of the strongly stable foliation and the extended stable manifold is transverse to a leaf of the strongly unstable foliation.

Observe that due to the invariance of the subspaces with respect to the linearized system, the transversality must be verified at one point on each trajectory Γ_s.

Different choices of the separation of the spectra of characteristic exponents lead to different manifolds and foliations involved in the above transversality conditions. For some of our choices transversality may hold, but for some it may not hold. So these conditions do make an additional selection among various possible trichotomic separations.

Theorem 6.6. *Let q and p be the maximal integers such that $\beta_i^{uu} > q\beta_i^u$, $\beta_i^{ss} > p\beta_i^s$ for any equilibrium state or periodic trajectory in the cycle (the β's are the separating constants from* (6.4.4)*).*

Let a $\mathbb{C}^r$-smooth ($r \geq 1$) system have a heteroclinic cycle C and let the trichotomy and transversality conditions be fulfilled. Then, in a small neighborhood U of C, the system has a smooth k-dimensional invariant manifold $\mathcal{W}^C$ which contains C, and which is tangent at each of the equilibrium states and periodic trajectories of C to the critical subspace $\mathcal{E} = \mathcal{E}^{sE} \cap \mathcal{E}^{ue}$ if and only if any leaf of the local strongly stable and strongly unstable foliations intersects the set C at not more than one point. *Under this condition, the manifold $\mathcal{W}^C$ exists for any system $\mathbb{C}^r$-close to the original system, and it varies continuously with the system.*

The manifold $\mathcal{W}^C$ is $\mathbb{C}^{\min(p,q,r)}$-smooth. It is the intersection of an $(m+k)$-dimensional invariant $\mathbb{C}^{\min(q,r)}$-smooth manifold $\mathcal{W}^{sC}$ and an $(n+k)$-dimensional invariant $\mathbb{C}^{\min(p,r)}$-smooth manifold $\mathcal{W}^{uC}$ which are tangent, respectively, to the extended stable and extended unstable invariant subspaces $\mathcal{E}^{sE}$ and $\mathcal{E}^{uE}$ at each of the equilibrium states and periodic trajectories in the heteroclinic cycle C. All trajectories which stay in the neighborhood U for all positive times belong to $\mathcal{W}^{sC}$ and all trajectories which stay in U for all negative times belong to $\mathcal{W}^{uC}$; hence, all trajectories which lie in U belong entirely to the invariant manifold $\mathcal{W}^C$.

We do not give the proof of this theorem. It includes the above non-local center manifold theorems of this chapter as a special cases. For example, when we consider a single homoclinic loop, the strongly unstable manifold is a particular leaf of the strongly unstable foliation. If the homoclinic trajectory lies in this leaf, it intersects the leaf, formally speaking, in continuum points which prevents of existence of the smooth invariant manifold under consideration. Thus, conditions (**C**) and (**C′**) are necessary for the theorems of Sec. 6.1 to be valid.

When we consider a pair of homoclinic loops, the leaves of the strongly unstable foliation are surfaces of the kind

$$(u, y) = \psi(w)$$

in the coordinates of Sec. 6.2. One may straighten the foliation so that the leaves are the intersections of the surfaces $\{y = \text{constant}\}$ with the unstable manifold. In the case of a homoclinic figure-eight, the leaves corresponding to $y > 0$ intersect the homoclinic trajectory Γ_1 at one point each, and the leaves corresponding to $y < 0$ intersect the homoclinic trajectory Γ_2 also at only one point each. The strongly unstable manifold — the leaf corresponding to $y = 0$

— intersects the homoclinic cycle $C = O \cup \Gamma_1 \cup \Gamma_2$ at the point O. Thus, Theorem 6.5 on the existence of the non-local center unstable manifold near a homoclinic figure-eight is consistent with the general result of Theorem 6.6.

On the contrary, in the case of a homoclinic butterfly, each of the leaves corresponding to a positive y intersects both homoclinic trajectories. Thus our previous conclusion on the absence of the smooth invariant manifold is in formal agreement with the latter theorem.

Appendix A

SPECIAL FORM OF SYSTEMS NEAR A SADDLE EQUILIBRIUM STATE

In the study of bifurcations of homoclinic loops and heteroclinic cycles composed of saddles and their connecting trajectories we run into the problem of getting a suitable asymptotic for the solutions of a system near a saddle equilibrium state. It is obvious that the simpler the form of a system near the equilibrium state the easier it is to study its behavior. The possibility of the reduction of a system near a saddle to a good form which is suitable for many bifurcational problems is established by Theorem 2.17 of Sec. 2.9, a complete proof of which we present here.

Consider a family $X(\mu)$ of dynamical systems which depends on some parameters μ. Assume that $X(\mu)$ is $\mathbb{C}^r$-smooth ($r \geq 2$) with respect to all variables and parameters. We may represent $X(\mu)$ in the form (see Chap. 2)

$$\begin{aligned} \dot{x} &= A_1(\mu)x + f_1(x,y,u,v,\mu)\,, \\ \dot{u} &= A_2(\mu)u + f_2(x,y,u,v,\mu)\,, \\ \dot{y} &= B_1(\mu)y + g_1(x,y,u,v,\mu)\,, \\ \dot{v} &= B_2(\mu)v + g_2(x,y,u,v,\mu)\,, \end{aligned} \tag{A.1}$$

where the eigenvalues of the block-diagonal matrix

$$A(0) \equiv \begin{pmatrix} A_1(0) & 0 \\ 0 & A_2(0)) \end{pmatrix}$$

lie to the left of the imaginary axis in the complex plane, and the eigenvalues of the block-diagonal matrix

$$B(0) \equiv \begin{pmatrix} B_1(0) & 0 \\ 0 & B_2(0) \end{pmatrix}$$

lie to the right of the imaginary axis.

Let us assume also that the eigenvalues $(\lambda_1, \dots, \lambda_{m_1})$ of the matrix $A_1(0)$ have the same real part, namely

$$\mathrm{Re}\lambda_1 = \cdots = \mathrm{Re}\lambda_{m_1} = \lambda, \qquad \lambda < 0\,,$$

and that the real parts of the eigenvalues $(\gamma_1, \dots, \gamma_{n_1})$ of the matrix $B_1(0)$ are equal to each other, *i.e.*,

$$\mathrm{Re}\gamma_1 = \cdots = \mathrm{Re}\gamma_{n_1} = \gamma, \qquad \gamma > 0\,.$$

With regard to the eigenvalues of the matrices $A_2(0)$ and $B_2(0)$, let us assume that the real parts of the eigenvalues of $A_2(0)$ are strictly less than λ, and those of $B_2(0)$ are strictly larger than γ. In this case x and y are the leading stable and unstable coordinates, respectively, and u and v are the non-leading coordinates.

Theorem A.1. *There exists a local transformation of coordinates of class* $\mathbb{C}^{r-1}$ *with respect to* (x, u, y, v) (*and the first derivative of the transformation with respect to* (x, u, y, v) *is* $\mathbb{C}^{r-2}$ *with respect to* (x, u, y, v, μ))[1] *which brings system* (A.1) *to the form*

$$\begin{aligned}
\dot{x} &= A_1(\mu)x + f_{11}(x, u, y, v, \mu)x + f_{12}(x, u, y, v, \mu)u\,, \\
\dot{u} &= A_2(\mu)u + f_{21}(x, u, y, v, \mu)x + f_{22}(x, u, y, v, \mu)u\,, \\
\dot{y} &= B_1(\mu)y + g_{11}(x, u, y, v, \mu)y + g_{12}(x, u, y, v, \mu)v\,, \\
\dot{v} &= B_2(\mu)v + g_{21}(x, u, y, v, \mu)y + f_{22}(x, u, y, v, \mu)v\,,
\end{aligned} \tag{A.2}$$

where f_{ij}, g_{ij} *are* $\mathbb{C}^{r-1}$ *with respect to* (x, u, y, v) *and their first derivatives with respect to* (x, u, y, v) *are* $\mathbb{C}^{r-2}$ *with respect to* (x, u, y, v, μ), *and*

$$\begin{aligned}
f_{ij}(0, 0, 0, 0, \mu) = 0, &\qquad g_{ij}(0, 0, 0, 0, \mu) = 0\,, \\
f_{1i}(x, u, 0, 0, \mu) \equiv 0, &\qquad g_{1i}(0, 0, y, v, \mu) \equiv 0\,, \\
f_{j1}(0, 0, y, v, \mu) \equiv 0, &\qquad g_{j1}(x, u, 0, 0, \mu) \equiv 0 \quad (i, j = 1, 2)\,.
\end{aligned} \tag{A.3}$$

Proof. System (A.1) may be reduced to the form (A.2) by a change of variables which straightens the invariant manifolds of the saddle point. Such

[1] At $r = \infty$, the transformation is $\mathbb{C}^\infty$ with respect to (x, u, y, v) but it has only finite smoothness with respect to μ.

a transformation has the form (see Sec. 2.7)

$$\begin{aligned}
\tilde{x} &= x - \varphi_{1s}(y, v, \mu)\,, \\
\tilde{u} &= u - \varphi_{2s}(y, v, \mu)\,, \\
\tilde{y} &= y - \psi_{1u}(x, u, \mu)\,, \\
\tilde{v} &= v - \psi_{2u}(x, u, \mu)\,,
\end{aligned} \tag{A.4}$$

where $\{x = \varphi_{1s}(y, v, \mu), u = \varphi_{2s}(y, v, \mu)\}$ and $\{y = \psi_{1u}(x, u, \mu), v = \psi_{2u}(x, u, \mu)\}$ are the equations of the stable and the unstable manifolds of the saddle point, respectively. This transformation does not give us the identities (A.3); by now we have that the functions f_{ij} and g_{ij} in (A.2) are $\mathbb{C}^{r-1}$-smooth and vanishing at the origin.

We can also recast system (A.2) into the form

$$\begin{aligned}
\dot{x} &= A_1(\mu)x + \underline{R_1(x, u, \mu)} + \underline{\varphi_1(y, v, \mu)x} + \varphi_2(y, v, \mu)u + \dots\,, \\
\dot{u} &= A_2(\mu)u + R_2(x, u, \mu) + \underline{\varphi_3(y, v, \mu)x} + \varphi_4(y, v, \mu)u + \dots\,, \\
\dot{y} &= B_1(\mu)y + \underline{P_1(y, v, \mu)} + \underline{\psi_1(x, u, \mu)y} + \psi_2(x, u, \mu)v + \dots\,, \\
\dot{v} &= B_2(\mu)v + P_2(y, v, \mu) + \underline{\psi_3(x, u, \mu)y} + \psi_4(x, u, \mu)v + \dots\,,
\end{aligned} \tag{A.5}$$

where

$$\begin{aligned}
R_i &= f_{i1}(x, u, 0, 0, \mu)x + f_{i2}(x, u, 0, 0, \mu)u\,, \\
P_i &= g_{i1}(0, 0, y, v, \mu)y + g_{i2}(0, 0, y, v, \mu)u\,, \\
\varphi_1 &= f_{11}(0, 0, y, v, \mu), \qquad \varphi_2 = f_{12}(0, 0, y, v, \mu)\,, \\
\varphi_3 &= f_{21}(0, 0, y, v, \mu), \qquad \varphi_4 = f_{22}(0, 0, y, v, \mu)\,, \\
\psi_1 &= g_{11}(x, u, 0, 0, \mu), \qquad \psi_2 = g_{12}(x, u, 0, 0, \mu)\,, \\
\psi_3 &= g_{21}(x, u, 0, 0, \mu), \qquad \psi_4 = g_{22}(x, u, 0, 0, \mu)\,,
\end{aligned}$$

and

$$\begin{aligned}
R_i(x, u, \mu) &= \tilde{R}_{i1}(x, u, \mu)x + \tilde{R}_{i2}(x, u, \mu)u\,, \\
P_i(y, v, \mu) &= \tilde{P}_{i1}(y, v, \mu)y + \tilde{P}_{i2}(y, v, \mu)v\,, \\
\tilde{R}_{ij}(0, 0, \mu) &\equiv 0, \qquad \tilde{P}_{ij}(0, 0, \mu) \equiv 0\,, \\
\varphi_j(0, 0, \mu) &\equiv 0, \qquad \psi_j(0, 0, \mu) \equiv 0\,,
\end{aligned}$$

and the ellipsis denotes the terms which we will hereafter call *negligible*: in the first two equations these are the terms of the form $\tilde{f}(x, u, y, v, \mu)x$ and

$\tilde{f}(x,u,y,v,\mu)u$ such that

$$\tilde{f}(0,0,y,v,\mu) \equiv 0 \quad \text{and} \quad \tilde{f}(x,u,0,0,\mu) \equiv 0\,,$$

and in the last two equations these are the terms of the form $\tilde{g}(x,u,y,v,\mu)y$ and $\tilde{g}(x,u,y,v,\mu)v$ such that

$$\tilde{g}(0,0,y,v,\mu) \equiv 0 \quad \text{and} \quad \tilde{g}(x,u,0,0,\mu) \equiv 0\,.$$

Obviously, the proof of this theorem is reduced to eliminating the underlined terms in (A.5). To kill these terms we will carry out a series of consecutive changes of variables

(1)

$$\begin{aligned} \xi_1 &= x + h_1(y,v,\mu)x, & \xi_2 &= u + h_2(y,v,\mu)x\,, \\ \eta_1 &= y, & \eta_2 &= v\,, \end{aligned}$$

where $h_i(0,0,\mu) = 0$;

(2)

$$\begin{aligned} \xi_1 &= x, & \xi_2 &= u\,, \\ \eta_1 &= y + s_1(x,u,\mu)y, & \eta_2 &= v + s_2(x,u,\mu)y\,, \end{aligned}$$

where $s_i(0,0,\mu) = 0$;

(3)

$$\begin{aligned} \xi_1 &= x + r_1(x,u,\mu)x + r_2(x,u,\mu)u, & \xi_2 &= u\,, \\ \eta_1 &= y, & \eta_2 &= v\,, \end{aligned}$$

where $r_1(0,0,\mu) = 0,\ r_2(0,0,\mu) = 0$;

(4)

$$\begin{aligned} \xi_1 &= x, & \xi_2 &= u\,, \\ \eta_1 &= y + p_1(y,v,\mu)y + p_2(y,v,\mu)v, & \eta_2 &= v\,, \end{aligned}$$

where $p_1(0,0,\mu) = 0,\ p_2(0,0,\mu) = 0\,.$

The change of variables (1) gets rid of the terms φ_1 and φ_3 in system (A.5). By a change of variables (2) we eliminate the terms ψ_1 and ψ_3. By a change of variables (3) we eliminate the terms R_1. Finally by a change of variables (4) we eliminate the terms P_1, thereby reducing the original system to the desired form.

Step 1. Let us make a change of coordinates (1). The first equation of system (A.5) is written as

$$
\begin{aligned}
\dot{\xi}_1 &= \dot{x} + \frac{\partial h_1}{\partial y}\dot{y}x + \frac{\partial h_1}{\partial v}\dot{v}x + h_1(y,v,\mu)\dot{x} \\
&= A_1(\mu)x + R_1(x,u,\mu) + \varphi_1(y,v,\mu)x + \varphi_2(y,v,\mu)u \\
&\quad + \frac{\partial h_1}{\partial y}\Big(B_1(\mu)y + P_1(y,v,\mu) + \underline{\psi_1(x,u,\mu)y} + \underline{\psi_2(x,u,\mu)v}\Big)x \\
&\quad + \frac{\partial h_1}{\partial v}\Big(B_2(\mu)v + P_2(y,v,\mu) + \underline{\psi_3(x,u,\mu)y} + \underline{\psi_4(x,u,\mu)v}\Big)x \\
&\quad + h_1(y,v,\mu)\Big(A_1(\mu)x + \underline{R_1(x,u,\mu)} + \varphi_1(y,v,\mu)x + \varphi_2(y,v,\mu)u\Big) + \dots\,.
\end{aligned}
\tag{A.6}
$$

Observe that the underlined summands

$$
\frac{\partial h_1}{\partial y}\psi_1(x,u,\mu)yx, \quad \frac{\partial h_1}{\partial y}\psi_2(x,u,\mu)vx, \quad \frac{\partial h_1}{\partial v}\psi_3(x,u,\mu)yx\,,
$$

$$
\frac{\partial h_1}{\partial v}\psi_4(x,u,\mu)vx,\ \ h_1(y,v,\mu)R_1(x,u,\mu)
$$

are negligible (*i.e.* they may be written as $\tilde{f}_1(x,u,y,v,\mu)x + \tilde{f}_2(x,u,y,v,\mu)u$, where $\tilde{f}_i(0,0,y,v,\mu) \equiv 0$ and $\tilde{f}_i(x,u,0,0,\mu) \equiv 0$). Note also that

$$
R_1(x,u,\mu) = R(\xi_1,\xi_2,\mu) + \dots
$$

where the dots, as above, stand for negligible terms. Since

$$
\begin{aligned}
x &= \xi_1 - h_1(y,v,\mu)x\,, \\
u &= \xi_2 - h_2(y,v,\mu)\,,
\end{aligned}
\tag{A.7}
$$

we obtain

$$
\begin{aligned}
\dot{\xi}_1 &= A_1(\mu)\xi_1 + R_1(\xi_1,\xi_2,\mu) + \varphi_2(\eta_1,\eta_2,\mu)\xi_2 \\
&\quad + \Big[-A_1(\mu)h_1(y,v,\mu) + \varphi_1(y,v,\mu) - \varphi_2(y,v,\mu)h_2(y,v,\mu) \\
&\quad + \frac{\partial h_1}{\partial y}\Big(B_1(\mu)y + P_1(y,v,\mu)\Big) + \frac{\partial h_1}{\partial v}\Big(B_2(\mu)v + P_2(y,v,\mu)\Big) \\
&\quad + h_1(y,v,\mu)A_1(\mu) + h_1(y,v,\mu)\varphi_1(y,v,\mu) \\
&\quad - h_1(y,v,\mu)\varphi_2(y,v,\mu)h_2(y,v,\mu)\Big]x + h_1(\eta_1,\eta_2,\mu)\varphi_2(\eta_1,\eta_2,\mu)\xi_2 + \dots\,.
\end{aligned}
\tag{A.8}
$$

Analogously, for the second equation in (A.5) we obtain

$$\begin{aligned}
\dot{\xi}_2 &= \dot{u} + \frac{\partial h_2}{\partial y}\dot{y}x + \frac{\partial h_2}{\partial v}\dot{v}x + h_2(y,v,\mu)\dot{x} \\
&= A_2(\mu)u + R_2(x,u,\mu) + \varphi_3(y,v,\mu)x + \varphi_4(y,v,\mu)u \\
&\quad + \frac{\partial h_2}{\partial y}\Big(B_1 y + P_1(y,v,\mu)\Big)x + \frac{\partial h_2}{\partial v}\Big(B_2 v + P_2(y,v,\mu)\Big)x \\
&\quad + h_2(y,v,\mu)\Big(A_1(\mu)x + \varphi_1(y,v,\mu)x + \varphi_2(y,v,\mu)u\Big) + \dots \\
&= A_2\xi_2 + R_2(\xi_1,\xi_2,\mu) + \varphi_4(\eta_1,\eta_2,\mu)\xi_2 \\
&\quad + \Bigg[-A_2(\mu)h_2(y,v,\mu) + \varphi_3(y,v,\mu) - \varphi_4(y,v,\mu)h_2 \\
&\quad + \frac{\partial h_2}{\partial y}\Big(B_1(\mu)y + P_1(y,v,\mu)\Big) + \frac{\partial h_2}{\partial v}\Big(B_2(\mu)v + P_2(y,v,\mu)\Big) \\
&\quad + h_2(y,v,\mu)A_1(\mu) + h_2(y,v,\mu)\varphi_1(y,v,\mu) \\
&\quad - h_2(y,v,\mu)\varphi_2(y,v,\mu)h_2(y,v,\mu)\Bigg]x \\
&\quad + h_2(\eta_1,\eta_2,\mu)\varphi_2(\eta_1,\eta_2,\mu)\xi_2 + \dots\,.
\end{aligned} \tag{A.9}$$

The form of the third and fourth equations is not affected by such change of variables.

We assume that the functions $h_1(y,v,\mu)$ and $h_2(y,v,\mu)$ satisfy the following conditions

$$\begin{aligned}
&A_1h_1 - h_1A_1 - \varphi_1 + \varphi_2h_2 - h_1\varphi_1 + h_1\varphi_2h_2 \\
&\qquad = \frac{\partial h_1}{\partial y}(B_1y + P_1) + \frac{\partial h_1}{\partial v}(B_2v + P_2)\,, \\
&A_2h_2 - h_2A_2 - \varphi_3 + \varphi_4h_2 - h_2\varphi_1 + h_2\varphi_2h_2 \\
&\qquad = \frac{\partial h_2}{\partial y}(B_1y + P_1) + \frac{\partial h_2}{\partial v}(B_2v + P_2)\,.
\end{aligned} \tag{A.10}$$

This implies that the expressions inside the square brackets in (A.8) and (A.9) vanish. Consider next the following system of matrix equations:

$$\begin{aligned}\dot{X} &= A_1X - XA_1 - \varphi_1 + \varphi_2U - X\varphi_1 + X\varphi_2U\,,\\ \dot{U} &= A_2U - UA_1 - \varphi_3 + \varphi_4U - U\varphi_1 + U\varphi_2U\,,\\ \dot{y} &= B_1y + P_1\,,\\ \dot{v} &= B_2v + P_2\,.\end{aligned} \tag{A.11}$$

Here, the matrices $X \in \mathbb{R}^{m_1\times m_1}$ and $U \in \mathbb{R}^{m_2\times m_1}$, where m_2 is the dimension of the vector u. System (A.11) has an equilibrium state $O_1(0,0,0,0)$. The linearized system is

$$\begin{aligned}\dot{X} &= A_1X - XA_1 - \frac{\partial\varphi_1}{\partial y}(0,0,\mu)y - \frac{\partial\varphi_1}{\partial v}(0,0,\mu)v\,,\\ \dot{U} &= A_2U - UA_1 - \frac{\partial\varphi_3}{\partial y}(0,0,\mu)y - \frac{\partial\varphi_3}{\partial v}(0,0,\mu)v\,,\\ \dot{y} &= B_1y\,,\\ \dot{v} &= B_2v\,.\end{aligned}$$

The spectrum of characteristic exponents of this system may be represented as a union of the spectra of the following associated linear operators

$$\begin{aligned}X &\mapsto A_1X - XA_1\,,\\ U &\mapsto A_2U - UA_1\,,\\ y &\mapsto B_1y\,,\\ v &\mapsto B_2v\,.\end{aligned}$$

Let us now recall a well-known fact from matrix theory (see [39]), namely, for any square matrices A and B the spectrum of the operator $Z \mapsto AZ - ZB$ (where Z is a rectangular matrix) belongs to the set of numbers generated all possible differences between the eigenvalues of the matrices A and B.

Then, since the eigenvalues of the matrix A_2 lie to the left of the eigenvalues of the matrix A_1, and the latter lie all on the line $\mathrm{Re}\cdot = \lambda$, it follows that when $\mu = 0$ the equilibrium state of system (A.11) possesses m_1^2 characteristic exponents on the imaginary axis, $m_1 \cdot m_2$ characteristic exponents in the open left-half plane, and $n_1 + n_2 = n$ characteristic exponents in the open right-half plane. Therefore, the equilibrium state of system (A.11) has an invariant n-dimensional strongly unstable manifold $\tilde{W}_1^{uu}$ defined by the equation

$\{X = h_1(y, v, \mu), U = h_2(y, v, \mu)\}$. Furthermore, the functions $h_1(y, v, 0)$ and $h_2(y, v, 0)$ satisfy conditions (A.10) because these are nothing but the condition of the invariance of the manifold $\{X = h_1, U = h_2\}$ with respect to (A.11).

The smoothness of h_1 and h_2 with respect to (y, v) coincides with the smoothness of the system (A.11). It equals $\mathbb{C}^{r-1}$ because by construction the functions φ_i and ψ_i are $\mathbb{C}^{r-1}$. The smoothness with respect to μ is one less; moreover, it is only finite even when $r = \infty$; see Sec. 5.4).

Thus, the smooth functions h_1, h_2 satisfying (A.10) exist due to the theorem on strong-unstable manifold. After making the change of variables (1) our system takes the form (A.5), where $\varphi_1 \equiv 0$ and $\varphi_3 \equiv 0$.

Step 2. Carrying out the transformation (2) we obtain

$$\dot{\xi}_1 = A_1(\mu)\xi_1 + R_1(\xi_1, \xi_2, \mu) + \varphi_2(\eta_1, \eta_2, \mu)\xi_2 + \dots,$$

$$\dot{\xi}_2 = A_2(\mu)\xi_2 + R_2(\xi_1, \xi_2, \mu) + \varphi_4(\eta_1, \eta_2, \mu)\xi_2 + \dots,$$

$$\begin{aligned}
\dot{\eta}_1 &= \dot{y} + \frac{\partial s_1}{\partial x}\dot{x}y + \frac{\partial s_1}{\partial u}\dot{u}y + s_1(x, u, \mu)\dot{y} \\
&= B_1(\mu)y + P_1(y, v, \mu) + \psi_1(x, u, \mu)y + \psi_2(x, u, \mu)v \\
&\quad + \frac{\partial s_1}{\partial x}\Big(A_1(\mu)x + R_1(x, u, \mu)\Big)y + \frac{\partial s_1}{\partial u}\Big(A_2(\mu)u + R_2(x, u, \mu)\Big)y \\
&\quad + s_1(x, u, \mu)\Big(B_1(\mu)y + \psi_1(x, u, \mu)y + \psi_2(x, u, \mu)v\Big) + \dots \\
&= B_1(\mu)\eta_1 + P_1(\eta_1, \eta_2, \mu) + \psi_2(\xi_1, \xi_2, \mu)\eta_2 \\
&\quad + \Big[-B_1(\mu)s_1(x, u, \mu) + \psi_1(x, u, \mu) \\
&\quad - \psi_2(x, u, \mu)s_2(x, u, \mu) + \frac{\partial s_1}{\partial x}\Big(A_1(\mu)x + R_1(x, u, \mu)\Big) \\
&\quad + \frac{\partial s_1}{\partial u}\Big(A_2(\mu)u + R_2(x, u, \mu)\Big) + s_1(x, u, \mu)B_1(\mu) \\
&\quad + s_1(x, u, \mu)\psi_1(x, u, \mu) - s_1(x, u, \mu)\psi_2(x, u, \mu)s_2(x, u, \mu)\Big]y \\
&\quad + s_1(\xi_1, \xi_2, \mu)\psi_2(\xi_1, \xi_2, \mu)\eta_2 + \dots,
\end{aligned}$$

$$
\begin{aligned}
\dot{\eta}_2 &= \dot{v} + \frac{\partial s_2}{\partial x}\dot{x}y + \frac{\partial s_2}{\partial u}\dot{u}y + s_2(x,u,\mu)\dot{y} \\
&= B_2(\mu)v + P_2(y,v,\mu) + \psi_3(x,u,\mu)y + \psi_4(x,u,\mu)v \\
&\quad + \frac{\partial s_2}{\partial x}\Big(A_1x + R_1(x,u,\mu)\Big)y + \frac{\partial s_2}{\partial u}\Big(A_2u + R_2(x,u,\mu)\Big)y \\
&\quad + s_2(x,u,\mu)v\Big(B_1(\mu)y + \psi_1(x,u,\mu)y + \psi_2(x,u,\mu)v\Big) + \dots \\
&= B_2\eta_2 + P_2(\eta_1,\eta_2,\mu) + \psi_4(\xi_1,\xi_2,\mu)\eta_2 \\
&\quad + \Big[-B_2(\mu)s_2(x,u,\mu) + \psi_3(x,u,\mu) - \psi_4(x,u,\mu)s_2 \\
&\quad + \frac{\partial s_2}{\partial x}\Big(A_1(\mu)x + R_1(x,u,\mu)\Big) + \frac{\partial s_2}{\partial u}\Big(A_2(\mu)u + R_2(x,u,\mu)\Big) \\
&\quad + s_2(x,u,\mu)B_1(\mu) + s_2(x,u,\mu)\psi_1(x,u,\mu) \\
&\quad - s_2(x,u,\mu)\psi_2(x,u,\mu)s_2(x,u,\mu)\Big]y \\
&\quad + s_2(\xi_1,\xi_2,\mu)\psi_2(\xi_1,\xi_2,\mu)\eta_2 + \dots .
\end{aligned}
$$

We choose the functions s_1 and s_2 such that the expressions inside the square brackets become identically equal to zero, *i.e.*

$$
\begin{aligned}
&B_1s_1 - s_1B_1 - \psi_1 + \psi_2s_2 - s_1\psi_1 + s_1\psi_2s_2 \\
&\qquad = \frac{\partial s_1}{\partial x}(A_1x + R_1) + \frac{\partial s_1}{\partial u}(A_2u + R_2)\,, \\
&B_2s_2 - s_2B_2 - \psi_3 + \psi_4s_2 - s_2\psi_1 + s_2\psi_2s_2 \\
&\qquad = \frac{\partial s_2}{\partial x}(A_1x + R_1) + \frac{\partial s_2}{\partial u}(A_2u + R_2)\,.
\end{aligned}
\tag{A.12}
$$

To show that such s_1 and s_2 exist, consider the matrix system

$$
\begin{aligned}
\dot{x} &= A_1x + R_1\,, \\
\dot{u} &= A_2u + R_2\,, \\
\dot{Y} &= B_1Y - YB_1 - \psi_1 + \psi_2V - Y\psi_1 + Y\psi_2V\,, \\
\dot{V} &= B_2V - VB_1 - \psi_3 + \psi_4V - V\psi_1 + V\psi_2V\,,
\end{aligned}
\tag{A.13}
$$

where $Y \in \mathbb{R}^{n_1^2}$ and $V \in \mathbb{R}^{n_1 n_2}$. For all small μ this system has an equilibrium state $O_2(0,0,0,0)$. The linearized system is

$$\begin{aligned}
\dot{x} &= A_1 x\,, \\
\dot{u} &= A_2 u\,, \\
\dot{Y} &= B_1 Y - Y B_1 - \frac{\partial \psi_1}{\partial x}(0,0,\mu)x - \frac{\partial \psi_1}{\partial u}(0,0,\mu)u\,, \\
\dot{V} &= B_2 V - V B_1 - \frac{\partial \psi_3}{\partial x}(0,0,\mu)x - \frac{\partial \psi_3}{\partial u}(0,0,\mu)u\,.
\end{aligned}$$

At $\mu = 0$ the characteristic exponents are ordered as follows: n_1^2 eigenvalues lie on the imaginary axis, $n_1 n_2$ eigenvalues lie to the left, and m eigenvalues lie to the right of it. Hence, system (A.13) possesses an m-dimensional invariant strongly stable manifold W_2^{ss} defined as $\{Y = s_1(x,u,\mu), V = s_2(x,u,\mu)\}$.

We have found the functions $s_1(x,u,\mu)$ and $s_2(x,u,\mu)$ which satisfy condition (A.12). Thus, the changes of variables (1) and (2) transform system (A.5) so that $\varphi_1 \equiv 0$, $\varphi_3 \equiv 0$, $\psi_1 \equiv 0$ and $\psi_3 \equiv 0$.

Step 3. To make the change of variables (3), let us introduce the notation

$$\mathrm{x} = \begin{pmatrix} x \\ u \end{pmatrix}, \qquad \mathrm{y} = \begin{pmatrix} y \\ v \end{pmatrix},$$

$$A(\mu) = \begin{pmatrix} A_1(\mu) & 0 \\ 0 & A_2(\mu) \end{pmatrix}, \quad B(\mu) = \begin{pmatrix} B_1(\mu) & 0 \\ 0 & B_2(\mu) \end{pmatrix},$$

$$\begin{aligned}
r(\mathrm{x},\mu) &= (r_1(\mathrm{x},\mu), r_2(\mathrm{x},\mu))\,, \quad R(\mathrm{x},\mu) = \begin{pmatrix} R_1(\mathrm{x},\mu) \\ R_2(\mathrm{x},\mu) \end{pmatrix}, \\
p(\mathrm{y},\mu) &= (r_1(\mathrm{y},\mu), r_2(\mathrm{y},\mu))\,, \quad P(\mathrm{y},\mu) = \begin{pmatrix} P_1(\mathrm{y},\mu) \\ P_2(\mathrm{y},\mu) \end{pmatrix}.
\end{aligned}$$

In terms of the above new notation, the change of variables (3) assumes the form

$$\xi_1 = x + r(\mathrm{x},\mu)\mathrm{x}, \quad \xi_2 = u, \quad \eta_1 = y, \quad \eta_2 = v$$

Let $R(\mathrm{x},\mu) = \tilde{R}(\mathrm{x},\mu)\mathrm{x}$, and, consequently, $R_1(\mathrm{x},\mu) = \tilde{R}_1(\mathrm{x},\mu)\mathrm{x}$ and $R_2(\mathrm{x},\mu) = \tilde{R}_2(\mathrm{x},\mu)\mathrm{x})$. After the change of variables we obtain

$$
\begin{aligned}
\dot{\xi}_1 &= \dot{x} + \frac{\partial r}{\partial \mathrm{x}}\dot{\mathrm{x}}\mathrm{x} + r(\mathrm{x},\mu)\dot{\mathrm{x}} = A_1(\mu)x + R_1(\mathrm{x},\mu) + \varphi_2(\mathrm{y},\mu)u \\
&\quad + \frac{\partial r}{\partial \mathrm{x}}\Big(A(\mu)\mathrm{x} + R(\mathrm{x},\mu)\Big)\mathrm{x} + r(\mathrm{x},\mu)\Big(A(\mu)\mathrm{x} + R(\mathrm{x},\mu)\Big) + \dots \\
&= A_1(\mu)\xi_1 + \varphi_2(\eta_1,\eta_2,\mu)\xi_2 + \Bigg[-A_1(\mu)r(\mathrm{x},\mu) + \tilde{R}_1(\mathrm{x},\mu) \\
&\quad + \frac{\partial r}{\partial \mathrm{x}}(A(\mu)\mathrm{x} + \tilde{R}(\mathrm{x},\mu)\mathrm{x}) + r(\mathrm{x},\mu)A(\mu) + r(\mathrm{x},\mu)\tilde{R}(\mathrm{x},\mu)\Bigg]\mathrm{x} + \dots\,, \\
\dot{\xi}_2 &= A_2(\mu)\xi_2 + \hat{R}_2(\xi_1,\xi_2,\mu) + \varphi_4(\eta_1,\eta_2,\mu)\xi_2 + \dots\,, \\
\dot{\eta}_1 &= B_1(\mu)\eta_1 + P_1(\eta_1,\eta_2,\mu) + \hat{\psi}_2(\xi_1,\xi_2,\mu)\eta_2 + \dots\,, \\
\dot{\eta}_2 &= B_2(\mu)\eta_2 + P_2(\eta_1,\eta_2,\mu) + \hat{\psi}_4(\xi_1,\xi_2,\mu)\eta_2 + \dots\,,
\end{aligned}
$$

where

$$
\hat{R}_2(0,0,\mu) \equiv 0, \qquad \frac{\partial \hat{R}_2}{\partial(\xi_1,\xi_2)}(0,0,\mu) \equiv 0\,,
$$

$$
\hat{\psi}_2(0,0,\mu) \equiv 0, \qquad \hat{\psi}_4(0,0,\mu) \equiv 0\,.
$$

Assume that $r(x,\mu)$ is such that the expression inside the square brackets vanishes, *i.e.* assume the following condition holds

$$
\begin{aligned}
&\frac{\partial r}{\partial \mathrm{x}}(A(\mu)\mathrm{x} + \tilde{R}(\mathrm{x},\mu)\mathrm{x}) \\
&\qquad = A_1(\mu)r(\mathrm{x},\mu) - r(\mathrm{x},\mu)A(\mu) - \tilde{R}_1(\mathrm{x},\mu) - r(\mathrm{x},\mu)\tilde{R}(\mathrm{x},\mu)\,.
\end{aligned}
\tag{A.14}
$$

Let us consider a matrix system of differential equations of the form

$$
\begin{aligned}
\dot{\mathrm{x}} &= A(\mu)\mathrm{x} + \tilde{R}(\mathrm{x},\mu)\mathrm{x}\,, \\
\dot{Y} &= A_1(\mu)Y - YA - \tilde{R}_1(\mathrm{x},\mu) - Y\tilde{R}(\mathrm{x},\mu)\,,
\end{aligned}
\tag{A.15}
$$

where $Y \in \mathbb{R}^{m_1 m}$ and $\mathrm{x} \in \mathbb{R}^m$. For all μ sufficiently small this system has an equilibrium state $O_3(0,0)$ whose characteristic exponents comprise the spectrum of the linear operator

$$
\begin{aligned}
&\mathrm{x} \mapsto A(\mu)\mathrm{x}\,, \\
&Y \mapsto A_1(\mu)Y - YA(\mu) - \frac{\partial \tilde{R}_1}{\partial \mathrm{x}}(0,\mu)\mathrm{x}\,.
\end{aligned}
$$

It follows that when $\mu = 0$ the point O_3 has m_1^2 characteristic exponents on the imaginary axis, $(mm_1 - m_1^2)$ and m characteristic exponents to the left and to the right of the imaginary axis, respectively. This implies that for sufficiently small μ system (A.15) possesses an m-dimensional invariant manifold (strongly stable) $Y = r(\mathrm{x}, \mu)$. This gives the existence of the function r which satisfies condition (A.14).

The transformation (3) with such $r(x, \mu)$ brings the system to the form (A.5) with $\varphi_1 \equiv 0$, $\varphi_3 \equiv 0$, $\psi_1 \equiv 0$, $\psi_3 \equiv 0$ and $R_1 \equiv 0$.

Step 4. Making the change of variables (4), we obtain

$$\begin{aligned}
\dot{\xi}_1 &= A_1(\mu)\xi_1 + R_1(\xi_1, \xi_2, \mu) + \hat{\varphi}_2(\eta_1, \eta_2, \mu)\xi_2 + \dots\,, \\
\dot{\xi}_2 &= A_2(\mu)\xi_2 + R_2(\xi_1, \xi_2, \mu) + \hat{\varphi}_4(\eta_1, \eta_2, \mu)\xi_2 + \dots\,, \\
\dot{\eta}_1 &= \dot{y} + \frac{\partial p}{\partial \mathrm{y}}\dot{\mathrm{y}}\mathrm{y} + p(\mathrm{y}, \mu)\dot{\mathrm{y}} = B_1(\mu)y + P_1(\mathrm{y}, \mu) + \psi_2(\mathrm{x}, \mu)v \\
&\quad + \frac{\partial p}{\partial \mathrm{y}}\Big(B(\mu)\mathrm{y} + P(\mathrm{y}, \mu)\Big)\mathrm{y} + p(\mathrm{y}, \mu)\Big(B(\mu)\mathrm{y} + P(\mathrm{y}, \mu)\Big) + \dots \\
&= B_1(\mu)\eta_1 + \psi_2(\xi_1, \xi_2, \mu)\eta_2 + \Bigg[-B_1(\mu)p(\mathrm{y}, \mu) + \tilde{P}_1(\mathrm{y}, \mu) \\
&\quad + \frac{\partial p}{\partial \mathrm{y}}(B(\mu)\mathrm{y} + \tilde{P}(\mathrm{y}, \mu)\mathrm{y}) + p(\mathrm{y}, \mu)B(\mu) + p(\mathrm{y}, \mu)\tilde{P}(\mathrm{y}, \mu)\Bigg]\mathrm{y} + \dots\,, \\
\dot{\eta}_2 &= B_2(\mu)\eta_2 + \hat{P}_2(\eta_1, \eta_2, \mu) + \psi_4(\xi_1, \xi_2, \mu)\eta_2 + \dots\,,
\end{aligned}$$

where $P(\mathrm{y}, \mu) = \tilde{P}(\mathrm{y}, \mu)\mathrm{y}$.

Choose the function p such that the expression inside the square brackets is equal identically to zero, *i.e.*

$$\begin{aligned}
&\frac{\partial p}{\partial \mathrm{y}}(B(\mu)\mathrm{y} + \tilde{P}(\mathrm{y}, \mu)\mathrm{y}) \\
&\qquad = B_1(\mu)p(\mathrm{y}, \mu) - p(\mathrm{y}, \mu)B(\mu) - \tilde{P}_1(\mathrm{y}, \mu) - p(\mathrm{y}, \mu)\tilde{P}(\mathrm{y}, \mu)\,.
\end{aligned} \tag{A.16}$$

Obviously, the system would finally take the desired form.

To assure the existence of such function p, consider a matrix system of differential equations of the form

$$\begin{aligned}
\dot{X} &= B_1(\mu)X - XB(\mu) - \tilde{P}_1(\mathrm{y}, \mu) - X\tilde{P}(\mathrm{y}, \mu)\,, \\
\dot{\mathrm{y}} &= B(\mu)\mathrm{y} + \tilde{P}(\mathrm{y}, \mu)\mathrm{y}\,,
\end{aligned} \tag{A.17}$$

where $X \in \mathbb{R}^{n_1 n}$ and $\mathrm{y} \in \mathbb{R}^n$. For all μ sufficiently small this system possesses the equilibrium state $O_4(0,0)$ whose characteristic exponents comprise the spectrum of the linear operator

$$\begin{aligned} &X \mapsto B_1(\mu)X - XB(\mu) - \frac{\partial \tilde{P}_1}{\mathrm{y}}(0,\mu)\mathrm{y}\,, \\ &\mathrm{y} \mapsto B(\mu)\mathrm{y}\,. \end{aligned}$$

When $\mu = 0$ the characteristic exponents of O_4 are as follow: n_1^2 eigenvalues lie on the imaginary axis, $(n_1 n - n_1^2)$ and n eigenvalues lie to the left and to the right of the imaginary axis, respectively. Therefore, for sufficiently small μ the system (A.17) possesses an m-dimensional invariant manifold W_4^{uu} (strongly unstable) of the form $X = p(y,\mu)$ where the function p satisfies condition (A.16). This completes the proof.

Appendix B

FIRST ORDER ASYMPTOTIC FOR THE TRAJECTORIES NEAR A SADDLE FIXED POINT

Consider a family of $\mathbb{C}^r$-smooth ($r \geq 2$) maps $T(\mu)$ of R^{m+n} in a neighborhood of a saddle fixed point with m-dimensional stable and n-dimensional unstable invariant manifolds.

Let the multipliers of the saddle be $(\lambda_1, \ldots, \lambda_m)$ and $(\gamma_1, \ldots, \gamma_n)$ where $|\lambda_k| < 1$ ($k = 1, \ldots, m$) and $|\gamma_k| > 1$ ($k = 1, \ldots, n$). Assume that the multipliers $(\lambda_1, \ldots, \lambda_{m_1})$ are equal in absolute values to some λ, $0 < \lambda < 1$ and the absolute values of the rest of the stable multipliers $(\lambda_{m_1+1}, \ldots, \lambda_m)$ are strictly less than λ. Concerning the unstable multipliers we assume that $|\gamma_1| = \cdots = |\gamma_{n_1}| = \gamma > 1$ and $|\gamma_k| > \gamma$ at $k > n_1$.

Absolutely analogously to systems near an equilibrium state (see Appendix A), the map $T(\mu)$ may be brought to the following form (by a $\mathbb{C}^{r-1}$ change of variables)

$$\begin{aligned}
\bar{x} &= A_1(\mu)x + f_{11}(x, y, v, \mu)x + f_{12}(x, u, y, v, \mu)u\,, \\
\bar{u} &= A_2(\mu)u + f_{21}(x, y, v, \mu)x + f_{22}(x, u, y, v, \mu)u\,, \\
\bar{y} &= B_1(\mu)y + g_{11}(x, u, y, \mu)y + g_{12}(x, u, y, v, \mu)v\,, \\
\bar{v} &= B_2(\mu)v + g_{21}(x, u, y, \mu)y + f_{22}(x, u, y, v, \mu)v\,,
\end{aligned} \tag{B.1}$$

where the eigenvalues of $A_1(0)$ are $(\lambda_1, \ldots, \lambda_{m_1})$, the eigenvalues of $A_2(0)$ are $(\lambda_{m_1+1}, \ldots, \lambda_m)$, those of $B_1(0)$ are $(\gamma_1, \ldots, \gamma_{n_1})$ and those of $B_2(0)$ are $(\gamma_{n_1+1}, \ldots, \gamma_n)$. Moreover, the $\mathbb{C}^{r-1}$-functions[1] f_{ij} and g_{ij} satisfy

[1]They have continuous derivatives with respect to all variables and μ up to the order $(r-1)$, except for the last $(r-1)$-th derivative with respect to μ alone which may not exist.

$$\begin{aligned} f_{ij}(0,0,0,0,\mu) &= 0, & g_{ij}(0,0,0,0,\mu) &= 0\,, \\ f_{11}(x,0,0,\mu) &\equiv 0, & g_{11}(0,0,y,\mu) &\equiv 0\,, \\ f_{12}(x,u,0,0,\mu) &\equiv 0, & g_{12}(0,0,y,v,\mu) &\equiv 0\,, \\ f_{j1}(0,y,v,\mu) &\equiv 0, & g_{j1}(x,u,0,\mu) &\equiv 0\,. \end{aligned} \tag{B.2}$$

We pay special attention to the reduction to this form because it enables one to obtain good estimates for the solutions of the boundary-value problem (see Sec. 3.7) near the saddle fixed point. Namely, let the functions $\xi_k^{1,2}$, $\eta_k^{1,2}$ define the solution of the boundary-value problem: the point (x^1,u^1,y^1,v^1) is the image of the point (x^0,u^0,y^0,v^0) by the map $T(\mu)^k$ (acting in a small neighborhood of the origin) if and only if $(x^1,u^1) = (\xi_k^1,\xi_k^2)(x^0,u^0,y^1,v^1)$ and $(y^0,v^0) = (\eta_k^1,\eta_k^2)(x^0,u^0,y^1,v^1)$. Let $\lambda_0(\mu)$ and $\gamma_0(\mu)$ be such that for all $j \geq 0$

$$\|A_1(\mu)^j\| \leq \text{const} \cdot \lambda_0(\mu)^j, \qquad \|B_1(\mu)^{-j}\| \leq \text{const} \cdot \gamma_0(\mu)^{-j}\,. \tag{B.3}$$

For example, when there is only one stable leading multipliers ($m_1 = 1$ and λ_1 is real), then $\lambda_0(\mu) = \lambda_1(\mu)$; if there is a pair of complex-conjugate stable leading multipliers ($m_1 = 2$ and $\lambda_1 = \lambda_2^*$ is not real), then $\lambda_0(\mu) = \operatorname{Re}\lambda_1(\mu)$. Analogously, $\gamma_0(\mu) = \gamma_1(\mu)$ if $n_1 = 1$; and $\gamma_0(\mu) = \operatorname{Re}\lambda_1(\mu)$ if $n_1 = 2$ and $\gamma_1 = \gamma_2^*$ is not real.

Since A and B depend smoothly on μ, we also have

$$\left\|\frac{\partial^q}{\partial\mu^q}\left(A_1(\mu)^j\right)\right\| \leq \text{const} \cdot j^q \lambda_0(\mu)^j, \quad \left\|\frac{\partial^q}{\partial\mu^q}\left(B_1(\mu)^{-j}\right)\right\| \leq \text{const} \cdot j^q \gamma_0(\mu)^{-j} \tag{B.4}$$

at $q = 1,\ldots,r-1$.

Let us also introduce the quantities λ' and γ', satisfying $\lambda_0^2 < \lambda' < \lambda_0$ and $\gamma_0 < \gamma' < \gamma_0^2$, such that for all $j \geq 0$

$$\|A_2(\mu)^j\| \leq \text{const} \cdot (\lambda')^j, \qquad \|B_2(\mu)^{-j}\| \leq \text{const} \cdot (\gamma')^{-j}\,, \tag{B.5}$$

and the same estimates hold true for all the derivatives with respect to μ.

Lemma B.1. *If identities* (B.2) *hold, then*

$$\xi_k^1 = A_1(\mu)^k x^0 + o(\lambda_0(\mu)^k), \qquad \eta_k^1 = B_1(\mu)^{-k} y^1 + o(\gamma_0(\mu)^{-k})\,, \tag{B.6}$$

$$\xi_k^2 = o(\lambda_0(\mu)^k), \qquad \eta_k^2 = o(\gamma_0(\mu)^{-k})\,. \tag{B.7}$$

where the terms $o(\lambda_0^k)$ and $o(\gamma_0^{-k})$ are $\mathbb{C}^{r-1}$-smooth and all their derivatives with respect to (x^0, u^0, y^1, v^1) are also of order $o(\lambda_0^k)$ and $o(\gamma_0^{-k})$ respectively, the derivatives which involve differentiation q times with respect to μ are estimated, respectively, as $o(k^q\lambda_0^k)$ and $o(k^q\gamma_0^{-k})$ $(q = 0, \ldots, r-2)$.

Proof. Let us denote

$$f_i = f_{i1}x + f_{i2}u \quad \text{and} \quad g_i = g_{i1}y + g_{i2}v\,. \tag{B.8}$$

It is sufficient to show (see Sec. 3.7) that the solution $\{(x_0, u_0, y_0, v_0), \ldots, (x_k, u_k, y_k, v_k)\}$ of the system

$$\begin{aligned}
x_j &= A_1^j x^0 + \sum_{s=0}^{j-1} A_1^{j-s-1} f_1(x_s, u_s, y_s, v_s, \mu)\,, \\
u_j &= A_2^j u^0 + \sum_{s=0}^{j-1} A_2^{j-s-1} f_2(x_s, u_s, y_s, v_s, \mu)\,, \\
y_j &= B_1^{-(k-j)} y^1 - \sum_{s=j}^{k-1} B_1^{-(s+1-j)} g_1(x_s, u_s, y_s, v_s, \mu)\,, \\
v_j &= B_2^{-(k-j)} v^1 - \sum_{s=j}^{k-1} B_2^{-(s+1-j)} g_2(x_s, u_s, y_s, v_s, \mu)
\end{aligned} \tag{B.9}$$

satisfies, given small (x^0, u^0, y^1, v^1), the following estimates:

$$\begin{aligned}
\|x_j - A_1^j x^0\| &\le \lambda_0^j \varphi_1(k)\,, \\
\|u_j\| &\le \lambda_0^j \varphi_2(j)\,, \\
\|y_j - B_1^{-(k-j)} y^1\| &\le \gamma_0^{-(k-j)} \psi_1(k)\,, \\
\|v_j\| &\le \gamma_0^{-(k-j)} \psi_2(k-j)
\end{aligned} \tag{B.10}$$

where φ_i and ψ_i are some positive sequences tending to zero.

Moreover, the analogous estimates must hold for all derivatives of the expressions in the left-hand side of (B.10) with respect to $(x^0, u^0, y^1, v^1, \mu)$ while φ_i and ψ_i may depend on the order of the derivative.

As we showed in Sec. 3.7, the solution of (B.9) is the limit of successive approximations $\{(x_0^{(n)}, u_0^{(n)}, y_0^{(n)}, v_0^{(n)}), \ldots, (x_k^{(n)}, u_k^{(n)}, y_k^{(n)}, v_k^{(n)})\}$ $(n \to +\infty)$

computed as

$$
\begin{aligned}
x_j^{(n+1)} &= A_1^j x^0 + \sum_{s=0}^{j-1} A_1^{j-s-1} f_1(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\,,\\
u_j^{(n+1)} &= A_2^j u^0 + \sum_{s=0}^{j-1} A_2^{j-s-1} f_2(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\,,\\
y_j^{(n+1)} &= B_1^{-(k-j)} y^1 - \sum_{s=j}^{k-1} B_1^{-(s+1-j)} g_1(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\,,\\
v_j^{(n+1)} &= B_2^{-(k-j)} v^1 - \sum_{s=j}^{k-1} B_2^{-(s+1-j)} g_2(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\,,
\end{aligned}
\tag{B.11}
$$

starting with the initial guess $(x_0^{(1)}, u_0^{(1)}, y_0^{(1)}, v_0^{(1)}) = 0$.

Thus, to prove some estimates on the solution of (B.9), we may assume that the n-th successive approximation satisfies these estimates and then, based on this assumption, we must check that the $(n+1)$-th approximation satisfies them too; of course, the estimators must be independent on n.

We have already proved in this way (see Lemma 3.3) that for any $\bar{\lambda} > \lambda_0$ and $\bar{\gamma} < \gamma_0$

$$
\|x_j, u_j\| \leq K\bar{\lambda}^j, \qquad \|y_j, v_j\| \leq K\bar{\gamma}^{j-k} \tag{B.12}
$$

where K is some positive constant (which depends on the specific choice of $\bar{\lambda}$ and $\bar{\gamma}$). Let us now check, that fulfillment of the identities (B.2) allows one to improve these estimates: namely, one can put $\bar{\lambda} = \lambda_0$ and $\bar{\gamma} = \gamma_0$ in (B.12).

Indeed, assume that the n-th approximation satisfies

$$
\begin{aligned}
\|x_j^{(n)}\| \leq K_x \lambda_0^j, &\quad \|u_j^{(n)}\| \leq K_u \lambda_0^j\,,\\
\|y_j^{(n)}\| \leq K_y \gamma_0^{j-k}, &\quad \|v_j^{(n)}\| \leq K_v \gamma_0^{j-k}\,.
\end{aligned}
\tag{B.13}
$$

We must verify that with the appropriate choice of the constants K_x, K_u, K_y, K_v the $(n+1)$-th approximation satisfies

$$
\begin{aligned}
\|x_j^{(n+1)}\| \leq K_x \lambda_0^j, &\quad \|u_j^{(n+1)}\| \leq K_u \lambda_0^j\,,\\
\|y_j^{(n+1)}\| \leq K_y \gamma_0^{j-k}, &\quad \|v_j^{(n+1)}\| \leq K_v \gamma_0^{j-k}\,.
\end{aligned}
\tag{B.14}
$$

Plugging (B.12), (B.13) into (B.11) gives

$$\begin{aligned}
\|x_j^{(n+1)}\| &\le \lambda_0^j\varepsilon + \sum_{s=0}^{j-1} \lambda_0^{j-s-1}(\delta K_x^2\lambda_0^{2s} + CK_u\gamma_0^{s-k}\lambda_0^s)\,, \\
\|u_j^{(n+1)}\| &\le (\lambda')^j\varepsilon + \sum_{s=0}^{j-1}(\lambda')^{j-s-1}(CK^2\bar\lambda^{2s} + \delta K_u\lambda_0^s)\,.
\end{aligned} \tag{B.15}$$

Here, C is some constant, ε bounds the norm of (x^0, u^0, y^1, v^1). Note that identities (B.2) were taken into account: we estimate $\|f_{22}\|$ by a constant δ which may be made arbitrarily small by decreasing the size of the neighborhood of the saddle fixed point under consideration, and $\|f_{21}\|$ is estimated as $\|f_{21}(x,y,v)\| \le \|f_{21}(0,y,v)\| + \sup\|f'_{21x}\| \cdot \|x\| \le C\|x\|$. In the same way we have $\|f_{11}(x,y,v)\| \le \sup\|f'_{11x}\| \cdot \|x\|$ and, since $f'_{11x} \equiv 0$ at $(y,v) = 0$, it follows that $\|f_{11}(x,y,v)\| \le \delta\|x\|$ where δ may be taken arbitrarily small. For the function f_{12} identities (B.2) imply $\|f_{12}\| \le C\|y,v\|$.

We have from (B.15) that $\|x_j^{(n+1)}\| \le \lambda_0^j\varepsilon + \lambda_0^{j-1}\delta K_x^2/(1-\lambda_0) + \lambda_0^{j-1}CK_u\gamma_0^{j-k}/(\gamma_0 - 1)$, and $\|u_j^{(n+1)}\| \le (\lambda')^j\varepsilon + (\lambda')^jCK^2/(\lambda' - \bar\lambda^2) + \lambda_0^j\delta K_u/(\lambda_0 - \lambda')$ from which the estimate (B.14) for $(x,u)_j^{(n+1)}$ follows, provided K_x and K_u are chosen such that

$$\begin{aligned}
K_x &\ge \varepsilon + \frac{\delta K_x^2}{\lambda_0(1-\lambda_0)} + \frac{CK_u}{\lambda_0(\gamma_0 - 1)}\,, \\
K_u &\ge \varepsilon + \frac{CK^2}{\lambda' - \bar\lambda^2} + K_u\frac{\delta}{\lambda_0 - \lambda'}\,.
\end{aligned}$$

The required estimates for $(y,v)_j^{(n+1)}$ are obtained in the same way, due to the symmetry of the problem.

Thus, the solution of (B.11) (as well as all the successive approximations) satisfies

$$(x_j, u_j) = O(\lambda_0^j), \quad (y_j, v_j) = O(\gamma_0^{-(k-j)})\,. \tag{B.16}$$

Let us now assume that the n-th approximation satisfies (B.10). Based on identities (B.2), the function f_1 is estimated as

$$\|f_1\| \le \sup_x \|f'_{11x}\| \cdot \|x\|^2 + \sup\|f'_{12(y,v)}\| \cdot \|u\| \cdot \|y,v\|\,. \tag{B.17}$$

Since $f'_{11x} \to 0$ as $(y,v) \to 0$, it follows from (B.16) and from the assumed validity of (B.10), that on the n-th approximation

$$\|f_1(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\| \le \tilde\varphi_1(k-s)\lambda_0^{2s} + C\gamma_0^{s-k}\lambda_0^s\varphi_2(s) \tag{B.18}$$

where C is some constant, φ_2 is an estimator for u in (B.10) and $\tilde{\varphi}_1$ is a positive function (independent on the choice of φ_i and ψ_i in (B.10)) which tends to zero as $k-s \to +\infty$.

Analogously,

$$\|f_2(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\| \leq \tilde{\varphi}_2(s)\lambda_0^s + \delta\varphi_2(s)\lambda_0^s \tag{B.19}$$

where δ may be taken as small as necessary by decreasing the size of the neighborhood of the saddle, and $\tilde{\varphi}_2(s) \to 0$ as $s \to +\infty$ ($\tilde{\varphi}_2$ gives an upper bound for $\|f_{21}\|$ at fixed $x = x_s^{(n)}$; by (B.2) it tends to zero as $x \to 0$).

By (B.18), (B.19) we obtain, respectively,

$$\left\| \sum_{s=0}^{j-1} \lambda_0^{-s} f_1(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu) \right\| \leq \sum_{s=0}^{j-1} \lambda_0^s \tilde{\varphi}_1(k-s) + C\sum_{s=0}^{j-1} \gamma_0^{s-k}\varphi_2(s)$$

and

$$\left\| \sum_{s=0}^{j-1} (\lambda')^{-s} f_2(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu) \right\| \leq \sum_{s=0}^{j-1} \left(\frac{\lambda_0}{\lambda'}\right)^s [\tilde{\varphi}_2(s) + \delta\varphi_2(s)] \,.$$

Thus (see (B.11)), $(x,u)^{(n+1)}$ will satisfy (B.10) if

$$\varphi_1(k) = \sum_{s=0}^{k-1} \lambda_0^s \tilde{\varphi}_1(k-s) + C\sum_{s=0}^{k-1} \gamma_0^{s-k}\varphi_2(s) \tag{B.20}$$

and

$$\varphi_2(j) = \left(\frac{\lambda'}{\lambda_0}\right)^j (\varepsilon + \frac{1}{\lambda'}\sum_{s=0}^{j-1} \left(\frac{\lambda_0}{\lambda'}\right)^s [\tilde{\varphi}_2(s) + \delta\varphi_2(s)]) \,. \tag{B.21}$$

It is well known, that the sum of the kind

$$\sum_{s=0}^{k-1} \alpha^s \varphi(k-s)$$

tends to zero as $k \to +\infty$ for any $\alpha < 1$ and any sequence φ which tends to zero as $k-s \to +\infty$. Therefore, Eq. (B.20) defines indeed a converging to zero sequence $\varphi_1(k)$ provided $\varphi_2(s)$ tends to zero as $s \to +\infty$.

The sequence $\varphi_2(j)$ is given by (B.21) which can be rewritten as

$$\varphi_2(j+1) = \frac{\lambda'}{\lambda_0}\left(1 + \frac{\delta}{\lambda'}\right) \cdot \varphi_2(j) + \frac{1}{\lambda_0} \cdot \tilde{\varphi}_2(j) \,.$$

Since $\frac{\lambda'}{\lambda_0}\left(1+\frac{\delta}{\lambda'}\right)<1$ for sufficiently small δ and since $\tilde{\varphi}_2(j)\to 0$ as $j\to\infty$, it follows from this formula that $\varphi_2(j)$ tends to zero indeed.

By the symmetry of the problem, appropriate functions ψ_1 and ψ_2 are found in an absolutely analogous way. We have proved the estimates (B.10). To complete the lemma we need to show that analogous estimates hold for all derivatives of the solution (x_j,u_j,y_j,v_j) of (B.9).

It is shown in Sec. 3.7 that the successive approximations converge to the solution of the boundary-value problem along with all derivatives. Thus, we may *assume* that the n-th approximation satisfies[2]

$$\begin{aligned}
&\|D_p x_j^{(n)} - D_p\left(A_1(\mu)^j x^0\right)\| \le k^{p_2}\lambda_0^j\varphi_1^{(p)}(k)\,,\\
&\|D_p u_j^{(n)}\| \le k^{p_2}\lambda_0^j\varphi_2^{(p)}(j)\,,\\
&\|D_p y_j^{(n)} - D_p\left(B_1(\mu)^{-(k-j)} y^1\right)\| \le k^{p_2}\gamma_0^{-(k-j)}\psi_1^{(p)}(k)\,,\\
&\|D_p v_j^{(n)}\| \le k^{p_2}\gamma_0^{-(k-j)}\psi_2^{(p)}(k-j)\,,
\end{aligned}\tag{B.22}$$

for some converging to zero sequences $\varphi_{1,2}$ and $\psi_{1,2}$ which are independent of n but may depend on the order $|p|$ of the derivative. Then, based on this assumption, we must show that the derivatives of the next approximation $\{(x_j^{(n+1)},u_j^{(n+1)},y_j^{(n+1)},v_j^{(n+1)})\}_{j=0}^k$ satisfy the same estimates.

In fact, we need to check these estimates only for $x_j^{(n+1)}$ and $u_j^{(n+1)}$; the analogous conclusion concerning $y_j^{(n+1)}$ and $v_j^{(n+1)}$ will follow from the symmetry of the problem.

The differentiation of (B.11) gives

$$D_p x_j^{(n+1)} = D_p\left(A_1(\mu)^j x^0\right) + \sum_{s=0}^{j-1} D_p\left(A_1(\mu)^{j-s-1} f_1(x_s^{(n)},u_s^{(n)},y_s^{(n)},v_s^{(n)},\mu)\right),$$

$$D_p u_j^{(n+1)} = D_p\left(A_2(\mu)^j u^0\right) + \sum_{s=0}^{j-1} D_p\left(A_2(\mu)^{j-s-1} f_2(x_s^{(n)},u_s^{(n)},y_s^{(n)},v_s^{(n)},\mu)\right).$$

[2]We use a notation $D_p=\frac{\partial^{p_1+p_2}}{\partial(x^0,u^0,y^1,v^1)^{p_1}\partial\mu^{p_2}}$ (here $p=(p_1,p_2)$).

By (B.4), (B.5) we have

$$\begin{aligned}
&\|D_p x_j^{(n+1)} - D_p\left(A_1(\mu)^j x^0\right)\| \le \text{const}\cdot\lambda_0^{j-1} \sum_{p_1'=p_1,\, p_2'=0,\dots,p_2} k^{p_2-p_2'} \\
&\qquad \times \sum_{s=0}^{j-1} \lambda_0^{-s} \left\| D_{p'} f_1(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\right\|, \\
&\|D_p u_j^{(n+1)}\| \le \text{const}\cdot(\lambda')^j \Bigg[1 + \sum_{p_1'=p_1,\, p_2'=0,\dots,p_2} \sum_{s=0}^{j-1} (\lambda')^{-s} \\
&\qquad \times \left\| D_{p'} f_2(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\right\|\Bigg].
\end{aligned} \tag{B.23}$$

Now, in the same way as before, to prove the lemma we must check that the estimates analogous to (B.18) and (B.19) hold for the derivatives $D_p f_{1,2}$ for any p:

$$\|D_p f_1(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\| \le [\beta_1(k-s)\lambda_0^{2s} + \beta_2(s)\gamma_0^{s-k}\lambda_0^s] k^{p_2} \tag{B.24}$$

and

$$\|D_p f_2(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\| \le [\beta_3(s) + \delta\varphi_2^{(p)}(s)]\lambda_0^s k^{p_2} \tag{B.25}$$

where δ may be taken arbitrarily small by decreasing the size of the neighborhood of the saddle fixed point under consideration and $\beta_{1,2,3}$ are some sequences converging to zero; moreover, β_3 is independent of the specific choice of the estimators $\varphi_{1,2}^{(p)}$ and $\psi_{1,2}^{(p)}$ in (B.22), and $\beta_{1,2}$ are independent of $\varphi_1^{(p)}$ and $\psi_1^{(p)}$ (nevertheless, $\beta_{1,2,3}$ may depend on φ and ψ corresponding to the derivatives of lower orders).

By the chain rule, the derivatives $D_p f_i(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)$ are estimated by the sum

$$\begin{aligned}
&\text{const}\cdot \sum_{q_1,q_2,q_3} \left\| \frac{\partial^{q_1+q_2+q_3} f_i}{\partial(x,u)^{q_1}\partial(y,v)^{q_2}\partial\mu^{q_3}}(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)\right\| \\
&\qquad \times \|D_{l_1}(x_s^{(n)}, u_s^{(n)})\| \cdots \|D_{l_{q_1}}(x_s^{(n)}, u_s^{(n)})\| \\
&\qquad \times \|D_{l_{q_1+1}}(y_s^{(n)}, v_s^{(n)})\| \cdots \|D_{l_{q_1+q_2}}(y_s^{(n)}, v_s^{(n)})\|
\end{aligned} \tag{B.26}$$

where $q_{1,2,3}$ are nonnegative integers such that $q_1+q_2+q_3 \le p_1+p_2$ and l's are pairs of nonnegative integers such that $l_{11}+\cdots+l_{q_1+q_2,1} = p_1$ and $l_{12}+\cdots+l_{q_1+q_2,2}+q_3 = p_2$.

By assumption, the estimates for the derivatives $\|D_l u_s^{(n)}\|$ and $\|D_l v_s^{(n)}\|$ are given by (B.22). Since φ_1 and ψ_1 are independent of j, there exists a constant C independent of the specific choice of φ and ψ such that when (B.22) is fulfilled

$$\|D_l x_s^{(n)}\| \leq C\lambda_0^s k^{l_2}, \qquad \|D_l y_s^{(n)}\| \leq C\gamma_0^{-(k-s)} k^{l_2} \tag{B.27}$$

for all sufficiently large k.

Thus, the estimate (B.26) is rewritten as

$$\begin{aligned} &\text{const} \cdot \sum_{q_1,q_2,q_3} \left\| \frac{\partial^{q_1+q_2+q_3} f_i}{\partial(x,u)^{q_1} \partial(y,v)^{q_2} \partial\mu^{q_3}} (x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu) \right\| \\ &\quad \times \lambda_0^{q_1 s} \gamma_0^{q_2(s-k)} k^{p_2 - q_3}\,. \end{aligned} \tag{B.28}$$

Obviously, in the estimate for f_1, the terms with $q_1 \geq 2$ and $q_2 \geq 1$ fit (B.24), and all terms with $q_1 \geq 2$ in the estimate for f_2 fit (B.25). Note also, that

$$\frac{\partial^{q_2+q_3} f_i}{\partial(y,v)^{q_2} \partial\mu^{q_3}} \equiv \frac{\partial^{q_2+q_3} f_{i1}}{\partial(y,v)^{q_2} \partial\mu^{q_3}} \cdot x_s^{(n)} + \frac{\partial^{q_2+q_3} f_{i2}}{\partial(y,v)^{q_2} \partial\mu^{q_3}} \cdot u_s^{(n)} = o(\lambda_0^s)$$

(we use (B.10), (B.16) and the identities (B.2) which give that $\frac{\partial^{q_2+q_3} f_{i1}}{\partial(y,v)^{q_2} \partial\mu^{q_3}} \to 0$ as $x \to 0$). Hence, the terms with $q_1 = 0$ and $q_2 \geq 1$ in the estimate (B.28) for f_1, and all terms with $q_1 = 0$ in the estimate for f_2 also fit (B.24) and (B.25), respectively.

The case $q_1 = 0$, $q_2 = 0$ corresponds to the differentiation with respect to μ alone (*i.e.* $p_1 = 0$ and $p_2 = q_3$). Recall that the $(r-1)$-th derivative with respect to μ may not exist, therefore we must estimate the derivatives $\frac{\partial^{q_3} f_1}{\partial\mu^{q_3}}$ only at $q_3 \leq r-2$. These derivatives are smooth with respect to (x,u,y,v), therefore we may write (using that $f_1 \equiv 0$ at $(y,v) = 0$; see (B.2))

$$\left\| \frac{\partial^{q_3} f_1}{\partial\mu^{q_3}} \right\| \leq \|y,v\| \cdot \sup \left\| \frac{\partial}{\partial(y,v)} \frac{\partial^{q_3} f_1}{\partial\mu^{q_3}} \right\|.$$

Thus, the term under consideration is estimated exactly like other terms with $q_1 = 0$.

The last remaining terms to examine in (B.26) are ($q_1 = 1$)

$$\left\| \frac{\partial}{\partial x} \frac{\partial^{q_2+q_3} f_i}{\partial(y,v)^{q_2} \partial\mu^{q_3}} \right\| \cdot \gamma_0^{q_2(s-k)} \lambda_0^s k^{p_2 - q_3}$$

and
$$\left\| \frac{\partial}{\partial u} \frac{\partial^{q_2+q_3} f_i}{\partial (y,v)^{q_2} \partial \mu^{q_3}} \right\| \cdot \gamma_0^{q_2(s-k)} o(\lambda_0^s) k^{p_2-q_3} \,.$$

Note that $f'_{ix} \to 0$ as $(x,u) \to 0$ (see (B.2)). Hence, both the terms above are estimated by $\gamma_0^{q_2(s-k)} o(\lambda_0^s) k^{p_2}$; *i.e.* they fit (B.25) and, if $q_2 \geq 1$, they fit (B.24).

It remains to consider the case $q_1 = 1$, $q_2 = 0$ for f_1. To satisfy (B.24) we have to show that
$$\left\| \frac{\partial}{\partial (x,u)} \frac{\partial^{q_3} f_1}{\partial \mu^{q_3}} \right\| \cdot \lambda_0^{-s}$$
tends to zero as $k - s \to +\infty$, but this obviously follows from (B.16) because $f_1 = f_{11} x + f_{12} y$ and both f_{1i} vanish at $(y,v) = 0$ (see (B.2)).

We have proved that the derivatives $D_p f_i(x_s^{(n)}, u_s^{(n)}, y_s^{(n)}, v_s^{(n)}, \mu)$ satisfy estimates (B.24) and (B.25). Note that for the derivatives of $x_s^{(n)}$ and $y_s^{(n)}$ we used only estimates (B.27) which are independent of the choice of φ and ψ in (B.22). Thus, the estimators $\beta_{1,2}$ in (B.24) are independent of φ_1 and ψ_1 indeed. The only terms in (B.26) which might give a contribution in (B.25) dependent on $\varphi_{1,2}^{(p)}$ and $\psi_{1,2}^{(p)}$ are
$$\|f'_{2u}\| \cdot \|D_p u_s^{(n)}\| \quad \text{ and } \quad \|f'_{2v}\| \cdot \|D_p v_s^{(n)}\| \,.$$

The first term here is estimated as $\delta \lambda_0^s \varphi_2^{(p)}(s) k^{p_2}$ where δ may be taken arbitrarily small. The second term is estimated as
$$k^{p_2} \psi_2^{(p)}(k-s) \cdot (\|f'_{21v}\| \|x_s^{(n)}\| + \|f'_{22v}\| \|u_s^{(n)}\|)$$
which gives, for sufficiently large k, the estimate $o(\lambda_0^s) k^{p_2}$ (see (B.16),(B.2)) independently of the choice of $\varphi_{1,2}^{(p)}$ and $\psi_{1,2}^{(p)}$. All this is in a complete agreement with (B.25).

Now, the validity of estimates (B.22) for the next approximation $\{(x_j^{(n+1)}, u_j^{(n+1)}, y_j^{(n+1)}, v_j^{(n+1)})\}_{j=0}^k$ follows from (B.24), (B.25) exactly in the same way like the validity of (B.10) follows from (B.18), (B.19). The lemma is proved.

Remark. In the same way, slightly better estimates where $o(\lambda_0^k)$ and $o(\gamma_0^{-k})$ terms are replaced, respectively, by $O((\lambda')^k)$ and $O((\gamma')^{-k})$ in (B.6) and (B.7) may be obtained for the functions ξ, η and their derivatives up to the order $(r-2)$ in case the map is at least $\mathbb{C}^3$-smooth (see Gonchenko and Shilnikov [27]).

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