Derivation of Binomial

We have independent trials with $\mathrm{P}(Y_i = 0) = 1 - p$ and $\mathrm{P}(Y_i = 1) = p$.

So

$$
\begin{aligned}
&\mathrm{P}(\text{a particular sequence of } x \text{ successes and } n-x \text{ failures}) \\
&= \mathrm{P}(Y_1 = 1 \cap \ldots \cap Y_x = 1 \cap Y_{x+1} = 0 \cap \ldots \cap Y_n = 0) \\
&= \mathrm{P}(Y_1 = 1) \times \ldots \times \mathrm{P}(Y_x = 1) \times \mathrm{P}(Y_{x+1} = 1) \times \ldots \times \mathrm{P}(Y_n = 0) \\
&= p^x (1-p)^{n-x}
\end{aligned}
$$

But, x successes and $n - x$ failures can be obtained in a number of ways.

e.g. $n = 5, x = 2$

SSFFF
SFSFF
SFFSF
SFFFS
FSSFF
FSFSF
FSFFS
FFSSF
FFSFS
FFFSS

Now, there are $5! = 5 \times 4 \times 3 \times 2 \times 1$ ways of arranging a sequency of 5 distinguishable objects. Now, we have 3 failures, so there are 3! ways of arranging these, and 2 successes, so there are 2! ways of arranging these.

Hence, total number of ways of obtaining 2 successes and 3 failures in a sequence of 5 is

$$\frac{5!}{2!3!} = \frac{5 \times 4}{2} = 10.$$

Extending this in general, we have

$$\mathrm{P}(X = k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where

$$\binom{n}{k} = \frac{n!}{r!(n-r)!}$$

– the binomial distribution.

Example 3.1

Toss a fair coin 5 times.

Let X = number of heads, then X is binomial, $p = \frac{1}{2}, n = 5$.

$$P(X = 0) = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(1-\frac{1}{2}\right)^5 = \frac{1}{2^5} = \frac{1}{32}$$

Similarly, we have

k	$P(X = k)$	$P(X \le k)$
0	$\frac{1}{32}$	$\frac{1}{32}$
1	$\binom{5}{1}\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^4 = \frac{5}{32}$	$\frac{6}{32}$
2	$\frac{10}{32}$	$\frac{16}{32}$
3	$\frac{10}{32}$	$\frac{26}{32}$
4	$\frac{5}{32}$	$\frac{31}{32}$
5	$\frac{1}{32}$	$\frac{32}{32}$

Distribution function:

$$P(\text{obtaining} \le 3 \text{ heads}) = \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} = \frac{26}{32}$$

$$\begin{aligned} P(\text{obtaining} \le 4 \text{ heads}) &= 1 - P(\text{obtaining} > 4 \text{ heads}) \\ &= 1 - P(X > 4) \\ &= 1 - P(X = 5) \\ &= 1 - \frac{1}{32} = \frac{31}{32} \end{aligned}$$

Example 3.2

Production line: P(defective item) $= 0.02 = p$.

We take a sample of size n.

X = number of defectives in the sample.

$S = \{0, 1, 2, 3, \ldots, n\}$.

e.g. $n = 10$, suppose 10 items are selected from a large batch and the batch is accepted if there are no defective items.

$$P(\text{batch is accepted}) = P(X = 0) = \binom{10}{0} 0.02^0 \times 0.98^{10} = 0.817.$$

Suppose now that it is accepted if there are not greater than 1 defectives.

$$\begin{aligned} P(\text{batch is accepted}) &= P(X \le 1) \\ &= P(X = 0) + P(X = 1) \\ &= \binom{10}{0} 0.02^0 \times 0.98^{10} + \binom{10}{1} 0.02^1 \times 0.98^9 \\ &= 0.817 + 10 \times 0.0167 = 0.984. \end{aligned}$$

Example 3.3

Suppose that the number, X, of telephone calls arriving at an exchange in 10 second periods is a Poisson random variable with $\mu = 2$. Then the rate, λ in 1 second is $\frac{2}{10} = 0.2$.

So,

$$P(X = x) = \begin{cases} \frac{e^{-2}2^x}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that there will be more than 3 calls in a particular 10 second period?

$$\begin{aligned} P(X > 3) &= 1 - P(X \leq 3) \\ &= 1 - \{P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\} \\ &= 1 - \left\{e^{-1} + 2e^{-2} + \frac{2^2 e^{-2}}{2!} + \frac{2^3 e^{-2}}{3!}\right\} \\ &= 1 - \{0.135 + 0.271 + 0.271 + 0.180\} \\ &= 1 - 0.857 = 0.143 \end{aligned}$$

Suppose now that we are interested in the number, Y, of calls in 20 second intervals. Now $\mu = \lambda t = 0.2 \times 20 = 4$ (since the rate is 2 per 10 second period), and

$$P(Y = y) = \begin{cases} \frac{e^{-4}4^y}{y!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that there are less than 2 calls in a 20 second period?

$$\begin{aligned} P(Y < 2) &= P(Y = 0) + P(Y = 1) \\ &= e^{-4} + 4e^{-4} \\ &= 5e^{-4} = 0.092 \end{aligned}$$

Uniform distribution

Suppose X is uniform on (a, b), *i.e.*

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

To show that this is a pdf, we must show:

1. $f(x) \geq 0 \; \forall \; x \in (a, b)$:

   $f(x) = \frac{1}{b-a} \geq 0$ as $b \geq a$.

2. $\int_a^b f(x)\,\mathrm{d}x = 1$:

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &= \int_a^b \frac{1}{b-a}\,\mathrm{d}x \\
&= \left[\frac{x}{b-a}\right]_a^b \\
&= \frac{b}{b-a} - \frac{a}{b-a} = 1
\end{aligned}$$

For the distribution function:

$$F(x_0) = \mathrm{P}(X \leq x_0) = \begin{cases} 0 & \text{if } \; x_0 \leq a, \\ 1 & \text{if } \; x_0 \geq b, \\ \int_a^{x_0} \frac{1}{b-a}\,\mathrm{d}x & \text{if } \; a \leq x_0 \leq b. \end{cases}$$

Now,

$$\int_a^{x_0} \frac{1}{b-a}\,\mathrm{d}x = \left[\frac{x}{b-a}\right]_a^{x_0} = \frac{x_0 - a}{b-a}$$

Exponential distribution

Suppose T is exponential with parameter λ, *i.e.*

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

To show that this is a pdf, we must show:

1. $f(t) \geq 0 \;\forall\, t \geq 0$:

   $f(t) = \lambda e^{-\lambda t} \geq 0$ as $\lambda \geq 0$.

2. $\int_0^\infty f(x)\,\mathrm{d}x = 1$:

$$\begin{aligned}
\int_0^\infty f(t)\,\mathrm{d}t &= \int_0^\infty \lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \left[-e^{-\lambda t}\right]_0^\infty = 1
\end{aligned}$$

For the distribution function:

$$F(t_0) \;=\; \mathrm{P}(T \leq t_0) = \begin{cases} 0 & \text{for } \; t_0 \leq 0, \\ \int_0^{t_0} f(t)\,\mathrm{d}t & \text{for } \; t_0 > 0. \end{cases}$$

Now,

$$\begin{aligned}
\int_0^{t_0} f(t)\,\mathrm{d}t &= \int_0^{t_0} \lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \left[-e^{-\lambda t}\right]_0^{t_0} \\
&= 1 - e^{-\lambda t_0}
\end{aligned}$$

Lack of memory property:

Consider $\mathrm{P}(T > t_0) = 1 - \mathrm{P}(T \leq t_0) = 1 - (1 - e^{-\lambda t_0}) = e^{-\lambda t_0}$.

Suppose we have a component which has lasted 5 hours, what is the probability it will last another 5 hours?

$$\begin{aligned}
\mathrm{P}(T > 10 | T > 5) &= \frac{\mathrm{P}(T > 10 \cap T > 5)}{\mathrm{P}(T > 5)} \\
&= \frac{\mathrm{P}(T > 10)}{\mathrm{P}(T > 5)} \\
&= \frac{e^{-10\lambda}}{e^{-5\lambda}} \\
&= e^{-\lambda(10-5)} = e^{-5\lambda} = \mathrm{P}(T > 5).
\end{aligned}$$

Similarly, we can show that for $t_0 + t > t_0 > 0$,

$$\begin{aligned}
\mathrm{P}(T > t_0 + t | T > t_0) &= \frac{\mathrm{P}(T > t_0 + t \cap T > t_0)}{\mathrm{P}(T > t_0)} \\
&= \frac{\mathrm{P}(T > t_0 + t)}{\mathrm{P}(T > t_0)} \\
&= \frac{e^{-\lambda(t_0+t)}}{e^{-\lambda t_0}} \\
&= e^{-\lambda(t_0+t-t_0)} = e^{-\lambda t} = \mathrm{P}(T > t)
\end{aligned}$$

Example 3.4

The standard normal distribution function is written as $\Phi(x)$, and is widely tabulated, *i.e.*,

$$\mathrm{P}(X \leq x_0) = \Phi(x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,\mathrm{d}x$$

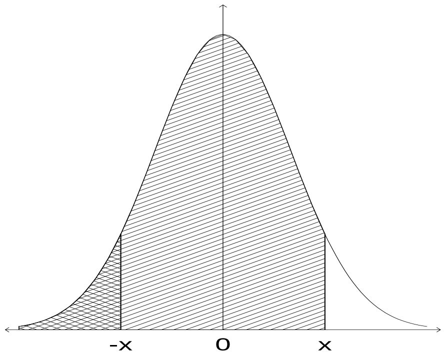
– not possible to solve this integral analytically.

If we want $\Phi(x)$ for $x \geq 0$ we look in the tables directly.

e.g., $\mathrm{P}(X \leq 1.6) = \Phi(1.6) = 0.9452$.

For $x < 0$ we use the fact that

$$\Phi(x) + \Phi(-x) = 1.$$

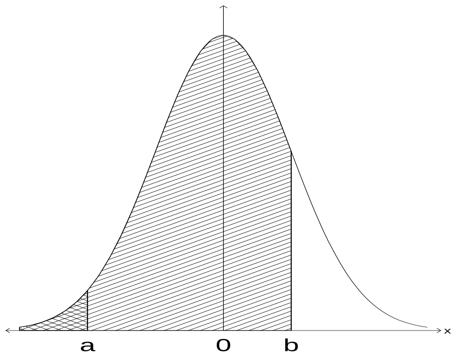


So, for $x < 0$

$$\Phi(x) = 1 - \Phi(-x)$$

e.g., $\mathrm{P}(X \leq -0.7) = 1 - \Phi(0.7) = 1 - 0.7580 = 0.2420.$

If we want $\mathrm{P}(a < X < b)$ we evaluate $\Phi(b) - \Phi(a)$.



e.g.,

$$\begin{aligned}\mathrm{P}(0.3 < X < 1.2) &= \Phi(1.2) - \Phi(0.3) \\ &= 0.8849 - 0.6179 = 0.2670\end{aligned}$$

$$\begin{aligned}\mathrm{P}(-2.0 < X < 1.0) &= \Phi(1.0) - \Phi(-2.0) \\ &= \Phi(1.0) - (1 - \Phi(2.0)) \\ &= 0.8413 - (1 - 0.9772) = 0.8185\end{aligned}$$

Example 3.5

Roll of a die, *i.e.*, uniform distribution on $1, \ldots 6$.

$$\begin{aligned}\mathrm{E}(X) &= \sum_{i=1}^{6} i\mathrm{P}(X = i) \\ &= \sum_{i=1}^{6} i \times \frac{1}{6} \\ &= \frac{1}{6}(1 + 23 + 4 + 5 + 6) = \frac{21}{6} = 3\frac{1}{2}\end{aligned}$$

Mean and Variance: Poisson distribution

Mean:

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{x=0}^{\infty} x\mathrm{P}(X=x) \\
&= \sum_{x=0}^{\infty} x\frac{e^{-\mu}\mu^x}{x!} \\
&= e^{-\mu}\left(\mu + 2\frac{\mu^2}{2!} + 3\frac{\mu^3}{3!} + 4\frac{\mu^4}{4!} + \dots\right) \\
&= \mu e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \dots\right) \\
&= \mu e^{-\mu}e^{\mu} = \mu
\end{aligned}$$

Variance:

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}(X-\mathrm{E}(X))^2 \\
&= \mathrm{E}\left(X^2 - 2X\mathrm{E}(X) + (\mathrm{E}(X))^2\right) \\
&= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\
&= \sum_{x=0}^{\infty} x^2\frac{e^{-\mu}\mu^x}{x!} - \mu^2 \\
&= e^{-\mu}\left(\mu + 4\frac{\mu^2}{2!} + 9\frac{\mu^3}{3!} + 16\frac{\mu^4}{4!} + \dots\right) - \mu^2 \\
&= \mu e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \dots\right. \\
&\qquad \left. + \mu + 2\frac{\mu^2}{2!} + 3\frac{\mu^3}{3!} + \dots\right) - \mu^2 \\
&= \mu e^{-\mu}\left(e^{\mu} + \mu\left[1 + \mu + \frac{\mu^2}{2!} + \dots\right]\right) - \mu^2 \\
&= \mu e^{-\mu}\left(e^{\mu} + \mu e^{\mu}\right) - \mu^2 \\
&= \mu + \mu^2 - \mu^2 = \mu.
\end{aligned}$$

Mean and Variance: Uniform distribution

Mean:

$$\begin{aligned}
\mathrm{E}(X) &= \int_a^b x f(x)\,\mathrm{d}x \\
&= \int_a^b \frac{x}{b-a}\,\mathrm{d}x \\
&= \left[\frac{x^2}{2(b-a)}\right]_a^b \\
&= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b^2-a^2}{2(b-a)} \\
&= \frac{(b-a)(b+a)}{2(b-a)} = \frac{(b+a)}{2}
\end{aligned}$$

Variance:

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\
&= \int_a^b \frac{x^2}{b-a}\,\mathrm{d}x - \frac{(b+a)^2}{4} \\
&= \left[\frac{x^3}{3(b-a)}\right]_a^b - \frac{(b+a)^2}{4} \\
&= \frac{b^3-a^3}{3(b-a)} - \frac{(b+a)^2}{4} \\
&= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \frac{(b+a)^2}{4} \\
&= \frac{4(b^2+ab+a^2) - 3(b^2+2ab+a^2)}{12} \\
&= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}$$

Mean and Variance: Exponential distribution

Mean:

$$\begin{aligned}
\mathrm{E}(X) &= \int_0^\infty x f(x)\,\mathrm{d}x \\
&= \int_0^\infty x\lambda e^{-\lambda x}\,\mathrm{d}x \qquad\qquad u = \lambda x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^{-\lambda x} \\
&\qquad\qquad\qquad\qquad\qquad\quad \frac{\mathrm{d}u}{\mathrm{d}x} = \lambda \quad v = -\tfrac{e^{-\lambda x}}{\lambda} \\
&= \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,\mathrm{d}x \\
&= 0 + \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty \\
&= \left[-0 - \left(-\frac{1}{\lambda}\right)\right] = \frac{1}{\lambda}
\end{aligned}$$

Variance:

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\
&= \int_0^\infty x^2\lambda e^{-\lambda x}\,\mathrm{d}x - \frac{1}{\lambda^2}
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^\infty x^2\lambda e^{-\lambda x}\,\mathrm{d}x \qquad\qquad u = \lambda x^2 \quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^{-\lambda x} \\
&\qquad\qquad\qquad\qquad\qquad\quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2\lambda x \quad v = -\tfrac{e^{-\lambda x}}{\lambda} \\
&= \left[-x^2e^{-\lambda x}\right]_0^\infty + 2\int_0^\infty xe^{-\lambda x}\,\mathrm{d}x \\
&= 0 + \frac{2}{\lambda} \times \frac{1}{\lambda} = \frac{2}{\lambda^2}
\end{aligned}$$

So,

$$\mathrm{var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Example 3.6

$$X \sim N(3, 5^2) \quad \text{so,} \;\; Z = \frac{X-3}{5} \sim N(0,1).$$

1.

$$\begin{aligned} \mathrm{P}(X < 5) &= \mathrm{P}\left(\frac{X-3}{5} < \frac{5-3}{5}\right) \\ &= \mathrm{P}(Z < 0.4) = \Phi(0.4) = 0.6554. \end{aligned}$$

2.

$$\begin{aligned} \mathrm{P}(X < 2) &= \mathrm{P}\left(\frac{X-3}{5} < \frac{2-3}{5}\right) \\ &= \mathrm{P}(Z < -0.2) = 1 - \Phi(0.2) = 1 - 0.5793 = 0.4207. \end{aligned}$$

3.

$$\begin{aligned} \mathrm{P}\left(\frac{1}{2} < X < 5\frac{1}{2}\right) &= \mathrm{P}\left(\frac{\frac{1}{2}-3}{5} < \frac{X-3}{5} < \frac{5\frac{1}{2}-3}{5}\right) \\ &= \mathrm{P}\left(-\frac{1}{2} < Z < \frac{1}{2}\right) = \Phi(0.5) - \Phi(-0.5) \\ &= \Phi(0.5) - (1 - \Phi(0.5)) \\ &= 2\Phi(0.5) - 1 = 2 \times 0.6915 - 1 = 0.3830. \end{aligned}$$