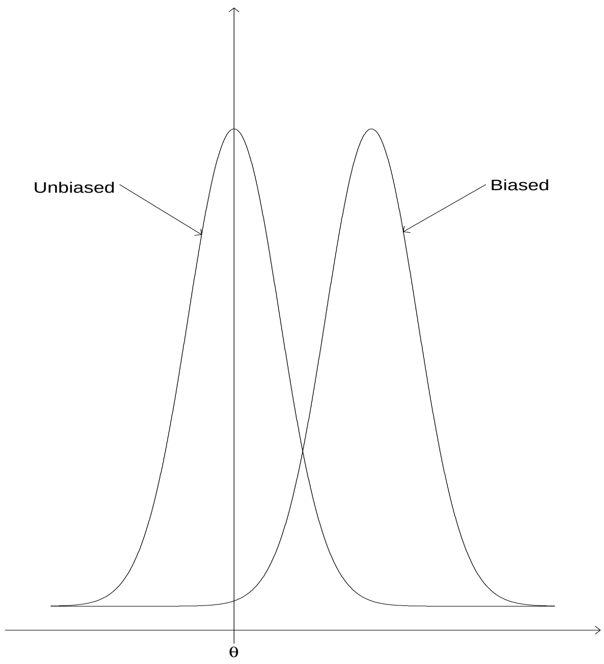
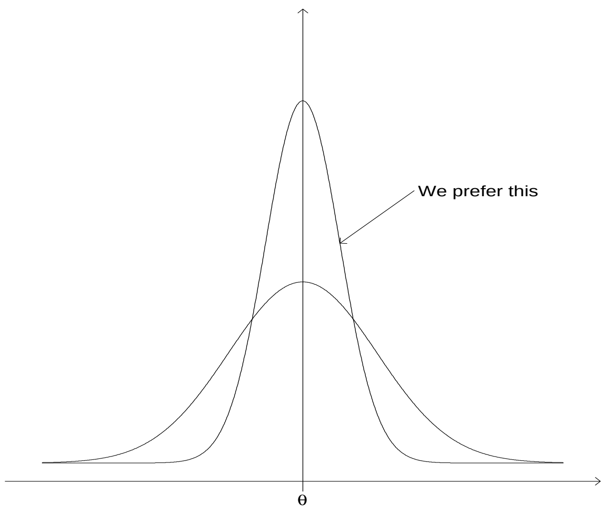
Bias

An estimator T is unbiased if $\mathrm{E}(T) = \theta$.



Variance

Note if we just use the first observation as our estimator i.e. $T = X_1$, then $T \sim \mathrm{N}(\mu, \sigma^2)$, so T is unbiased but has variance $= \sigma^2 > \sigma^2/n$, the variance of $\bar{X}$. So, we prefer to use $\bar{X}$.



Example 4.1 revisited

$$\bar{x} = 5.58 \quad \text{and} \quad \text{var}(\bar{x}) = \frac{0.5^2}{10} = 0.025.$$

Showing s^2 is an unbiased estimator of σ^2

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \quad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\begin{aligned}
\sum(X_i - \bar{X}) &= \sum(X_i^2 - 2\bar{X}X_i + \bar{X}^2) \\
&= \sum X_i^2 - 2\bar{X}\sum X_i + \sum \bar{X}^2 \\
&= \sum X_i^2 - 2\bar{X}(n\bar{X}) + n\bar{X}^2 \\
&= \sum X_i^2 - n\bar{X}^2
\end{aligned}$$

$$\mathrm{E}(s^2) = \frac{1}{n-1}\left(\sum \mathrm{E}(X_i^2) - n\mathrm{E}(\bar{X}^2)\right)$$

$$\text{var}(X_i) = \sigma^2 = \mathrm{E}(X_i^2) - [E(X_I)]^2 \Rightarrow \mathrm{E}(X_i^2) = \sigma^2 + \mu^2$$

$$\text{var}(\bar{X}) = \frac{\sigma^2}{n} = \mathrm{E}(\bar{X}^2) - [\mathrm{E}(\bar{X})^2] \Rightarrow \mathrm{E}(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$$

So,

$$\begin{aligned}
\mathrm{E}(s^2) &= \frac{1}{n-1}\left[\sum(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right] \\
&= \frac{1}{n-1}\left(\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2\right) \\
&= \frac{1}{n-1}(\sigma^2(n-1)) = \sigma^2
\end{aligned}$$

Derivation of CI: known variance

$$\mathrm{E}(\bar{X}) = \mu \quad \text{var}(\bar{X}) = \sigma^2$$

$$\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right) \qquad \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim \mathrm{N}(0, 1)$$

$$\begin{aligned}
& \mathrm{P}\left(-1.96 \leq \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq 1.96\right) = 0.95 \\
\Rightarrow\ & \mathrm{P}\left(1.96 \geq \frac{-\bar{X} + \mu}{\frac{\sigma}{\sqrt{n}}} \geq -1.96\right) = 0.95 \\
\Rightarrow\ & \mathrm{P}\left(-1.96\frac{\sigma}{\sqrt{n}} \leq -\bar{X} + \mu \leq 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95 \\
\Rightarrow\ & \mathrm{P}\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95
\end{aligned}$$

We say,

$$\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}$$

is a 95% confidence interval (CI) for μ. The endpoints of the interval are random, the probability that it contains the unknwon value μ is 0.95.

If our required level of confidence is $1-\alpha$, i.e.

P(interval covers unknown parameter) $= 1-\alpha$, then the $100(1-\alpha)\%$ CI for μ is

$$\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$

e.g. 90% = 100(1-0.1)% CI for μ is given by

$$\bar{X} - 1.645\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.645\frac{\sigma}{\sqrt{n}}$$

Example 4.2

data from Example 4.1, $\bar{x} = 5.58$ and $\frac{\sigma^2}{n} = \frac{0.5^2}{10} = 0.025$.

95% CI for μ is given by

$$\left(\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

where $z_{\frac{\alpha}{2}} = 1.96$.

$$\begin{aligned} &= (5.58 - 1.96\sqrt{0.025}, 5.58 + 1.96\sqrt{0.025}) \\ &= (5.58 - 0.301, 5.58 + 0.301) \\ &= (5.279, 5.881) \end{aligned}$$

Derivation of CI: unknown variance

If T has a t distribution with $n-1$ degrees of freedom, write t_α for the point such that

$$\mathrm{P}(T > t_\alpha) = \alpha.$$

e.g.

sample size, n	$t_{0.025}$
5	2.776
10	2.262
30	2.045
∞	1.96 $(= z_{0.025})$.

$$\Rightarrow \mathrm{P}\left(\frac{|\bar{X} - \mu|}{s/\sqrt{n}} < t_{\frac{\alpha}{2}}\right) = 1 - \alpha.$$

Then, a $100(1-\alpha)\%$ CI for μ when σ^2 is NOT known is

$$\bar{X} - \frac{s}{\sqrt{n}}t_{\frac{\alpha}{2}} \leq \mu \leq \bar{X} + \frac{s}{\sqrt{n}}t_{\frac{\alpha}{2}}$$

Example 4.3 revisited

Suppose we know that $\sigma^2 = 0.4^2$ – the variance of the distribution of the weights of the packets, and that $n = 10$.

Suppose we choose our rejection region as $c = 99.8$, i.e. we reject the null hypothesis if $\bar{X} < 99.8$ – what are the properties of this test?

$$\begin{aligned}
\alpha &= \mathrm{P}(\bar{X} < 99.8 | H_0 \text{ is true}) \\
&= \mathrm{P}(\bar{X} < 99.8 | \mu = 100) \\
&= \mathrm{P}\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{99.8 - \mu}{\sigma/\sqrt{n}} | \mu = 100\right) \\
&= \mathrm{P}\left(\frac{\bar{X} - 100}{0.4/\sqrt{10}} < \frac{99.8 - 100}{0.4/\sqrt{10}}\right) \\
&= \mathrm{P}(Z < -1.58) = 1 - \Phi(1.58) \\
&= 1 - 0.9429 = 0.0571.
\end{aligned}$$

Often, α is specified, e.g. $\alpha = 0.10$, and we must determine c.

$$\begin{aligned}
0.1 &= \mathrm{P}\left(Z < \frac{c - 100}{0.4/\sqrt{10}}\right) \\
\Rightarrow \quad \frac{c - 100}{0.4/\sqrt{10}} &= -1.2816 \\
\Rightarrow \quad c &= -1.2816 \times \frac{0.4}{\sqrt{10}} + 100 \\
&= 99.83.
\end{aligned}$$

So, to obtain a Type I error of 0.10, we would reject H_0 if the sample mean was less than 99.83.

Example 4.4

Steel bolts are produced by a manufacturer. The length of the bolts (in mm) is assumed to be normally distributed $\mathrm{N}(\mu, \sigma^2)$ with $\sigma^2 = 0.1$ mm^2. The nominal specification is $\mu = 50$ mm and the manufacturer wished to see whether this specification is being met. The manufacturer takes a random sample of size 20, and calculates the sample mean as 49.2.

Here we have

$$\begin{aligned}
H_0 &: \mu = 50 \\
H_A &: \mu \neq 50
\end{aligned}$$

So the test statistic is of the form

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{49.2 - 50}{\sqrt{0.1/20}} = -11.314.$$

With H_0 being rejected if $Z < -c$ or $Z > c$.

Suppose we want significance level $\alpha = 0.05$. We therefore have $c = z_{\alpha/2} = z_{0.025} = 1.96$

As $-11.314 < -1.96$, we reject H_0 at the 5% level.

Exam Question (on Exercises III)

The unbiased estimators of the mean and variance are:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Giving,

$$\bar{x} = 9.926 \quad \text{and} \quad s^2 = 0.1072^2.$$

We have,

$$H_0 \,:\, \mu = 10$$
$$H_A \,:\, \mu \neq 10$$

The observed value of the test statistic is

$$T = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{9.926 - 10}{0.1072/\sqrt{10}} = -2.18.$$

Here, we have a 2-sided test. If $\alpha = 0.05$, we have $t^9_{0.025} = 2.26$. So, we fail to reject the null hypothesis at the 5% level (as $-2.26 < -2.18 < 2.26$)

For $\alpha = 0.1$ we have $t^9_{0.05} = 1.83$. So we reject the null hypothesis at the 10% level (as $-2.18 < -1.83$).

Example 4.5

An oil company develops an additive that may change the average mileage per gallon in cars.

Two random samples of mileages per gallon were collected giving the following sample statistics:

The mileages in each group are assumed to be normally distributed with known variances $\sigma_1^2 = 10^2$ mpg^2 and $\sigma_2^2 = 12^2$ mpg^2.

Is there evidence that the miles per gallon are different?

We have,

Without additive	With additive
$n_1 = 25$	$n_2 = 25$
$\bar{x}_1 = 28.4$	$\bar{x}_2 = 32.1$

$$H_0 \; : \; \mu_1 = \mu_2$$
$$H_A \; : \; \mu_1 \neq \mu_2$$

with

$$Z = \frac{28.4 - 32.1}{\sqrt{\frac{10^2}{25} + \frac{12^2}{25}}} = \frac{-3.7}{3.12} = -1.19$$

For $\alpha = 0.1$, we have rejection region

$$Z > 1.6449 \quad \text{or} \quad Z < -1.6449.$$

Since -1.19 is not contained in this region, we have no evidence to reject H_0 at the 10% level.

Example 4.6

A computer centre wishes to compare the average response times of two disk drives. We assume that the response times for the disk drives are normally distributed with means, μ_1 and μ_2, and common variance σ^2.

We wish to test

$$H_0 \; : \; \mu_1 = \mu_2$$
$$H_A \; : \; \mu_1 \neq \mu_2$$

Two random samples of response times (in millisecs) were collected, giving the following summary statistics:

Disk 1	Disk2
$n_1 = 13$	$n_2 = 15$
$\bar{y}_1 = 68$	$\bar{y}_2 = 53$
$s_1 = 18$	$s_2 = 16$

Test H_0 with $\alpha = 0.05$.

We must evaluate the observed test statistic:

$$T = \frac{\bar{y}_1 - \bar{y}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where

$$s^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2}{n_1+n_2-2} = \frac{12\times 18^2 + 14\times 16^2}{13+15-2} = 287.38.$$

Rejection region is of the form

$$T < -t_{0.025}^{26} = -2.056 \quad \text{and} \quad T > t_{0.025}^{26} = 2.056.$$

We have

$$T = \frac{68-53}{16.95\sqrt{\frac{1}{13}+\frac{1}{15}}} = 2.34.$$

So we reject H_0 at the 5% level.

Example 4.7

A bakery implements a new leavening process to decrease the number of calories in its loaves. The table below gives summary statistics of two randomly collected samples of loaves from the new and old process.

New	Old
$n_1 = 20$	$n_2 = 18$
$\bar{y}_1 = 1255$ calories	$\bar{y}_2 = 1330$ calories
$s_1 = 245$ calories	$s_2 = 238$ calories

We assume that the new and old process have calories per loaf which are normally distributed with means, μ_1 and μ_2, and common variance σ^2.

Here we have a 1-sided test:

$$H_0 : \mu_1 = \mu_2$$
$$H_A : \mu_1 < \mu_2.$$

Test this hypothesis at the 5% level.

We have the observed test statistic

$$T = \frac{\bar{y}_1 - \bar{y}_2}{s\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

where

$$s^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2}{n_1+n_2-2} = \frac{19\times 245^2 + 17\times 238^2}{20+18-2} = 58428.42.$$

Rejection region is of the form

$$T < -t_{0.05}^{36} = -1.688$$

We have

$$T = \frac{1255 - 1330}{241.72\sqrt{\frac{1}{18} + \frac{1}{20}}} = -0.96.$$

So, there is no evidence to reject H_0 at the 5% level.