Subtle relations: prime numbers, complex functions, energy levels and Riemann.

Prof. Henrik J. Jensen, Department of Mathematics, Imperial College London.

At least since the old Greeks mathematics the nature of the prime numbers has puzzled people. To understand how the primes are distributed Gauss studied the number $\pi(x)$ of primes less than a given number x. Gauss fund empirically that $\pi(x)$ is approximately given by $x/\log(x)$. In 1859 Riemann published a short paper where he established an exact expression for $\pi(x)$. However, this expression involves a sum over the zeroes of a certain complex function. The properties of the zeroes out in the complex plane determine the properties of the primes! Riemann conjectured that all the relevant zeroes have real part ½. This has become known as the Riemann hypothesis and is considered to be one of the most important unsolved problems in mathematics. Some statistical properties of the zeroes are known. Surprisingly it has become clear that the zeroes are distributed according to the same probability function that describes the energy levels of big atomic nucleuses. Hence we arrive at a connection between the prime numbers and quantum energy levels in nuclear physics.

Subtle relations: Prime numbers, Complex functions, Energy levels and Riemann

Henrik Jeldtoft Jensen,

Dept of Mathematics

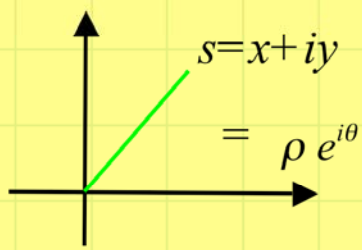
Background from: http://www.math.ucsb.edu/~stopple/zeta.html

Even numbers: 2, 4, 6, 8 , …

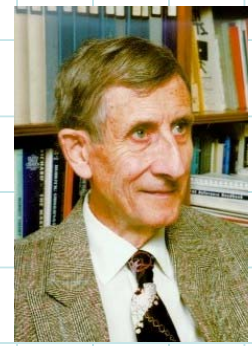
Odd numbers: 1, 3, 5, 7, …

- The primes: 2, 3, 5, 7, 11, 13, … what's the pattern?

Complex numbers



Energy levels of heavy nucleuses



Dyson

Riemann

Zeroes of $\zeta(s)$

Quantum mechanics of chaotic systems

Hugh L Montgomery

Odlyzko

Motto of the talk:

Why it is a good idea to know about complex numbers and complex functions.

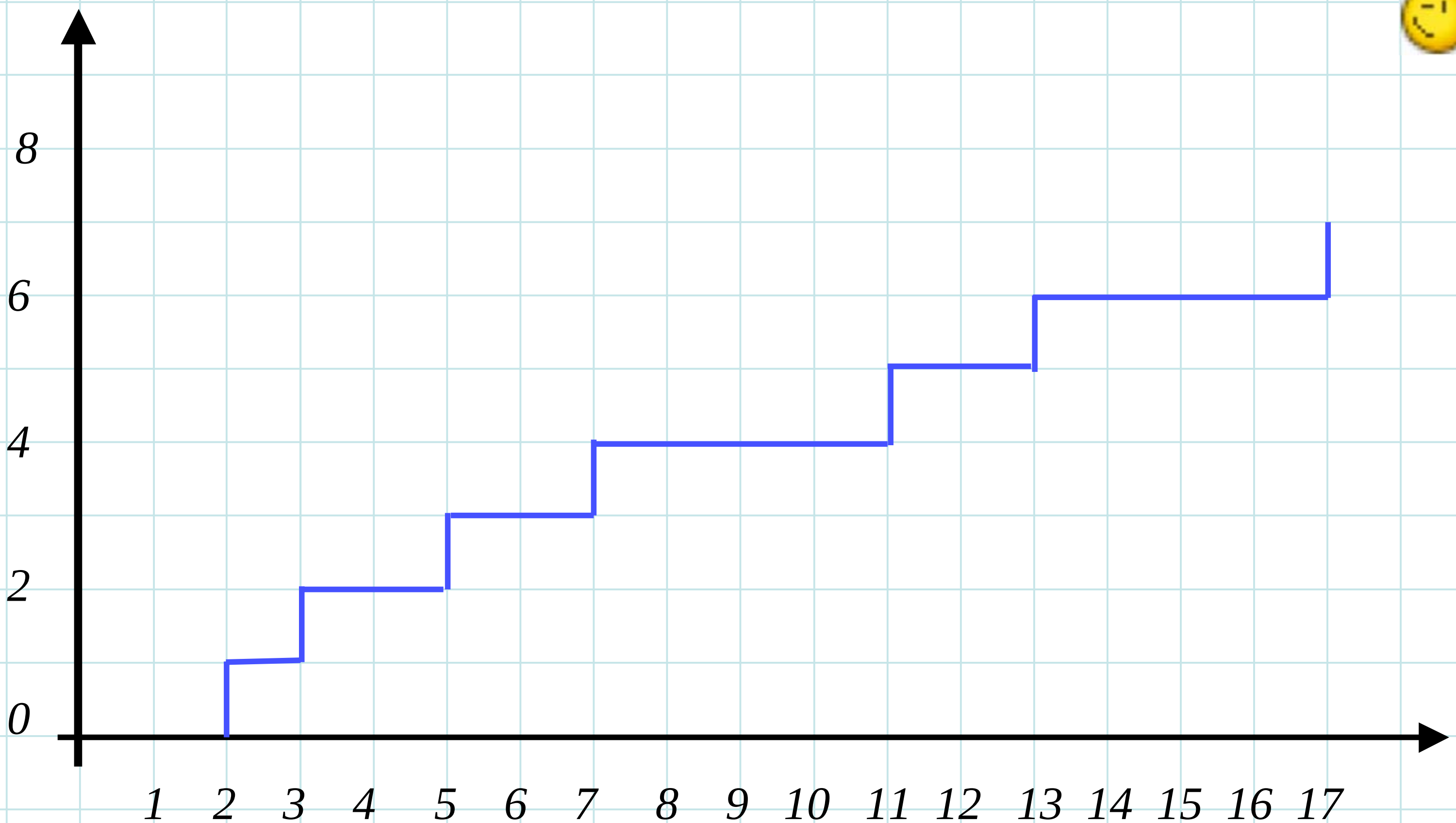
Copied from http://www.union.ic.ac.uk/dsc/physoc/index.php

Content:

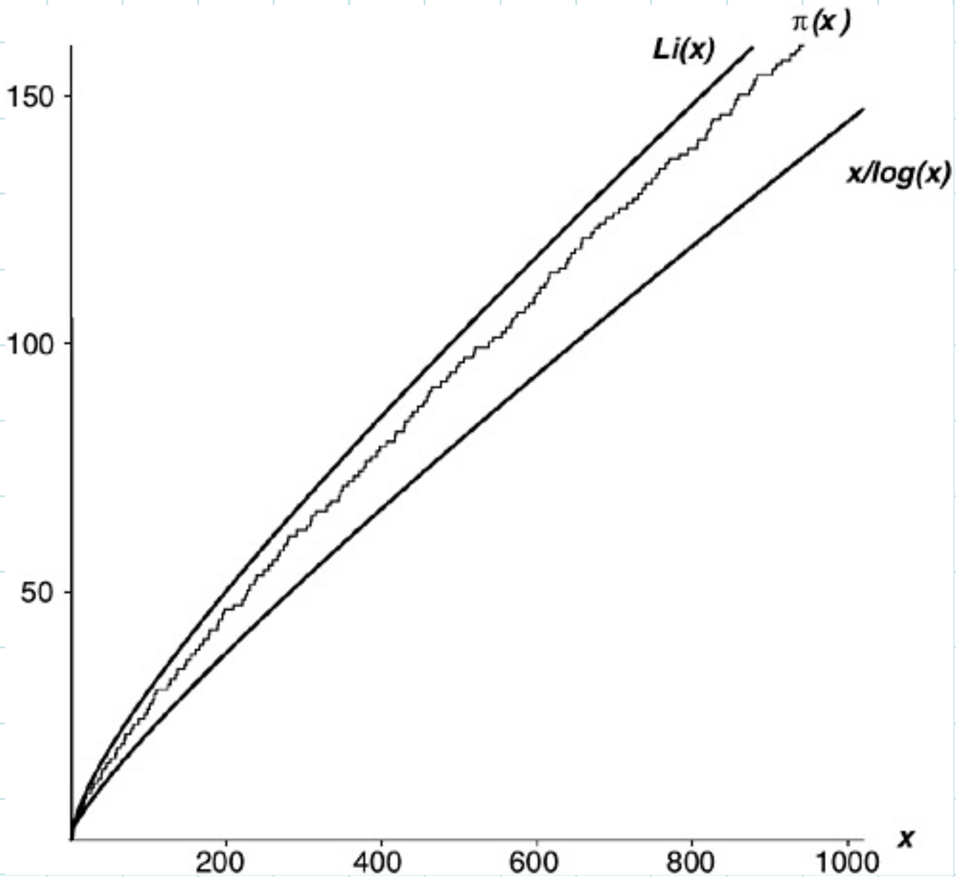
1. Number of primes less than x
2. Riemann's 1859 paper
3. Analytic continuation
4. Energy levels of large nucleuses
5. Random matrix theory
6. The same statistics

Number theory:

$\pi(x)$ = number of primes less than x



Gauss' estimate: $\pi(x) \approx \dfrac{x}{\log(x)}$



From J Derbyshire: *Prime Obsession*

Riemann's analytic approach.

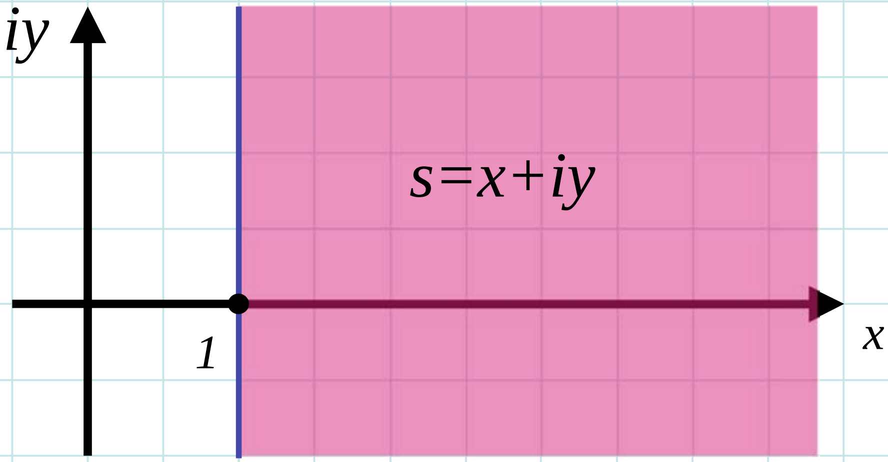
The zeta function: $$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Examples:

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$$

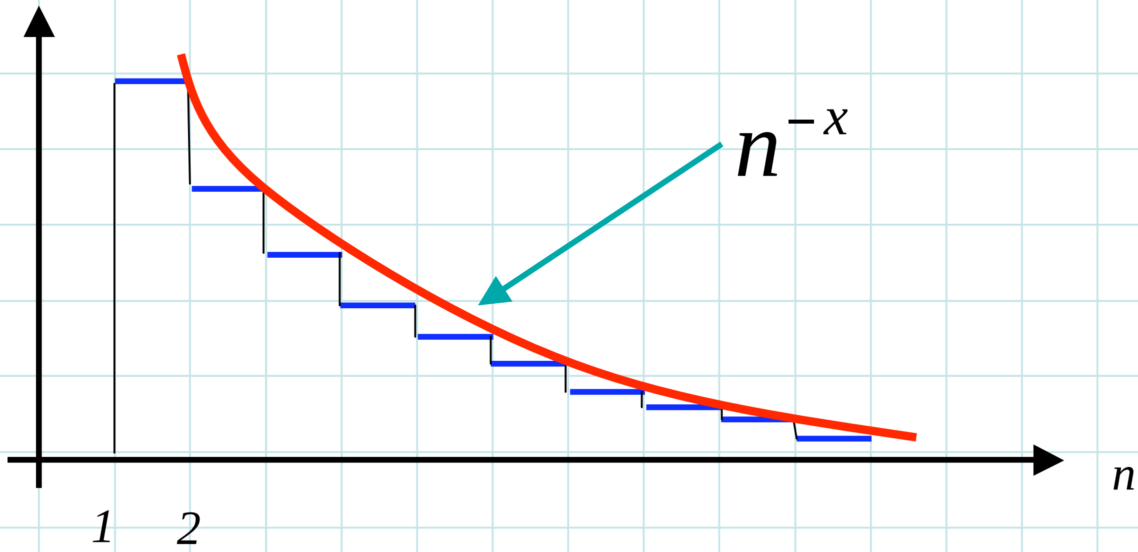
$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \infty$$

Domain of definition:



$$|\zeta(s)| = \left|\sum_{n=1}^{\infty}\frac{1}{n^s}\right| \leq \sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty}\frac{1}{\left|n^{x+iy}\right|}$$

$$= \sum_{n=1}^{\infty}\frac{1}{\left|n^{x}\right|\left|n^{iy}\right|} = \sum_{n=1}^{\infty}\frac{1}{n^{x}}$$



Area under blue steps < Area under red curve

$$\sum_{n=1}^{\infty} \frac{1}{n^x} < 1 + \int_2^{\infty} n^{-x} dn < \infty$$

When $x>1$

And what about the primes?

According to Euler:

$$\zeta(s) = \prod_{p \in prime} \left(1 - \frac{1}{p^s}\right)^{-1}$$

because:

$$\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots$$

then:

$$\frac{1}{\left(p_1^{m_1} p_2^{m_2} \cdots p_q^{m_q}\right)^s} = \frac{1}{n^s}$$

From Euler’s product we learn:

1. Prime numbers are hidden inside $\zeta(s)$

2. And $\zeta(s) \neq 0$ when product formula valid

Since

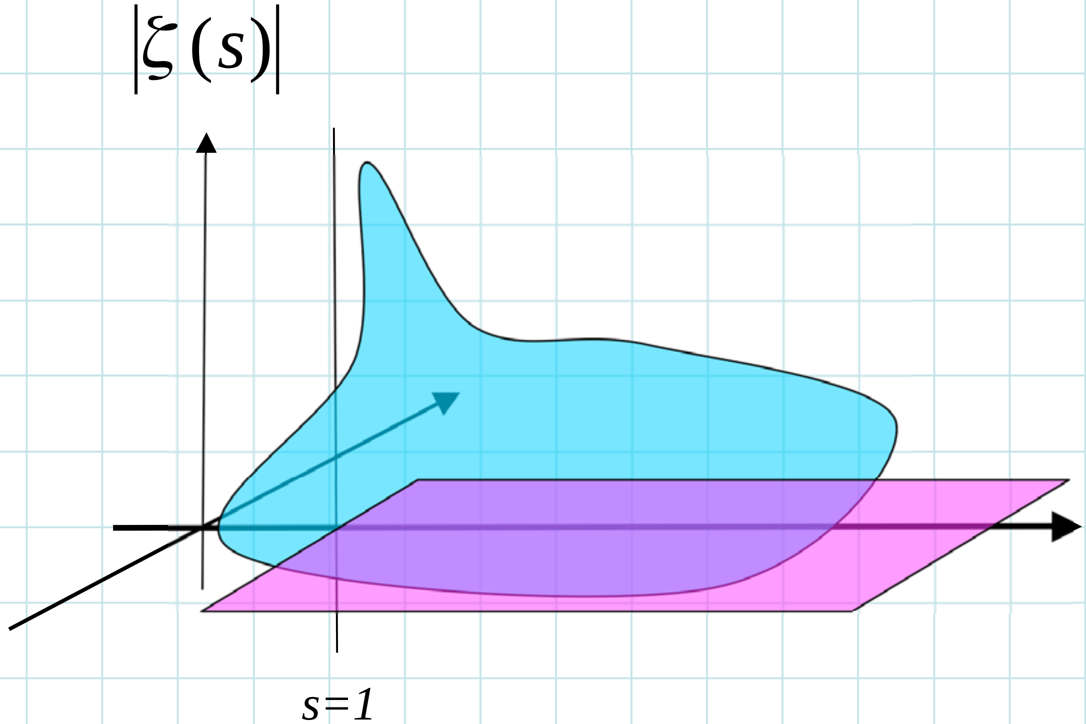
$$0 = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \Rightarrow \exists p \text{ such that } \left(1 - \frac{1}{p^s}\right)^{-1} = 0$$

Not possible

Summary so far

Structure of $\zeta(s)$:

$\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$



Where is the relation to the primes?

Can one make sense of $\zeta(s)$ for $Re(s)<1$?

We start with the *Re(s)* <1 issue.

To illustrate the nature of what Riemann did consider the function:

$$f(z) = 1 + z + z^2 + z^3 + \cdots = \begin{cases} \dfrac{1}{1-z} & \text{for } |z| < 1 \\ \infty & \text{for } |z| < 1 \end{cases}$$

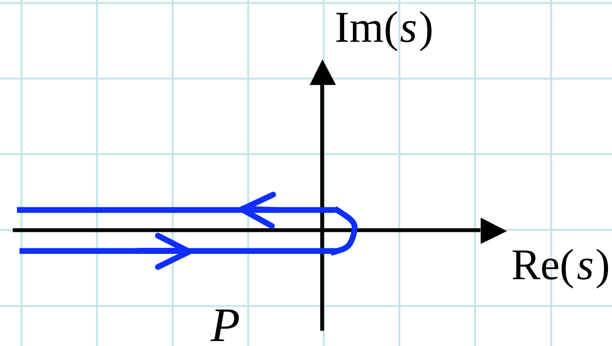
Cannot be used for
$|z|>1$

Well defined for all $z \neq 1$

The extension beyond $|z|<1$ is **UNIQUE!**

Riemann's extension of $\zeta(z)$

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\oint_P \frac{u^{s-1}}{e^{-u}-1}\,du$$



Valid for all $z \neq 1$

Where $\Gamma(z)$ is the Gamma function originating from the usual $n! = n(n-1)(n-2)\cdots 2\cdot 1$

The functional equation for the zeta function

$$\zeta(s) = \frac{(2\pi)^s \sin(\pi s/2)\zeta(1-s)}{\sin(\pi s)\Gamma(s)}$$

From this follows that

$$\zeta(s) = 0 \text{ for } s = -2, -4, -6, \ldots$$

and that all other zeroes MUST lie in the strip

$$0 \le \mathrm{Re}(s) \le 1$$

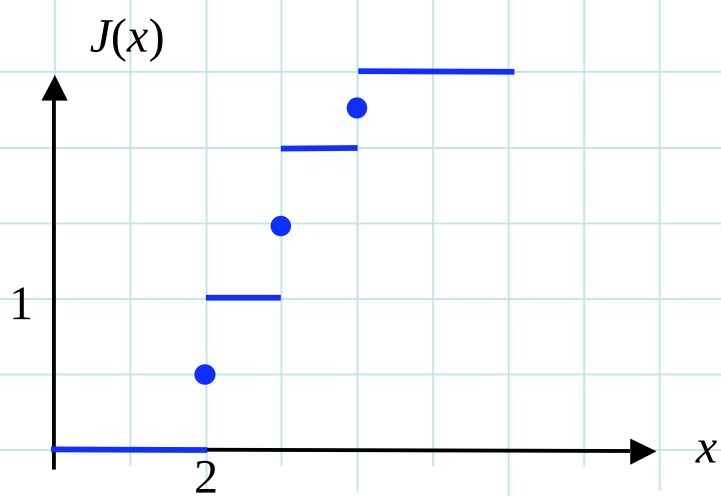
the critical strip

Why is it interesting?

Because the primes can be found from the zeroes of $\zeta(s)$.

Consider

$$J(x) = \begin{cases} 0 \quad \text{for } x = 0 \\ \text{jump by } 1 \text{ when } x = \text{prime } p \\ \text{jump by } 1/2 \text{ when } x = \text{prime squares } p^2 \\ \text{jump by } 1/3 \text{ when } x = \text{prime cubes } p^3 \\ ect. \end{cases}$$



Defined to be "halfway" at jumps

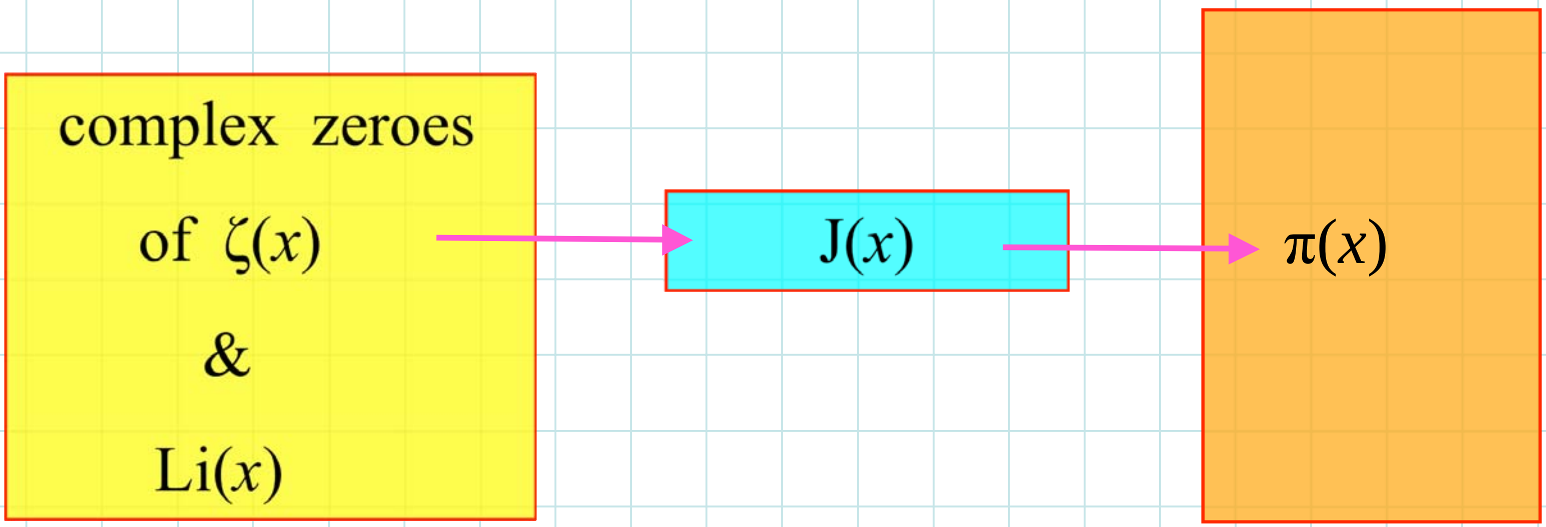
Express $\pi(x)$ in terms of $J(x)$

express $J(x)$ in terms of $Li(x)$

evaluated at the complex zeroes of

$\zeta(x)$

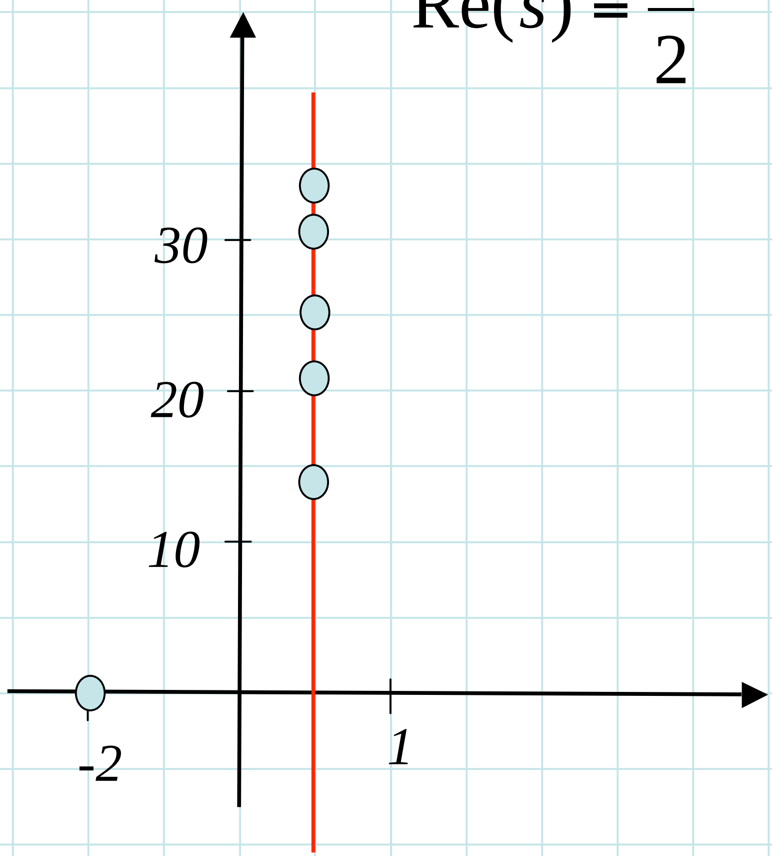
Procedure:



Riemann's hypothesis

All complex zeroes are confined to

$$\mathrm{Re}(s) = \frac{1}{2}$$



With

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

continued to

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \oint_P \frac{u^{s-1}}{e^{-u}-1} du$$

Why is it interesting?

Because the primes can be found from the zeroes of $\zeta(s)$.

According to Riemann:

$$\pi(x) = \sum_{n=1}^{M} \frac{\mu(n)}{n} J(x^{1/n}),$$

with $\mu = -1,0,1$ and $M = \left\lceil \log x \big/ \log 2 \right\rceil$

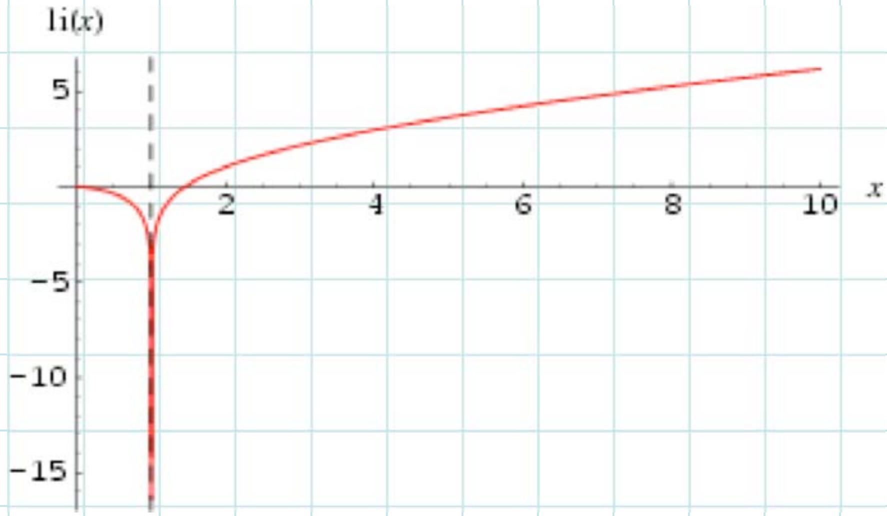
and

$$J(x) = Li(x) - \sum_{\operatorname{Im}\rho>0} Li(x^{\rho}) - \log 2 + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\log t}$$

here

$$J(x) = Li(x) - \sum_{\operatorname{Im}\rho>0} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

$$Li(x) = \int_0^x \frac{dt}{\log t}$$



From:http://mathworld.wolfram.com/LogarithmicIntegral.html

$$J(x) = Li(x) - \sum_{\operatorname{Im}\rho>0} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

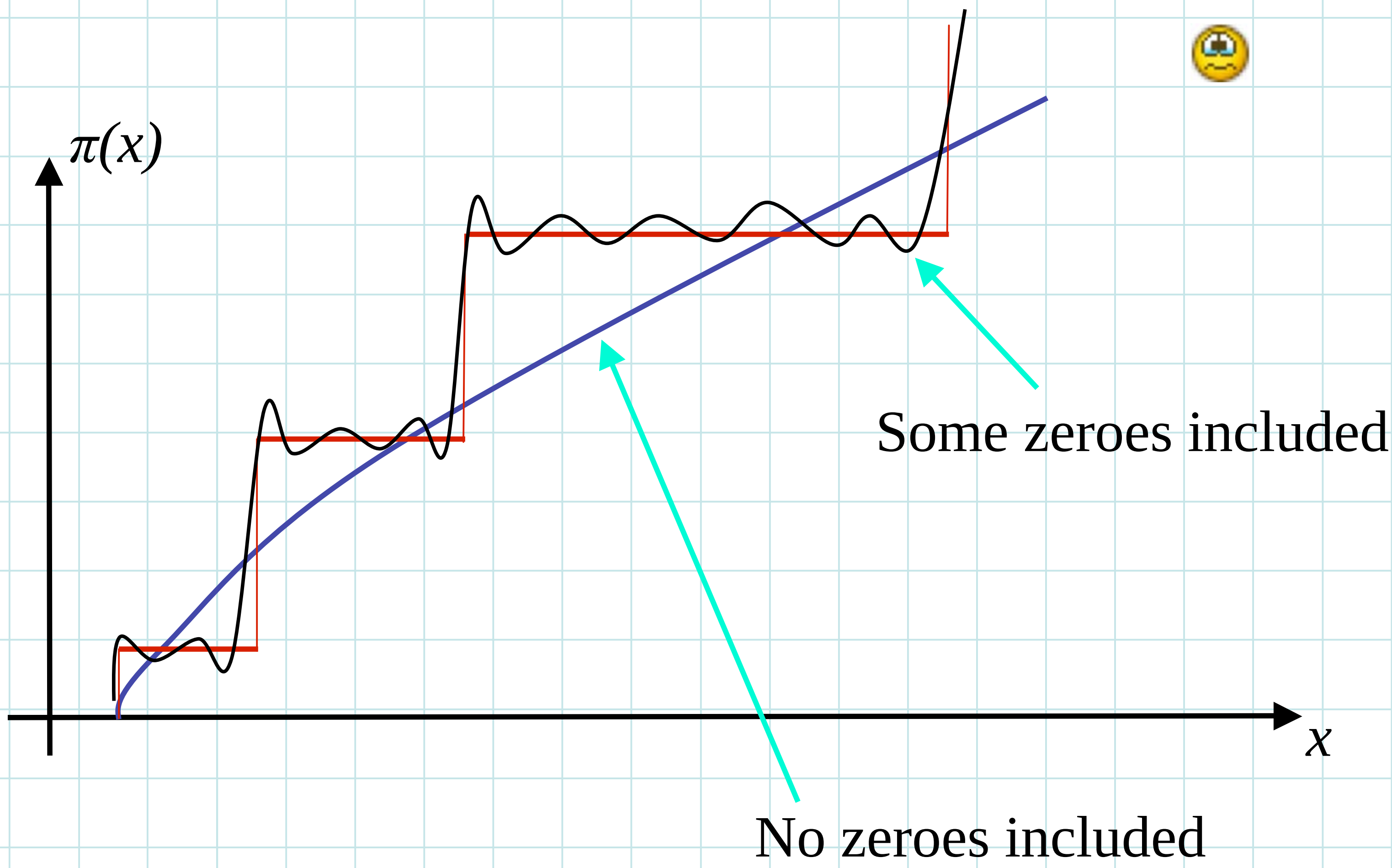
The sum over the zeroes contains oscillatory terms since

$$x^{\rho} = x^{\frac{1}{2}+it} = \sqrt{x}\cdot x^{it} = \sqrt{x}\cdot\left(e^{\ln x}\right)^{it}$$

$$= \sqrt{x}e^{it\ln x} = \sqrt{x}\left[\cos(t\ln x) + i\sin(t\ln x)\right]$$

Adding up the contribution from the zeroes

From (http://www.maths.ex.ac.uk/~mwatkins/zeta/encoding2.htm)

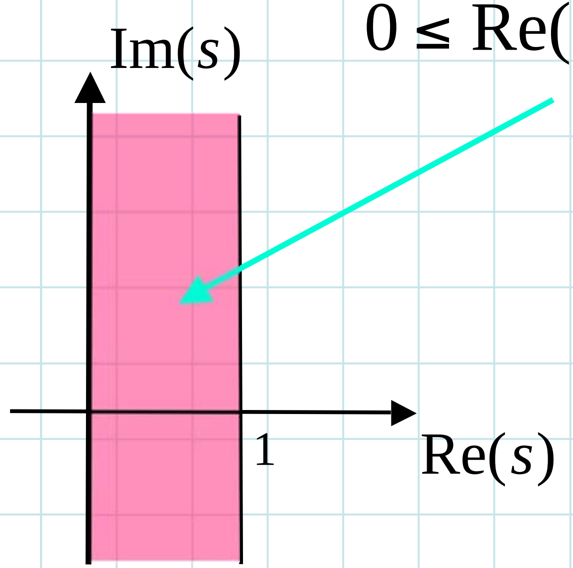


Hunting the zeroes

From functional equation $\zeta(s) = \dfrac{(2\pi)^s \sin(\pi s/2)\zeta(1-s)}{\sin(\pi s)\Gamma(s)}$

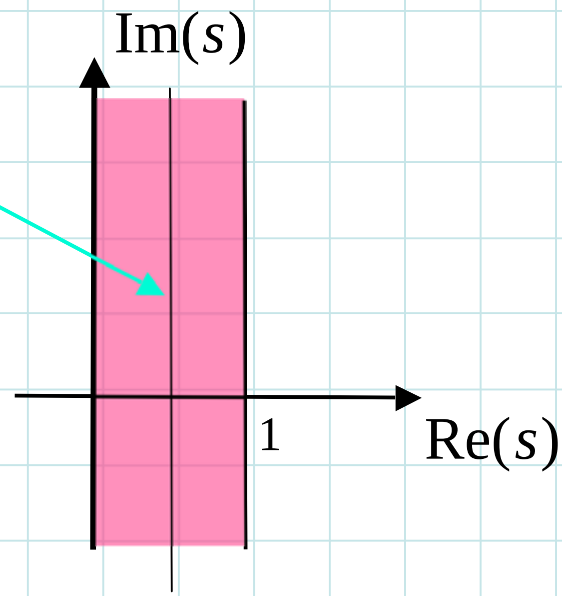
Follows that all zeroes must be confined in the strip

$$0 \le \mathrm{Re}(s) \le 1$$



Riemann conjectured in 1859 that zeroes are all on the line

$$\mathrm{Re}(s) = \frac{1}{2}$$



But he couldn't prove it!

Nor has anyone else been able to ever since. Now it is the *Riemann Hypothesis.*

Prove it and get 10^6 \$ from the Clay Institute.

Numerical results support the Riemann Hypothesis:

- Andrew M. Odlyzko

  The 10^{20}-th zero of the Riemann zeta function and 175 million of its neighbours

- *Sebastian Wedeniwski*

  The first 10^{11} nontrivial zeros of the Riemann zeta function lie on the line $Re(s) = 1/2$. Thus, the Riemann Hypothesis is true at least for all

  $$|Im(s)| < 29,538,618,432.236$$

  which required $1.3 \cdot 10^{18}$ floating-point operations

A zero off the critical line would induce a pattern in the distribution of the primes:

$$J(x) = Li(x) - \sum_{\operatorname{Im}\rho>0} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

$$x^{\operatorname{Re}(s)}\left[\cos(t \ln x) + i \sin(t \ln x)\right]$$

Zeroes with different Re(s) would contribute with different weights

- The distribution along the critical line determines the distribution of the primes.

- So how are they distributed?

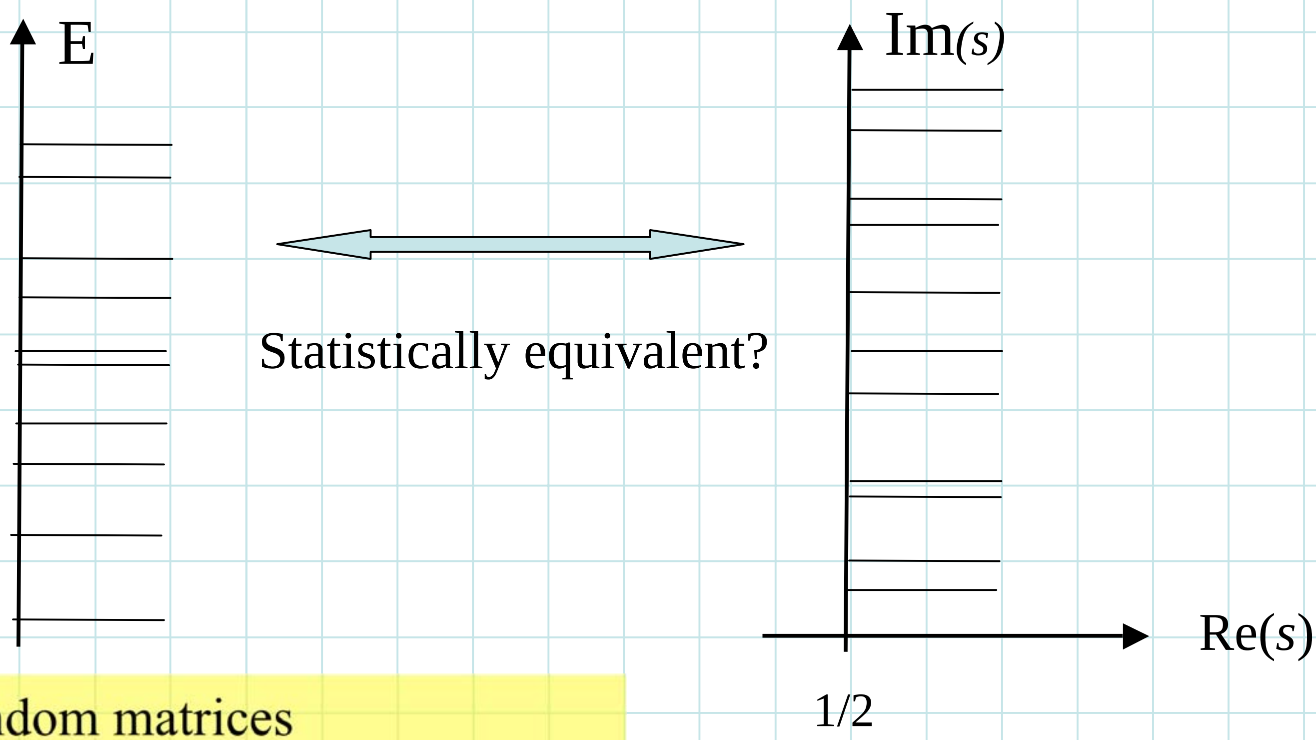
- Hugh L Montgomery's Pair Correlation Conjecture:

  Derived from the Riemann Hypothesis plus conjectures concerning twin primes.

  Spacing between zeroes controlled by

$$1-\left(\frac{\sin \pi u}{\pi u}\right)^2$$

Dyson spotted that the same function describes the spacing between energy levels, i.e., eigenvalues of Hamiltonians describing big nucleuses:



Random matrices

(Gaussian unitary ensemble)

The imaginary zeroes of $\zeta(s)$

Connection between random matrices and ζ(*s*) confirmed numerically:

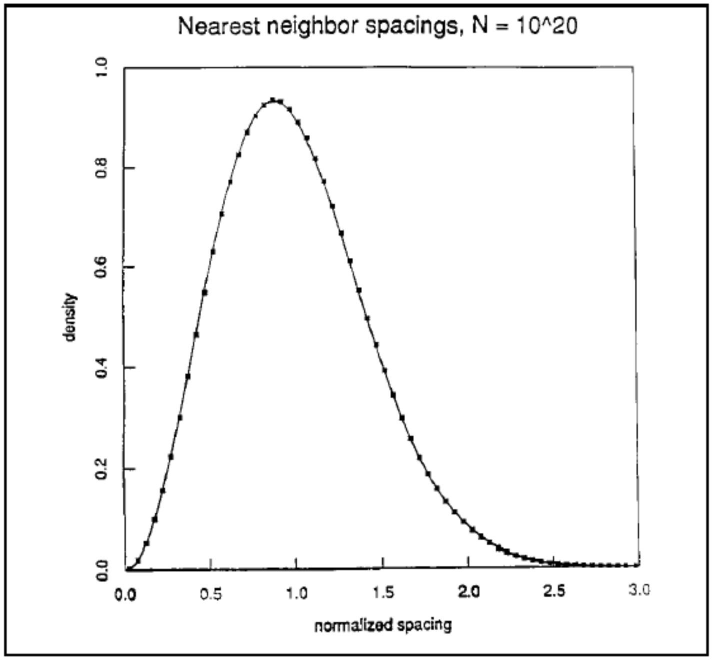


Figure 8a. The nearest neighbor spacing for GUE (solid) and for 7.8×10^7 zeros of $\zeta(s)$ near the 10^{20} zero (scatterplot). Graphic by A. Odlyzko.

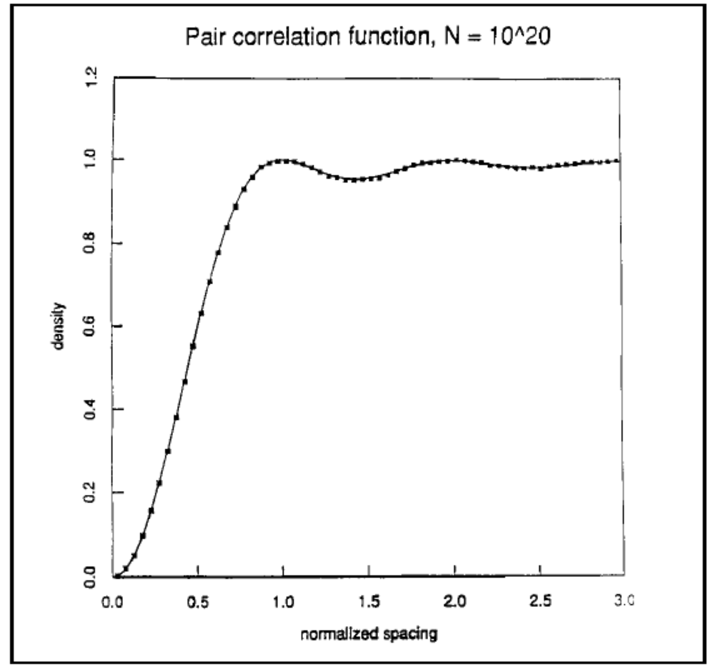


Figure 8b.The pair-correlation function for GUE (solid) and for 8×10^6 zeros of $\zeta(s)$ near the 10^{20} zero (scatterplot). Graphic by A. Odlyzko.

From: J. Brian Conrey, *Notices of the AMS*, March 2003, p. 341

Puzzle:

- However as one study the correlations between the N^{th} and the $N^{th}+K$ zeroes for large N and K the statistics doesn't exactly fit the GUE.

- M. Berry pointed out that the correlations between the zeroes of $\zeta(s)$ are like the correlations of the energy levels of a Quantum Chaotic system.

Conclusion

- Complex analysis is impressively powerful

- Profoundly surprising connections:

    primes,   energy levels,   quantum chaos

Background from: http://www.math.ucsb.edu/~stopple/zeta.html

Some references:

Technical:

H.M. Edwards: *Riemann's Zeta Function*

M.L. Mehta: *Random Matrices*

Non-technical:

J. Derbyshire: *Prime Obsession*

M. du Sautoy: *The Music of the Primes*

A very useful link that contains a wealth of references:

Matthew R. Watkins' home page namely:

http://www.maths.ex.ac.uk/~mwatkins/

What to see the slides again?

Can be found at: www.ma.imperial.ac.uk/~hjjens