Question I. Determine if the following systems are topologically equivalent or not:

$$\text{(a)} \begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = x(x^2+y^2) \\ \dot{y} = -y(x^2+y^2) \end{cases}$$

$$\text{(b)} \begin{cases} \dot{x} = x+1 \\ \dot{y} = -y \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = (x+1)(x^2+y^2) \\ \dot{y} = -y(x^2+y^2) \end{cases}$$

$$\text{(c)} \begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

$$\text{(d)} \begin{cases} \dot{x} = y \\ \dot{y} = x - x^2 \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = y \\ \dot{y} = x^2 - x^3 \end{cases}$$

Solutions (5 points each, all unseen). I(a): The right-hand sides of the equations coincide up to a multiplication to a non-negative scalar that vanish only at the equilibrium state. Therefore they have identical sets of phase curves, hence they are topologically equivalent.

I(b). First system has one equilibrium (at $(x=-1, y=0)$ the second one has two equilibria (at $(x=-1, y=0)$ and at at $(x=0, y=0)$). Hence, they are not topologically equivalent.

I(c). The first system has no periodic solutions (it is a linear saddle), the second system (a harmonic oscillator) has infinitely many periodic solutions. Hence, there is no topological equivalence.

I(d). First system has an integral $H = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$. The solution that corresponds $H=0$ and $x<0, y<0$ satisfies the equation $\dot{x} = y = x\sqrt{1+2|x|/3}$, and it tends to infinity as t grows. The second system has an integral $H = \frac{y^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}$, all level sets of which are bounded, so this system has no unbounded solutions. Hence, there is no topological equivalence.

Question II. (a) Prove that given any $(n \times n)$-matrix $A(t)$ and an n-vector $b(t)$ that depend continuously on t, every solution $x(t)$ of the equation

$$\frac{dx}{dt} = A(t)x + b(t), \qquad x \in R^n,$$

is defined for all $t \in (-\infty, +\infty)$.

(b) Let a bounded region U be defined by condition $F(x) < 0$ where $F : R^n \to R^1$ is a smooth scalar function. The boundary ∂U of the region U is given by $F(x) = 0$. Assume that the right-hand side f of the system $\frac{dx}{dt} = f(x), \quad x \in R^n$, is a smooth function such that

$$F'(x) \cdot f(x) < 0$$

everywhere on ∂U. Prove that for every $x_0 \in U$ the solution of the system that starts at the point $x = x_0$ at $t = 0$ exists for all $t \geq 0$.

(c) Consider a differential equation on the straight line $\frac{dx}{dt} = f(x), \quad x \in R^1$. Prove that every bounded solution of this system tends to an equilibrium state as $t \to +\infty$.

Solutions (a- 6 points, b,c - 7 points each, all seen or seen similar). II(a): Define $u = x^2$, note that u is a nonnegative scalar. We have

$$\frac{du}{dt} = 2x \cdot \frac{dx}{dt} = 2x \cdot A(t)x + 2x \cdot b(t),$$

so

$$\frac{du}{dt} \leq 2\|A(t)\|\|x\|^2 + 2\|x\| \ \|b(t)\| = 2\|A(t)\|u + 2\|b(t)\|\sqrt{u} \leq (2\|A(t)\| + \|b(t)\| + 1)u.$$

By comparison principle, $u(t) \leq v(t)$ at $t \geq 0$ where v solves

$$\frac{dv}{dt} = (2\|A(t)\| + \|b(t)\| + 1)v,$$

i.e.

$$x^2(t) = u(t) \leq C \exp[\int_0^t (2\|A(s)\| + \|b(s)\| + 1)ds].$$

Thus, $x(t)$ cannot tend to infinity at a finite positive time. By the change $t \to -t$ we obtain an equation of the same form, so $x(t)$ cannot tend to infinity at any finite negative time too. Hence, $x(t)$ remains defined for all t.

II(b) For any initial condition x_0 on the boundary of U, we have $\frac{d}{dt}F(x(t)) = F'(x) \cdot f(x) < 0$, hence $F(x_t) < F(x_0) = 0$ for $t > 0$ small enough, and $F(x_t) > 0$ at $t < 0$ small enough, i.e. the orbit of x_0 must enter U as t grows and get outside of U as t decreases. In particular, it shows that once the phase point is inside U its forward orbit cannot leave U: to do this, it must hit the boundary, which would mean, as we just proved, that the orbit was outside of U before, a contradiction. Hence the orbit cannot tend to infinity, hence it exists for all $t \geq 0$.

II(c). If the initial condition is at an equilibrium state, then it remains at the equilibrium state and there is nothing to prove. If the initial condition x_0 for an orbit x_t is not at the equilibrium state, i.e. $f(x_0) \neq 0$, then $f(x_t) \neq 0$ for all times (if $f(x_\tau) = 0$ at some time τ, that would mean that the point x_τ is an equilibrium, but then x_t would be a constant function of t, so $f(x_t)$ will also be a constant, hence $f(x_0) = f(x_\tau) = 0$, a contradiction). Thus, $dx/dt = f(x_t)$ will keep a constant sign for all t, i.e. x_t will be a monotone function. So, if x_t is bounded, there exists a limit $x^* = \lim_{t\to+\infty} x_t$. If x^* is not an equilibrium, then $f(x^*) \neq 0$. By continuity of f we have $f(x^*) = \lim_{t\to+\infty} f(x_t) = \lim_{t\to+\infty} \frac{dx}{dt}$, so $f(x^*) \neq 0$ implies dx/dt stays bounded away from zero for all large t. This implies at least linear increase (or decrease) of x_t with time, which contradicts to the assumption x_t is bounded. Hence, the limit x^* of x_t must be an equilibrium state.

Question III. Consider the following system on a plane

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \\ \dfrac{dy}{dt} = x + x^2 - y \end{cases}$$

(a) Prove that every bounded orbit of this system tends to an equilibrium both as $t \to +\infty$ and as $t \to -\infty$.

(b) Find all equilibria and determine their stability.

(c) Describe the set of all bounded orbits. How many orbits does this set contain?

Solutions (seen similar, parts unseen; a - 7 points, b - 5 points, c - 8 points). III(a). It is enough to check that the energy $H = \frac{y^2}{2} - \frac{x^2}{2} - \frac{x^3}{3}$ is a Lyapunov function (i.e. it strictly decreases along any orbit which is not an equilibrium). We have

$$\frac{dH}{dt} = -y^2 < 0 \quad \text{if} \quad y \neq 0,$$

$$\frac{d^3H}{dt^3}|_{y=0} = -2(\dot{y})^2 = -2x^2(x+1)^2 < 0 \quad \text{if} \quad x \notin \{0, -1\},$$

i.e. H indeed is strictly increasing unless $(x = 0, y = 0)$ or $(x = -1, y = 0)$.

III(b). The equilibria have just been found: $O_1(0,0)$ and $O_2(-1,0)$. The linearisation matrix of the system at O_1 is $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, its determinant is negative, so O is a saddle. The linearisation matrix of the system at O_2 is $A_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$; we have $det(A) > 0$, $tr(A) < 0$, hence O_2 is a stable equilibrium.

III(c). By III(a), the bounded orbits are equilibria O_1 and O_2 and, possibly, unstable separatrices of the saddle O_1 unless they tend to infinity. As $t \to -\infty$, the function H on these separatrices tends to $H(O_1) = 0$, hence the separatrices near O_1 are tangent to the line $y = x$; one of the separatrices, W^+, goes towards $x > 0, y > 0$ and the other, W^-, goes towards $x < 0, y < 0$. The region $x > 0, y > 0$ is forward invariant (since $\dot{x} > 0$ at $x = 0, y > 0$ and $\dot{y} > 0$ at $y = 0, x > 0$), therefore W^+ cannot tend to O_2 as $t \to +\infty$, hence W^+ is unbounded. The separatrix W^- must stay in the region $H < 0$, hence it cannot cross the line $x = 0$, hence it must stay at $x < 0$, hence it cannot tend to infinity ($H > 0$ at large y or at large $|x|$ if $x < 0$). Thus W^- is a bounded orbit, so it tends to O_2. We see that the set of bounded orbits consist of 3 orbits: the points O_1, O_2 and the separatrix W^- that connects O_1 and O_2.

Question IV.Consider the following map of a plane $T : (x, y) \mapsto (\bar{x}, \bar{y})$

$$\bar{x} = y, \qquad \bar{y} = 7 - x - 8 \sin y.$$

(a) Prove that this map has infinitely many bounded orbits.

(b) How many points of period k does this map have?

(c) Prove that this map has an uncountable set of bounded non-periodic orbits.

Solutions (a - 10 points; b,c -5 points each; a,b - seen similar, c- unseen). IV(a): Take the square $\Pi : \{0 \leq x \leq \pi,\ 0 \leq y \leq \pi\}$ and write the map in the cross-form on Π:

$$\bar{x} = f_j(x, \bar{y}), \qquad y = f_j(x, \bar{y}),$$

where $j = 1$ or 2, and

$$f_1 = arcsin(\frac{1}{8}(7 - x - \bar{y})) \qquad f_2 = \pi - arcsin(\frac{1}{8}(7 - x - \bar{y})).$$

As $(x, \bar{y})$ runs Π, the value of f_1 stays inside $(0, \pi/2)$, and f_2 stays inside $(\pi/2, \pi)$, which implies that $T^{-1}(\Pi) \cap \Pi$ consists of 2 connected components, Π_1 and Π_2, where Π_j corresponds to $y \in range\{f_j\}$. The map is hyperbolic on these: to check this one must verify that

$$\|\frac{\partial f_j}{\partial x}\| + \|\frac{\partial f_j}{\partial \bar{y}}\| < 1$$

on Π_j . This inequality reduces to

$$\frac{1}{4|\cos f|} < 1$$

or

$$|\sin f| < \frac{\sqrt{15}}{4} \quad \Longleftrightarrow \quad \frac{1}{8}|7 - x - \bar{y}| < \frac{\sqrt{15}}{4},$$

which is indeed true for $(x, \bar{y}) \in \Pi$: the maximum of $|7 - x - \bar{y}|$ on Π is achieved at $(, 0)$ and equals to$7 < 2\sqrt{15}$. Thus, we have a hyperbolic map on a Markov partition with two components, i.e. a Smale horseshoe. The set of all orbits that never leave Π is in one-to-one correspondence with the set of all bi-infinite sequences of 0's and 1's; this set is infinite.

IV(b): By IV(b), the number of points of period k equals to the number of periodic sequences of zeros and ones of period k, i.e. to the number of all binary sequences of length k, i.e. to 2^k.

IV(c): The number of all infinite binary sequences is uncountable, hence the set of all bounded orbits is uncountable. The number of periodic orbits is countable, so the set of all bounded non-periodic orbits is uncountable.

Mastery Question. Prove chaotic behaviour for the equation

$$\ddot{x} - x + 2x^3 = A \sin t$$

at all small $A > 0$.

Solution. (20 points, seen similar). The equation is a time periodic perturbation of a Hamiltonian system with a Hamiltonian function $H = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{2}x^4$. At $A = 0$ the equation has a homoclinic loop in the energy level $H = 0$; the equation of the loop is found from

$$H = 0 \Longleftrightarrow \dot{x} = \pm x\sqrt{1 - x^2},$$

which gives

$$x_0(t) = \frac{1}{\cosh(t)}.$$

The Melnikov function is given by

$$M(\theta) = 2A \int_{-\infty}^{+\infty} \dot{x}_0(t) \sin(t + \theta) dt = 2AC \cos\theta,$$

where

$$C = \int_{-\infty}^{+\infty} \dot{x}_0 \sin t dt = -\int_{-\infty}^{+\infty} x_0(t) \cos t dt,$$

i.e.

$$C = -\int_{-\infty}^{+\infty} \frac{\cos t}{\cosh t} dt.$$

At $A \neq 0$ and $C \neq 0$ we have $M(\pi/2) = 0$ and $M'(\pi/2) \neq 0$, which will prove that the system has a transverse homoclinic (hence chaotic behaviour), if we show that $C \neq 0$ indeed. The integral C is computed by the method of residues:

$$C = -\mathrm{Re} \int_{-\infty}^{+\infty} \frac{e^{it}}{\cosh t} dt = -\mathrm{Re} \sum_{k=0}^{\infty} \frac{e^{iz_k}}{\sinh(z_k)} \oint_{|z - z_k| = \rho} \frac{dz}{z - z_k},$$

where $z_k = i\frac{\pi}{2}(2k + 1)$ are the roots of $\cosh z$ which lie in the upper half-plane.This gives

$$C = -2\pi \sum_{k=0}^{\infty} (-1)^k e^{-\frac{\pi}{2}(2k+1)} = -\frac{\pi}{\cosh \pi} \neq 0.$$

End of the proof.