UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc and MSci EXAMINATIONS (MATHEMATICS) MAY-JUNE 2000

This paper is also taken for the relevant examination for the Associateship

M3S8/M4S8 TIME SERIES

DATE: Monday, 5th June 2000 TIME: 2pm — 4pm

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Note: throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables having zero mean and variance σ_ϵ^2, unless stated otherwise.

1. a) What is meant by saying that a stochastic process is second-order stationary?

   b) Let $\{X_t\}$ be a stationary autoregressive process of order one, i.e.,

   $$X_t = \phi_{1,1} X_{t-1} + \epsilon_t,$$

   with the initial condition $X_{-\infty} = 0$.

   (i) What condition on $\phi_{1,1}$ guarantees that $\{X_t\}$ will be stationary?

   (ii) Show that $\{X_t\}$ can be written as a linear combination of ϵ_t, $\epsilon_{t-1}, \ldots$.

   (iii) What does (ii) say about the mean of X_t?

   (iv) Show that the autocorrelation sequence of $\{X_t\}$ is given by

   $$\rho_\tau = \phi_{1,1}^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \ldots$$

   (v) Suppose now that $\{X_t\}$ is observed with additive error, i.e., we observe

   $$Y_t = X_t + a_t,$$

   where $\{a_t\}$ is a white noise process having mean zero and constant finite variance σ_a^2, and is uncorrelated with $\{\epsilon_t\}$. Derive the autocorrelation sequence of $\{Y_t\}$. How does it differ from the autocorrelation sequence of $\{X_t\}$?

2. Let $\{U_t\}$ be a zero mean second-order stationary process having an autocovariance sequence $\{s_{U,\tau}\}$ and a spectral density function $S_U(f)$.

a)

(i) Show that the first-order backward difference process

$$V_t = U_t - U_{t-1}$$

has the spectral density function $S_V(f)$ given by

$$S_V(f) = 4\sin^2(\pi f)\, S_U(f)\,.$$

(ii) Does a first-order backward difference filter resemble a low-pass or high-pass filter?

(iii) If the process $\{W_t\}$ is the second-order backward difference of $\{U_t\}$ given by

$$W_t = U_t - 2U_{t-1} + U_{t-2},$$

deduce the form of $S_W(f)$, the spectral density function of $\{W_t\}$, in terms of $S_U(f)$.

b) Consider the sum of a linear trend and the stationary process $\{U_t\}$ defined by

$$X_t = a + b\,t + U_t,$$

where a and b are real-valued non-zero constants. Show that

(i) $\{X_t\}$ is a non-stationary process,

(ii) the first-order backward difference of $\{X_t\}$, the process $\{Y_t\}$ say, is a stationary process with mean b, and give the form of the autocovariance sequence $\{s_{Y,\tau}\}$ in terms of $\{s_{U,\tau}\}$, and

(iii) the second-order backward difference of $\{X_t\}$, the process $\{Z_t\}$ say, is a stationary process with mean zero, and give the form of the autocovariance sequence $\{s_{Z,\tau}\}$ in terms of $\{s_{U,\tau}\}$.

3. Let $\{X_t\}$ be a zero mean stationary stochastic processes, with spectral density function $S_X(f)$.

a) Specify the three conditions which must be satisfied by a linear time-invariant (LTI) digital filter.

b) Suppose we define the zero mean stationary process $\{Y_t\}$ as

$$Y_t = \overline{X}_t - \overline{X}_{t-K}, \quad \text{where } \overline{X}_t = \frac{1}{K}\sum_{j=0}^{K-1} X_{t-j}.$$

Demonstrate that $Y_t = L\{X_t\}$ defines a LTI digital filter $L\{\cdot\}$.

c) Show that

$$\sum_{j=0}^{K-1} e^{i2\pi fj} = \begin{cases} Ke^{i(K-1)\pi f}\mathcal{D}(f), & \text{if } f \neq 0, \pm1, \pm2, ..; \\ K, & \text{if } f = 0, \pm1, \pm2, .. \end{cases}$$

where $\mathcal{D}(f)$ is Dirichlet's kernel defined by

$$\mathcal{D}(f) = \frac{\sin(K\pi f)}{K\sin(\pi f)}.$$

d) Deduce that $S_Y(f) = |G(f)|^2 S_X(f)$, where

$$|G(f)|^2 = \begin{cases} \frac{4\sin^4(K\pi f)}{K^2\sin^2(\pi f)} & \text{if } f \neq 0, \pm1, \pm2, ..; \\ 0, & \text{if } f = 0, \pm1, \pm2, .. \end{cases}$$

where $S_Y(f)$ is the spectral density function of $\{Y_t\}$.

4. Let $X_1, \ldots, X_N$ be a sample of size N from a stationary process with mean μ and spectral density function $S(f)$. At lag $\tau = 0$ both the unbiased and biased estimators of the autocovariance sequence reduce to

$$\hat{s}_0 \equiv \frac{1}{N} \sum_{t=1}^{N} \left(X_t - \overline{X}\right)^2 .$$

a) Show that $E\{\hat{s}_0\} = s_0 - \text{var}\,\{\overline{X}\}$, where $s_0 = \text{var}\,\{X_t\}$.

b) Define the spectral estimator where the exact mean is known and subtracted as

$$\widehat{S}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} (X_t - \mu) e^{-i2\pi f t} \right|^2 .$$

Use the spectral representation $X_t - \mu = \int_{-1/2}^{1/2} e^{i2\pi f t} dZ(f)$, to show that the mean of the spectral estimator $\widehat{S}(f)$ is given by

$$E\{\widehat{S}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f') S(f') df' ,$$

where $\mathcal{F}(f)$ denotes Fejer's kernel given by $\mathcal{F}(f) = (1/N) \left| \sum_{t=1}^{N} e^{-i2\pi f t} \right|^2$.

c) Demonstrate that

$$\text{var}\,\{\overline{X}\} = (1/N) E\{\widehat{S}(0)\},$$

and hence that

$$E\{\hat{s}_0\} = \int_{-1/2}^{1/2} \left(1 - \frac{1}{N}\mathcal{F}(f)\right) S(f)\, df.$$

5. a) Give one advantage and one disadvantage accruing from the use of a single data taper in spectrum analysis. What benefits arise from multitapering?

b) Let $\{h_{t,0}\}, \{h_{t,1}\}, ..., \{h_{t,N-1}\}$ be N real-valued orthonormal data tapers, each of length N, of the type used in multitaper spectrum estimation, so that

$$\sum_{t=1}^{N} h_{t,j} h_{t,k} = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases}$$

Let $\mathbf{V}$ be the $N \times N$ orthonormal matrix with jth column $\{h_{t,j}\}$. By considering the matrix $\mathbf{V}$ show that

$$\sum_{k=0}^{N-1} h_{t,k} h_{u,k} = \begin{cases} 1, & \text{if } u = t; \\ 0, & \text{if } u \neq t. \end{cases}$$

c) Let $\{X_t\}$ be a zero mean stationary process with variance σ_X^2 and spectral density function $S(f)$. Consider a multitaper spectrum estimator of $S(f)$ which uses all N orthonormal tapers:

$$\hat{S}^{(mt)}(f) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{S}_k^{(mt)}(f) \quad \text{with} \quad \hat{S}_k^{(mt)}(f) = \left| \sum_{t=1}^{N} h_{t,k} X_t e^{-i2\pi ft} \right|^2.$$

Show that

$$E\{\hat{S}^{(mt)}(f)\} = \sigma_X^2.$$

d) Since

$$E\{\hat{S}^{(mt)}(f)\} = \int_{-1/2}^{1/2} \overline{\mathcal{H}}(f - f') S(f') df',$$

where $\overline{\mathcal{H}}(f) = (1/N) \sum_{k=0}^{N-1} \mathcal{H}_k(f)$ and $\mathcal{H}_k(f) = \left| \sum_{t=1}^{N} h_{t,k} e^{-i2\pi ft} \right|^2$, what does the result in (c) tell us about $\overline{\mathcal{H}}(f)$?

SOLUTIONS

M3S8/M4S8 TIME SERIES

DATE: Monday, 5th June 2000 TIME: 2pm — 4pm

Soln. 1.

a) $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, $\text{var}\{X_t\}$ is a finite constant for all t, and $\text{cov}\{\mathrm{X_t, X_{t+\tau}}\} = \mathrm{s}_\tau$, a finite quantity depending only on τ and not on t.

b)

(i) We have

$$X_t - \phi_{1,1} X_{t-1} = \epsilon_t.$$

The characteristic polynomial for this AR(1) process is

$$\Phi(z) = 1 - \phi_{1,1} z.$$

The root is $z = 1/\phi_{1,1}$ and is outside the unit circle provided $|\phi_{1,1}| < 1$, whence the process is stationary.

(ii) By repeated substitution we obtain

$$X_t = \sum_{k=0}^{\infty} \phi_{1,1}^k \epsilon_{t-k} + \text{ term in } X_{-\infty},$$

and the last term is zero because of the initial condition. Hence X_t is a linear combination of $\epsilon_t, \epsilon_{t-1}, \ldots$

(iii) Since

$$E\{X_t\} = \sum_{k=0}^{\infty} \phi_{1,1}^k E\{\epsilon_{t-k}\},$$

we have that $E\{X_t\} = 0$. (Alternatively we could observe that by taking expectations in the defining equation we get

$$(1 - \phi_{1,1})\mu_X = E\{\epsilon_t\} = 0,$$

and since this must be true for any $|\phi_{1,1}| < 1$, then $\mu_X = 0$. Also since $E\{X_{-\infty}\} = 0$, for stationarity the mean must be zero everywhere.)

(iv) Take $\tau > 0$, and multiply through the defining equation by $X_{t-\tau}$. Since $X_{t-\tau}$ is a function of $\epsilon_{t-\tau}, \epsilon_{t-\tau-1}, \ldots$ (part (ii)) we have $E\{X_{t-\tau}\epsilon_t\} = 0$, and hence we obtain

$$s_{X,\tau} = \phi_{1,1} s_{X,\tau-1},$$

which when iterated gives

$$s_{X,\tau} = \phi_{1,1}^{\tau} s_{X,0}$$

and hence, since the autocovariance is symmetric about 0, the autocorrelation is given by

$$\rho_{X,\tau} = \phi_{1,1}^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \ldots$$

(v) Now $E\{Y_t Y_{t-\tau}\} = E\{(X_t + a_t)(X_{t-\tau} + a_{t-\tau})\}$ and since Y_t, X_t and a_t all have mean zero, and the processes $\{a_t\}$ and $\{X_t\}$ are uncorrelated, (since X_t is a linear combination of $\epsilon_t, \epsilon_{t-1}, \ldots$ and $\{a_t\}$ and $\{\epsilon_t\}$ are uncorrelated) we get

$$s_{Y,\tau} = s_{X,\tau} + s_{a,\tau},$$

so that

$$s_{Y,\tau} = s_{X,\tau}, \text{ for } \tau \neq 0; \; \sigma_Y^2 = \sigma_X^2 + \sigma_a^2 \text{ for } \tau = 0.$$

Hence

$$\rho_{Y,\tau} = \begin{cases} \frac{s_{X,\tau}}{\sigma_X^2\left(1+\frac{\sigma_a^2}{\sigma_X^2}\right)}, & \text{if } \tau \neq 0; \\ 1, & \text{if } \tau = 0. \end{cases}$$

Since $\{X_t\}$ is AR(1),

$$\rho_{Y,\tau} = \begin{cases} \phi_{1,1}^{|\tau|}\left(1 + \frac{\sigma_a^2}{\sigma_X^2}\right)^{-1}, & \text{if } \tau \neq 0; \\ 1, & \text{if } \tau = 0. \end{cases}$$

Assuming $\sigma_a^2 > 0$, the effect of the term in brackets is to attenuate the autocorrelation of $\{X_t\}$, namely $\rho_{X,\tau} = \phi_{1,1}^{|\tau|}$.

Soln. 2.

a) (i) The filter is defined by $L\{U_t\} = U_t - U_{t-1}$. The transfer function is obtained by inputing $U_t = \exp(i2\pi ft)$:

$$L\{e^{i2\pi ft}\} = e^{i2\pi ft} - e^{i2\pi f(t-1)} = e^{i2\pi ft}\left(1 - e^{-i2\pi f}\right) = e^{i2\pi ft}G(f),$$

where $G(f) \equiv 1 - \exp(-i2\pi f)$ is the transfer function. Now

$$|G(f)|^2 = \left|1 - e^{-i2\pi f}\right|^2 = \left|e^{-i\pi f}(e^{i\pi f} - e^{-i\pi f})\right|^2 = \left|e^{-i\pi f}2i\sin(\pi f)\right|^2 = 4\sin^2(\pi f).$$

Hence, $S_W(f) = 4\sin^2(\pi f)S_U(f)$.

(ii) Now $|G(f)|^2$ increases from 0 to Nyquist (1/2) so that the first difference filter resembles a high-pass filter.

(iii) Second differencing is the same as applying first differencing twice. Hence $S_W(f) = 4\sin^2(\pi f)S_V(f) = 16\sin^4(\pi f)S_U(f)$.

b)

(i) Since $E\{U_t\} = 0$ for all t, it follows that

$$E\{X_t\} = E\{a + b\,t + U_t\} = a + b\,t,$$

which is not independent of t, so $\{X_t\}$ is a nonstationary process.

(ii) Let $\{s_{U,\tau}\}$ denote the acvs for $\{U_t\}$. Note that the first order backward difference

$$Y_t \equiv X_t - X_{t-1} = a + b\,t + U_t - (a + b\,(t-1) + U_{t-1}) = b + U_t - U_{t-1}$$

has mean value

$$E\{Y_t\} = E\{b + U_t - U_{t-1}\} = b + E\{U_t\} - E\{U_{t-1}\} = b \neq 0$$

and covariance between Y_t and $Y_{t+\tau}$ of

$$\begin{aligned}
s_{Y,\tau} = \operatorname{cov}\{Y_t, Y_{t+\tau}\} &= E\{(Y_t - b)(Y_{t+\tau} - b)\} \\
&= E\{(U_t - U_{t-1})(U_{t+\tau} - U_{t+\tau-1})\} \\
&= E\{U_tU_{t+\tau}\} - E\{U_tU_{t+\tau-1}\} \\
&\quad - E\{U_{t-1}U_{t+\tau}\} + E\{U_{t-1}U_{t+\tau-1}\} \\
&= s_{U,\tau} - s_{U,\tau-1} - s_{U,\tau+1} + s_{U,\tau} \\
&= 2s_{U,\tau} - s_{U,\tau-1} - s_{U,\tau+1}.
\end{aligned}$$

Since $E\{Y_t\}$ and $\text{cov}\,\{Y_t, Y_{t+\tau}\}$ are both independent of t and finite, the process $E\{Y_t\}$ is stationary with a nonzero mean.

(iii) Let $\{Z_t\}$ be the first backward difference of $\{Y_t\}$, which is the same as the second backward difference of $\{X_t\}$. We have

$$E\{Z_t\} = E\{Y_t - Y_{t-1}\} = E\{Y_t\} - E\{Y_{t-1}\} = 0,$$

and we can use the same argument as above to establish that

$$\begin{aligned} s_{Z,\tau} = \text{cov}\,\{Z_t, Z_{t+\tau}\} &= 2s_{Y,\tau} - s_{Y,\tau-1} - s_{Y,\tau+1} \\ &= 2(2s_{U,\tau} - s_{U,\tau-1} - s_{U,\tau+1}) \\ &\quad - (2s_{U,\tau-1} - s_{U,\tau-2} - s_{U,\tau}) \\ &\quad - (2s_{U,\tau+1} - s_{U,\tau} - s_{U,\tau+2}) \\ &= 6s_{U,\tau} - 4s_{U,\tau-1} - 4s_{U,\tau+1} + s_{U,\tau-2} + s_{U,\tau+2}, \end{aligned}$$

which depends on τ, but not t. Thus $\{Z_t\}$ is a stationary process with mean zero.

Soln. 3.

a) A digital filter L that transforms an input sequence $\{X_t\}$ into an output sequence $\{Y_t\}$ is called a linear time-invariant digital filter if it has the following three properties:

[1] Scale preservation:

$$L\{\alpha X_t\} = \alpha L\{X_t\}.$$

[2] Superposition:

$$L\{X_{t,1} + X_{t,2}\} = L\{X_{t,1}\} + L\{X_{t,2}\}.$$

[3] Time invariance:

$$\text{if } L\{X_t\} = \{Y_t\}, \text{ then } L\{X_{t+\tau}\} = \{Y_{t+\tau}\},$$

where τ is integer-valued and the notation $\{X_{t+\tau}\}$ refers to the sequence whose tth element is $X_{t+\tau}$.

b) We can write $Y_t = (1/K)\sum_{j=0}^{K-1}(X_{t-j} - X_{t-K-j})$. Then,

[1]

$$L\{\alpha X_t\} = (1/K)\sum_{j=0}^{K-1}(\alpha X_{t-j} - \alpha X_{t-K-j}) = \alpha L\{X_t\}.$$

[2]

$$\begin{aligned}
L\{X_{t,1} + X_{t,2}\} &= (1/K)\sum_{j=0}^{K-1}(X_{t-j,1} + X_{t-j,2} - X_{t-K-j,1} - X_{t-K-j,2}) \\
&= (1/K)\sum_{j=0}^{K-1}(X_{t-j,1} - X_{t-K-j,1}) + (1/K)\sum_{j=0}^{K-1}(X_{t-j,2} - X_{t-K-j,2}) \\
&= L\{X_{t,1}\} + L\{X_{t,2}\}.
\end{aligned}$$

[3] Let $Y_t = L\{X_t\}$ and let $X_{t;\tau} \equiv X_{t+\tau}, Y_{t;\tau} \equiv Y_{t+\tau}$. Then

$$L\{X_{t;\tau}\} = (1/K)\sum_{j=0}^{K-1}(X_{t+\tau-j} - X_{t+\tau-K-j}) = Y_{t+\tau} = Y_{t;\tau}.$$

c) If $f \neq 0, \pm 1, \pm 2, ..,$ then

$$\begin{aligned}
\sum_{j=0}^{K-1} e^{i2\pi fj} &= (1 - e^{i2\pi fK})/(1 - e^{i2\pi f}) \\
&= [e^{iK\pi f}(e^{-iK\pi f} - e^{iK\pi f})]/[e^{i\pi f}(e^{-i\pi f} - e^{i\pi f})] \\
&= e^{i(K-1)\pi f}\sin(K\pi f)/\sin(\pi f) = Ke^{i(K-1)\pi f}\mathcal{D}(f).
\end{aligned}$$

If $f = 0, \pm 1, \pm 2, ..$ then $\sum_{j=0}^{K-1} e^{i2\pi fj} = K$.

d) Letting $X_t = \exp(i2\pi ft)$, we get

$$Y_t = \frac{1}{K}\left(\sum_{j=0}^{K-1} e^{i2\pi f(t-j)} - \sum_{j=0}^{K-1} e^{i2\pi f(t-K-j)}\right) = e^{i2\pi ft}\frac{1}{K}\left(1 - e^{-i2\pi fK}\right)\sum_{j=0}^{K-1} e^{-i2\pi fj},$$

so the transfer function is

$$G(f) = \frac{1}{K}\left(1 - e^{-i2\pi fK}\right)\sum_{j=0}^{K-1} e^{-i2\pi fj}.$$

The summation may be reduced using the result in part c) so that

$$G(f) = \begin{cases} \frac{1}{K}\left(1 - e^{-i2\pi fK}\right) e^{-i\pi f(K-1)} \frac{\sin(K\pi f)}{\sin(\pi f)} & \text{if } f \neq 0, \pm 1, \pm 2, ..; \\ 0, & \text{if } f = 0, \pm 1, \pm 2,.. \end{cases}$$

i.e.,

$$G(f) = \begin{cases} \frac{1}{K}\left(e^{i\pi fK} - e^{-i\pi fK}\right) e^{-i\pi f(2K-1)} \frac{\sin(K\pi f)}{\sin(\pi f)} & \text{if } f \neq 0, \pm 1, \pm 2, ..; \\ 0, & \text{if } f = 0, \pm 1, \pm 2,.. \end{cases}$$

giving

$$G(f) = \begin{cases} \frac{2i}{K} e^{-i\pi f(2K-1)} \frac{\sin^2(K\pi f)}{\sin(\pi f)} & \text{if } f \neq 0, \pm 1, \pm 2, ..; \\ 0, & \text{if } f = 0, \pm 1, \pm 2,.. \end{cases}$$

Hence, as required,

$$|G(f)|^2 = \begin{cases} \frac{4\sin^4(K\pi f)}{K^2\sin^2(\pi f)} & \text{if } f \neq 0, \pm 1, \pm 2, ..; \\ 0, & \text{if } f = 0, \pm 1, \pm 2,.. \end{cases}$$

Soln. 4.

a) Here $\{X_t\}$ is a stationary process with mean value $\mu = E\{X_t\}$, and variance s_0. By definition,

$$\begin{aligned}
\hat{s}_0 &= \frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})^2 = \frac{1}{N}\sum_{t=1}^{N}([X_t - \mu] - [\overline{X} - \mu])^2 \\
&= \frac{1}{N}\sum_{t=1}^{N}\left([X_t - \mu]^2 - 2[X_t - \mu][\overline{X} - \mu] + [\overline{X} - \mu]^2\right) \\
&= \frac{1}{N}\sum_{t=1}^{N}[X_t - \mu]^2 - 2[\overline{X} - \mu][\overline{X} - \mu] + [\overline{X} - \mu]^2 \\
&= \frac{1}{N}\sum_{t=1}^{N}[X_t - \mu]^2 - [\overline{X} - \mu]^2.
\end{aligned}$$

Taking the expectation of both sides and noting that $E\{\overline{X}\} = \mu$ yields

$$\begin{aligned}
E\{\hat{s}_0\} &= \frac{1}{N}\sum_{t=1}^{N} E\{[X_t - \mu]^2\} - E\{[\overline{X} - \mu]^2\} \\
&= \text{var}\,\{X_t\} - \text{var}\,\{\overline{X}\} = s_0 - \text{var}\,\{\overline{X}\},
\end{aligned}$$

the desired result.

b) Let

$$J(f) \equiv (1/\sqrt{N})\sum_{t=1}^{N}(X_t - \mu)e^{-i2\pi ft}.$$

By the spectral representation theorem

$$X_t - \mu = \int_{-1/2}^{1/2} e^{i2\pi f't}\, dZ(f'),$$

where $\{Z(\cdot)\}$ is a process with orthogonal increments, and $E\{dZ(f)\} = 0$. Thus

$$\begin{aligned}
J(f) &= (1/\sqrt{N})\sum_{t=1}^{N}\left(\int_{-1/2}^{1/2} e^{i2\pi f't}\, dZ(f')\right) e^{-i2\pi ft} \\
&= (1/\sqrt{N})\int_{-1/2}^{1/2}\sum_{t=1}^{N} e^{-i2\pi(f-f')t}\, dZ(f') \\
&= \int_{-1/2}^{1/2} F(f - f')\, dZ(f'),
\end{aligned}$$

where

$$F(f) = (1/\sqrt{N}) \sum_{t=1}^{N} e^{-i2\pi ft}$$

Now it is given that,

$$\widehat{S}(f) \equiv |J(f)|^2 = (1/N) \left| \sum_{t=1}^{N} (X_t - \mu) e^{-i2\pi ft} \right|^2 .$$

Because $\{Z(\cdot)\}$ has orthogonal increments, we therefore have

$$E\{\widehat{S}(f)\} = \int_{-1/2}^{1/2} \mathcal{F}(f - f') S(f') \, df',$$

where

$$\mathcal{F}(f) \equiv |F(f)|^2 = (1/N) \left| \sum_{t=1}^{N} e^{-i2\pi ft} \right|^2 .$$

c) Now

$$\begin{aligned} \text{var}\,\{\overline{X}\} &= E\{(\overline{X} - \mu)^2\} \\ &= (1/N^2) E \left\{ \left(\sum_{t=1}^{N} (X_t - \mu) \right)^2 \right\} \\ &= (1/N) E\{\widehat{S}(0)\}, \end{aligned}$$

and from b) $E\{\widehat{S}(0)\} = \int_{-1/2}^{1/2} \mathcal{F}(f) S(f) \, df$ (by symmetry of spectral density function), and of course $s_0 = \int_{-1/2}^{1/2} S(f) \, df$, so that the result follows from part a), i.e.,

$$E\{\hat{s}_0\} = s_0 - \text{var}\,\{\overline{X}\} = \int_{-1/2}^{1/2} \left(1 - \frac{1}{N} \mathcal{F}(f) \right) S(f) \, df.$$

Soln. 5.

a) The main point of data tapering in spectrum analysis is to reduce sidelobe leakage by changing the default Fejér blurring kernel into something with smaller sidelobes. There are two disadvantages: (i) the main lobe of the blurring kernel is made wider, reducing resolution, and (ii) some good data at the ends is downweighted relative to that in the middle of the series. Multitapering is useful in that each orthogonal taper catches some of the data deleted by the previous tapers. The result is that the variance of a multitaper spectral estimator is reduced by a factor equal to the number of tapers, compared to using a single taper: also the estimator is consistent.

b) The matrix $\mathbf{V}$ is orthonormal. Hence $\mathbf{V}^T = \mathbf{V}^{-1}$. The fact that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$ gives us

$$\sum_{t=1}^{N} h_{t,j} h_{t,k} = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases}$$

But the orthonormality also means that $\mathbf{V}\mathbf{V}^T = \mathbf{I}$, so that

$$\sum_{k=0}^{N-1} h_{t,k} h_{u,k} = \begin{cases} 1, & \text{if } u = t; \\ 0, & \text{if } u \neq t. \end{cases}$$

c) The multitaper spectrum estimator which uses N orthonormal tapers is given by

$$\begin{aligned} \hat{S}^{(mt)}(f) &= \frac{1}{N} \sum_{k=0}^{N-1} \left| \sum_{t=1}^{N} h_{t,k} X_t e^{-i2\pi f t} \right|^2 \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{t=1}^{N} h_{t,k} X_t e^{-i2\pi f t} \right) \left(\sum_{u=1}^{N} h_{u,k} X_u e^{i2\pi f u} \right) \\ &= \frac{1}{N} \sum_{t=1}^{N} \sum_{u=1}^{N} X_t X_u \left(\sum_{k=0}^{N-1} h_{t,k} h_{u,k} \right) e^{-i2\pi f (t-u)}. \end{aligned}$$

But from the result in (a), $\sum_{k=0}^{N-1} h_{t,k} h_{u,k}$ is unity if $t = u$ and zero otherwise. Hence,

$$E\{\hat{S}^{(mt)}(f)\} = \frac{1}{N} E\{\sum_{t=1}^{N} X_t^2\} = \sigma_X^2.$$

d) Since the integral of the spectrum is equal to σ_X^2 and the result in c) is true for all f, we must have that $\overline{\mathcal{H}}(f) = 1$ for all f, a result that can easily be verified directly.