Dual Control Monte-Carlo Method for Tight Bounds of Value Function under Heston Stochastic Volatility Model

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Abstract

The aim of this paper is to study the fast computation of the lower and upper bounds on the value function for utility maximization under the Heston stochastic volatility model with general utility functions. It is well known there is a closed form solution to the HJB equation for power utility due to its homothetic property. It is not possible to get closed form solution for general utilities and there is little literature on the numerical scheme to solve the HJB equation for the Heston model. In this paper we propose an efficient dual control Monte-Carlo method for computing tight lower and upper bounds of the value function. We identify a particular form of the dual control which leads to the closed form upper bound for a class of utility functions, including power, non-HARA and Yaari utilities. Finally, we perform some numerical tests to see the efficiency, accuracy, and robustness of the method. The numerical results support strongly our proposed scheme.

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1 Introduction

Dynamic portfolio optimization is one of the most studied research areas in mathematical finance. Stochastic control and convex duality are two standard methods to solve utility maximization problems. For a complete market such as the Black-Scholes model, the problem has already been solved. With the convex duality method, one may first solve a static convex optimization problem for the optimal terminal wealth and then derive the optimal control (trading strategy) to replicate it with the martingale representation theorem. With the stochastic control method, one may first solve the dynamic programming equation (a nonlinear partial differential equation (PDE), called the Hamilton-Jacobi-Bellman (HJB) equation) for the optimal value function, and then find the optimal control, see many excellent books for expositions, e.g., Karatzas and

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Shreve (1998), Pham (2009). For an incomplete market model, one cannot use the standard convex duality method to solve the problem as not all risks can be hedged, a key requirement for finding a replicating strategy of the optimal terminal wealth with the martingale representation theorem.

One well-known incomplete market model is the Heston stochastic volatility model which was first introduced by Heston (1993) for a European options pricing problem. The Heston model is also called the mean reversion square root model which is used to describe dynamically the variance of the underlying stock and ensures the variance process (a stochastic process) is nonnegative and moves towards a long term average variance level at certain speed. The Heston model is widely used in financial industry due to its ability to characterize the market volatility phenomenon of traded assets, such as smiles and skews, and its analytic tractability for European options pricing. The option value (price) satisfies a linear PDE with two state variables (asset price and variance) and a closed form solution (in terms of the Fourier transform of the price variable) can be found, see Gatheral (2006) for details.

Utility maximization with the Heston stochastic volatility model is much more difficult to solve, compared with European options pricing with the Heston model or utility maximization with the Black-Scholes model. One may still use the standard stochastic control method to derive the HJB equation for the value function. However, the resulting PDE is fully nonlinear with two state variables (wealth and variance) and is highly difficult to solve.

For a power (or exponential) utility and the Heston stochastic volatility model, one may apply the separation principle to decompose the solution to the HJB equation and get a simplified nonlinear PDE with one state variable (variance). Thanks to the affine structure of the Heston model, Zariphopoulou (2001) uncovers a clever transformation that simplifies the nonlinear PDE further into an equivalent linear PDE and derives a closed-form solution, see also Kraft (2005). Kallsen and Muhle-Karbe (2010) extend the Heston model to general univariate affine stochastic volatility models and Richter (2014) to multivariate case, both are solved with the probabilistic method. The model framework with the PDE approach is extended further by Noh and Kim (2011) to stochastic volatility and stochastic interest rate on an infinite time horizon, Zeng and Taksar (2013) to general stochastic volatility with the Heston and 3/2 models as special cases, Zhang and Ge (2016) to optimal asset allocation and consumption models, and Boguslavskaya and Muravey (2016) to general factor models. All aforementioned papers discuss only power (or exponential) utilities and have used the separation principle to reduce the dimensionality of state variables by one.

The success of finding a closed-form solution for utility maximization with the Heston model (or general affine stochastic processes) crucially depends on the underlying utility being a power utility if the wealth process is exponential (or exponential utility if the wealth process is additive). Such combination of utility and wealth process would keep the utility of the terminal wealth still in the exponential form of a linear combination of underlying stochastic processes, which in turn helps one decouple wealth and variance variables in the optimal value function and find a solution with the help of the special affine structure of the Heston model, whether using the HJB equation or the quadratic backward stochastic differential equation (SDE). For general utilities, all benefits associated with the power utility in expressing the utility of the terminal wealth in exponential form disappear and, consequently, there are no results for the existence of a classical solution to the HJB equation, let alone a closed form solution. One may contemplate the difficult problem of solving the HJB equation for the Heston model with some numerical methods, such as the finite difference method (see e.g., Forsyth and Labahn (2008)), however, due to high nonlinearity of the HJB equation with two state variables and unspecified boundary conditions when variance variable is at zero or infinite, there is no known numerical method in the literature to solve the HJB equation for the Heston model, to the best knowledge of the authors. In fact, Zariphopoulou (2009) points out that one of open problems in utility maximization with stochastic volatility models is to develop effective numerical schemes for the value function and the optimal feedback policies for general utility functions.

The aim of this paper is to develop an effective Monte-Carlo method to find tight lower and upper bounds of the value function and bypass the almost-impossible task of finding the exact solution of the HJB equation for the Heston model for general utility functions. The intuition and ideas are described as follows.

It is known that the dual control method can help to solve convex control problems. For the Black-Scholes market with closed convex cone trading constraints and general continuous increasing concave utility functions, Bian and Zheng (2015) (see also Bian et al. (2011)) first find a solution to the dual HJB equation and then use it to construct a classical solution to the HJB equation and verify it is indeed the optimal value function. The approach works due to the Black-Scholes market being a complete market model and the dual HJB equation being a linear PDE. It does not work for an incomplete market model such as the Heston model. The main reason is that the stochastic volatility is not a traded asset (unless an additional volatility related security is introduced) and the dual HJB equation is an equally difficult nonlinear PDE with two state variables. It is virtually impossible to find an exact solution to the dual HJB equation for general utilities except for power utility. Despite this limitation, the dual control method still provides highly useful information for the optimal value function. This is because the dual objective function with any fixed dual control supplies a natural upper bound for the primal value function due to the weak duality relation and a feasible control for the primal problem may be constructed using the information of the upper bound and the strong duality relation to give a good lower bound for the primal value function. There is no need to show the existence of an optimal control to the dual problem nor to find the exact dual value function. If one can make the gap between the lower and upper bounds small, then one has found a good approximation to the primal value function, which would be impossible without using the dual control method. This idea has been applied successfully to find the approximate optimal value function for regime switching asset price models with general utility functions, see Ma et al. (2017).

In this paper we adopt a similar line of attack to utility maximization with the Heston stochastic volatility model. We, however, cannot apply any results of Ma et al. (2017) in this paper as model formulations are different, which results in completely different dual control problems and lower and upper bounds. We derive the dual control problem and recover the optimal solution for power utility in Zariphopoulou (2001) and Kraft (2005) using the dual control approach. For general utilities, we propose a Monte-Carlo method to compute the lower and upper bounds for the primal value function. We identify a class of dual controls that are linear functions of the square root of the variance process, for which the upper bounds can be computed efficiently with the closed form formula or the fast Fourier-cosine method thanks to the affine structure of the Heston model. Numerical tests for power, non-HARA and Yaari utilities show that these bounds are tight, which provides a good approximation to the primal value function. To the best of our knowledge, this is the first time an effective dual control Monte-Carlo method is proposed to find the tight lower and upper bounds for the value function with the Heston stochastic volatility model and general utility functions. The significance of our dual control Monte-Carlo method for utility maximization with the Heston model is that it provides a simple and reliable way of estimating the lower and upper bounds of the value function and finding the approximate optimal feedback control when the gap is tight. All these would be impossible if one wants to solve the original problem directly with the HJB equation.

The rest of the paper is arranged as follows. In Section 2 we discuss the dual control method, prove the strong duality theorem, and recover the same closed-form solution for power utility as that in Kraft (2005) with the dual approach. In Section 3 we present the dual control Monte-Carlo method for computing tight lower and upper bounds of the value function. In Section 4 we derive the closed-form upper bound for a specific form of the dual control and a class of utility functions, including power, non-HARA and Yaari utilities. In Section 5 we perform numerical tests to see the efficiency, accuracy, and robustness of the method, including an example with regime-switching Heston model. Section 6 concludes. Appendix A gives the closed form solution to the Riccati equation associated with the Heston model and Appendix B explains the COS method in computing the upper bound for Yaari utility.

2 The Heston model and the dual control method

Assume that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a given probability space with filtration $\mathcal{F}_t$ generated by standard Brownian motions W^s and W^v with correlation coefficient ρ and completed with all P-null sets. The market is composed of two traded assets, one savings account B with riskless interest rate r and one risky asset S satisfying the following SDE (see Heston (1993)):

$$dS_t = S_t[(r + Av_t)dt + \sqrt{v_t}dW_t^s],$$

where A is a positive constant representing the market price of risk, v is an asset variance process satisfying a mean-reverting square-root process:

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^v,$$

θ is the long-run average volatility, κ the rate that v_t reverts to θ, ξ the variance of $\sqrt{v_t}$, and all parameters are positive constants.

Let X be the wealth process. At time $t \in [0, T]$ the investor allocates a proportion π_t of wealth X_t in risky asset S and the remaining wealth in savings account B. Then the wealth process X satisfies the SDE:

$$dX_t = X_t[(r + \pi_t Av_t)dt + \pi_t\sqrt{v_t}dW_t^s], \ X_0 = x_0, \tag{2.1}$$

where x_0 is the initial wealth and π is a progressively measurable control process.

The utility maximization problem is defined by

$$\sup_{\pi} E[U(X_T)] \text{ subject to (2.1)}, \tag{2.2}$$

where U is a utility function that is continuous, increasing and concave (but not necessarily strictly increasing and strictly concave) on $[0, \infty)$, and $U(0) = 0$. To solve (2.2) with the stochastic control method, we define the value function

$$\mathcal{W}(t, x, v) := \sup_{\pi} E_{t,x,v}[U(X_T)], \tag{2.3}$$

where $E_{t,x,v}$ is the conditional expectation operator given $X_t = x, v_t = v$. Since U is concave, the value function $\mathcal{W}(t, x, v)$ is concave in x for fixed t and v. The dynamic programming principle states that for $h > 0$, sufficiently small, the value function $\mathcal{W}$ satisfies the following dynamic programming equation:

$$\mathcal{W}(t, x, v) = \sup_{\pi} E_{t,x,v}[\mathcal{W}(t + h, X_{t+h}, v_{t+h})],$$

where π is a feasible control process defined on $[t, t+h]$. Furthermore, if $\mathcal{W}$ is in $C^{1,2,2}$ satisfying some growth conditions, then $\mathcal{W}$ satisfies the following HJB equation:

$$\frac{\partial \mathcal{W}}{\partial t} + \sup_{\pi} \left\{ (rx + \pi x A v)\mathcal{W}_x + \kappa(\theta - v)\mathcal{W}_v + \frac{1}{2}\pi^2 x^2 v \mathcal{W}_{xx} + \frac{1}{2}\xi^2 v \mathcal{W}_{vv} + \rho \pi x \xi v \mathcal{W}_{xv} \right\} = 0 \quad (2.4)$$

with the terminal condition $\mathcal{W}(T, x, v) = U(x)$, where $\mathcal{W}_x$ is the partial derivative of $\mathcal{W}$ with respect to x and evaluated at (t, x, v), the other derivatives are similarly defined. The maximum in (2.4) is achieved at

$$\pi = -\frac{A\mathcal{W}_x}{x\mathcal{W}_{xx}} - \frac{\xi \rho \mathcal{W}_{xv}}{x\mathcal{W}_{xx}}. \quad (2.5)$$

Inserting (2.5) into (2.4) gives a nonlinear PDE

$$\frac{\partial \mathcal{W}}{\partial t} + rx\mathcal{W}_x + \kappa(\theta - v)\mathcal{W}_v + \frac{1}{2}\xi^2 v \mathcal{W}_{vv} - \frac{1}{2\mathcal{W}_{xx}}[A\sqrt{v}\mathcal{W}_x + \xi\rho\sqrt{v}\mathcal{W}_{xv}]^2 = 0. \quad (2.6)$$

The HJB equation (2.6) is crucially important in characterizing the value function $\mathcal{W}$. Furthermore, if there exists a $C^{1,2,2}$ solution $\bar{\mathcal{W}}$ to the HJB equation (2.6), then the verification theorem (see Pham (2009)), under some mild integrability conditions, states that $\bar{\mathcal{W}}$ coincides with the value function $\mathcal{W}$ and π in (2.5) is the optimal feedback control. Since the HJB equation (2.6) is a nonlinear PDE with two state variables, it is in general difficult to prove the existence of a $C^{1,2,2}$ solution. Without assuming the differentiability of the value function $\mathcal{W}$, one may show that $\mathcal{W}$ is a viscosity solution of (2.6) but may not be able to find the optimal control. However, for a power utility $U(x) = x^p/p$, $0 < p < 1$, the solution to (2.6) can be decomposed as

$$\mathcal{W}(t, x, v) = U(x)f(t, v)$$

for some function f which satisfies

$$\frac{\partial f}{\partial t} + prf + \kappa(\theta - v)f_v + \frac{1}{2}\xi^2 v f_{vv} - \frac{pv}{2(p-1)f}[Af + \xi\rho f_v]^2 = 0 \quad (2.7)$$

with the terminal condition $f(T, v) = 1$. The optimal control is given by

$$\pi = -\frac{A}{(p-1)} - \frac{\xi\rho f_v}{(p-1)f}.$$

The equation (2.7) is simpler than the equation (2.6) but is still a nonlinear PDE. Thanks to the affine structure of the Heston model, the solution f of the equation (2.7) has an analytical form as

$$f(t, v) = \exp(C(t) + D(t)v),$$

where C and D are solutions to some Riccati-type ordinary differential equations (ODEs) with terminal conditions $C(T) = 0$ and $D(T) = 0$ and can be easily solved, and the optimal control is given by $\pi = (A + \xi\rho D(t))/(1 - p)$, see Kraft (2005) for details.

The success of simplifying the HJB equation (2.6) to a solvable nonlinear PDE (2.7) crucially depends on the assumption that the utility function is a power utility. For general utility functions (e.g., non-HARA and Yaari utilities), it is virtually impossible one can postulate the form of the solution to the HJB equation (2.6) and then find the closed-form solution.

The dual function of U is defined by

$$\widetilde{U}(y) = \sup_{x \geq 0}[U(x) - xy], \quad (2.8)$$

for $y \geq 0$. The function $\widetilde{U}(y)$ is a continuous, decreasing and convex function on $[0, \infty)$ and satisfies $\widetilde{U}(\infty) = 0$. The dual process has the following form:

$$dY_t = Y_t[\alpha_t dt + \beta_t dW_t^s + \gamma_t dW_t^v], \ Y_0 = y,$$

where α, β, γ are some stochastic processes and y is a positive number. α, β, γ, y are to be determined such that XY is a super-martingale for any control process π, which, together with the definition of the dual function $\widetilde{U}$ in (2.8), leads to

$$\sup_{\pi} E[U(X_T)] \leq \inf_{y}(\inf_{\alpha,\beta,\gamma} E[\widetilde{U}(Y_T)] + xy), \tag{2.9}$$

and we have a weak duality relation. To make XY a super-martingale, we can use Itô's formula to get

$$\alpha_t \leq -r, \ \beta_t = -A\sqrt{v_t} - \rho\gamma_t.$$

Furthermore, since $\widetilde{U}$ is a decreasing convex function, we must have $\alpha_t = -r$. Therefore, the dual process is given by

$$dY_t = Y_t[-rdt - (\rho\gamma_t + A\sqrt{v_t})dW_t^s + \gamma_t dW_t^v], \ Y_0 = y, \tag{2.10}$$

where γ is a dual control process and y is also a dual control variable. The solution to (2.10) at time T, with initial condition $Y_t = y$, can be written as

$$Y_T = y\exp(M_{t,T}),$$

where

$$M_{t,T} = -\int_t^T \left(r + \frac{1}{2}(1-\rho^2)\gamma_u^2 + \frac{1}{2}A^2 v_u\right) du - \int_t^T (\rho\gamma_u + A\sqrt{v_u})dW_u^s + \int_t^T \gamma_u dW_u^v.$$

Define the dual value function as

$$\widetilde{\mathcal{W}}(t, y, v) := \inf_{\gamma} E_{t,y,v}[\widetilde{U}(Y_T)].$$

Since U is concave and $U(0) = 0$, the dual function $\widetilde{U}$ is convex and nonnegative, which implies that the dual value function $\widetilde{\mathcal{W}}$ is well-defined and $\widetilde{\mathcal{W}}(t, y, v)$ is convex in y for fixed t and v. Similar to the derivations of the HJB equation (2.4), assuming that $\widetilde{\mathcal{W}}$ is in $C^{1,2,2}$, then $\widetilde{\mathcal{W}}$ satisfies the following dual HJB equation

$$\begin{aligned}\frac{\partial\widetilde{\mathcal{W}}}{\partial t} + \inf_{\gamma}\Bigg\{ & -ry\widetilde{\mathcal{W}}_y + \kappa(\theta - v)\widetilde{\mathcal{W}}_v + \frac{1}{2}y^2[A^2 v + \gamma^2(1-\rho^2)]\widetilde{\mathcal{W}}_{yy} \\ & + y[\gamma\xi\sqrt{v}(1-\rho^2) - Av\xi\rho]\widetilde{\mathcal{W}}_{yv} + \frac{1}{2}\xi^2 v\widetilde{\mathcal{W}}_{vv}\Bigg\} = 0\end{aligned} \tag{2.11}$$

with the terminal condition $\widetilde{\mathcal{W}}(T, y, v) = \widetilde{U}(y)$. The minimum in (2.11) is achieved at

$$\gamma = -\frac{\xi\sqrt{v}\widetilde{\mathcal{W}}_{yv}}{y\widetilde{\mathcal{W}}_{yy}}. \tag{2.12}$$

Inserting (2.12) into (2.11) gives

$$\frac{\partial\widetilde{\mathcal{W}}}{\partial t} - ry\widetilde{\mathcal{W}}_y + \kappa(\theta - v)\widetilde{\mathcal{W}}_v + \frac{1}{2}\widetilde{\mathcal{W}}_{yy}y^2A^2v - \frac{\xi^2 v\widetilde{\mathcal{W}}_{yv}^2}{2\widetilde{\mathcal{W}}_{yy}}(1-\rho^2) - Av\xi\rho y\widetilde{\mathcal{W}}_{yv} + \frac{1}{2}\widetilde{\mathcal{W}}_{vv}\xi^2 v = 0. \tag{2.13}$$

A similar discussion to that after the HJB equation (2.6) applies here, that is, if there exists a $C^{1,2,2}$ solution to the dual HJB equation (2.13) then that solution is the dual value function, otherwise, one may show that $\widetilde{\mathcal{W}}$ is a viscosity solution to (2.13). From the discussions above and the inequality (2.9), we have the following dynamic version of the weak duality relation:

$$\mathcal{W}(t,x,v) \le \inf_{y>0}[\widetilde{\mathcal{W}}(t,y,v) + xy]. \tag{2.14}$$

The next theorem states that there is a strong duality relation between the classical solutions of the primal HJB equation (2.6) and the dual HJB equation (2.13), which implies that if one can solve the dual problem, then one can use the dual optimal solution to construct the primal optimal solution without having to solve the primal HJB equation directly.

Theorem 2.1. *Assume that there exists a $C^{1,2,2}$ solution $\widetilde{\mathcal{W}}$ to the dual HJB equation* (2.13) *and $\widetilde{\mathcal{W}}(t,y,v)$ is strictly convex in y for fixed t and v and satisfies $\widetilde{\mathcal{W}}_y(t,0,v) = -\infty$ and $\widetilde{\mathcal{W}}_y(t,\infty,v) = 0$. Then the primal value function is given by*

$$\mathcal{W}(t,x,v) = \widetilde{\mathcal{W}}(t,y^*,v) + xy^*,$$

where $y^* = y(t,x,v)$ is the solution to the equation

$$\widetilde{\mathcal{W}}_y(t,y,v) + x = 0.$$

Furthermore, $\mathcal{W} \in C^{1,2,2}$ is the solution to the HJB equation (2.6) *with the boundary condition $\mathcal{W}(T,x,v) = U(x)$ and the optimal feedback control is given by*

$$\pi(t,x,v) = \frac{A}{x} y^* \widetilde{\mathcal{W}}_{yy}(t,y^*,v) - \frac{\xi\rho}{x}\widetilde{\mathcal{W}}_{yv}(t,y^*,v). \tag{2.15}$$

Proof. Define

$$\widehat{\mathcal{W}}(t,x,v) = \inf_{y>0}\left[\widetilde{\mathcal{W}}(t,y,v) + xy\right]. \tag{2.16}$$

Since $\widetilde{\mathcal{W}} \in C^{1,2,2}$ and is strictly convex in y and $\widetilde{\mathcal{W}}_y(t,0,v) = -\infty$ and $\widetilde{\mathcal{W}}_y(t,\infty,v) = 0$, we have

$$\widehat{\mathcal{W}}(t,x,v) = \widetilde{\mathcal{W}}(t,y^*,v) + xy^*,$$

where $y^* = y(t,x,v)$ satisfies $\widetilde{\mathcal{W}}_y(t,y,v) + x = 0$. Using the Implicit Function Theorem, we have $y \in C^{1,2,2}$ and therefore $\widehat{\mathcal{W}} \in C^{1,2,2}$. Simple calculus shows that

$$\frac{\partial \widehat{\mathcal{W}}}{\partial t} = \frac{\partial \widetilde{\mathcal{W}}}{\partial t},\ \widehat{\mathcal{W}}_x = y^*,\ \widehat{\mathcal{W}}_v = \widetilde{\mathcal{W}}_v,$$

and

$$\widehat{\mathcal{W}}_{xx} = -\frac{1}{\widetilde{\mathcal{W}}_{yy}},\ \widehat{\mathcal{W}}_{xv} = -\frac{\widetilde{\mathcal{W}}_{yv}}{\widetilde{\mathcal{W}}_{yy}},\ \widehat{\mathcal{W}}_{vv} = -\frac{(\widetilde{\mathcal{W}}_{yv})^2}{\widetilde{\mathcal{W}}_{yy}} + \widetilde{\mathcal{W}}_{vv}.$$

Substituting these relations into (2.13) gives that $\widehat{\mathcal{W}}$ satisfies the HJB equation (2.6). Moreover it follows from the conjugate equation (2.16) and $\widetilde{W}(T,y,v) = \widetilde{U}(y)$ that $\widehat{\mathcal{W}}(T,x,v) = U(x)$. The verification theorem then gives $\mathcal{W}(t,x,v) = \widehat{\mathcal{W}}(t,x,v)$. The optimal feedback control is derived from (2.5) and the dual relations of the derivatives. $\square$

Remark 2.2. *Theorem 2.1 shows that there is no duality gap if there exists a* $C^{1,2,2}$ *solution* $\widetilde{\mathcal{W}}$ *to the dual HJB equation* (2.13) *and the dual control* γ *takes the form* (2.12). *This is interesting in theory and is useful if one knows* $\widetilde{\mathcal{W}}$. *In general, it is highly unlikely one can find* $\widetilde{\mathcal{W}}$ *as it requires to solve an equally difficult nonlinear PDE* (2.13). *At first sight this seems to indicate that the usefulness of the dual formulation is limited. A closer look would convince us that the dual formulation is very useful in helping solve the primal problem. Since the dual problem is a minimization problem, for any fixed dual control* γ, *one automatically gets an upper bound for the primal value function using the weak duality relation (2.14). Furthermore, if one can choose a dual control* γ *which gives a good approximation to* $\widetilde{\mathcal{W}}$, *then one may use (2.15) to construct a feasible control for the primal problem and get a good lower bound for the primal value function. All these would be impossible without using the dual formulation. This is essentially the idea we use to design a Monte-Carlo method for computing tight lower and upper bounds of the primal value function in the next section. Since we are only interested in finding the bounds of the primal value function, there is no need to show the existence of an optimal control to the dual problem nor to find the exact dual value function. If the gap between the lower and upper bounds is sufficiently tight, we have found a good approximate value, which would be practically adequate for many portfolio optimization problems.*

Remark 2.3. *There are two expressions for optimal control* π. *One is in the original space* (2.5), *the other in the dual space* (2.15). *They are equivalent by the derivative relations between* $\widehat{\mathcal{W}}$ *and* $\widetilde{\mathcal{W}}$ *in the proof of Theorem 2.1. We normally approximate the dual space optimal control* π *with the Monte-Carlo method, but if* π *has an explicit expression in the original or dual space, we choose* (2.5) *or* (2.15) *respectively to save the computational time.*

Remark 2.4. *Zeng and Taksar (2013) investigate a general stochastic volatility model of the following form*

$$\begin{aligned} dS_t &= S_t\left[\mu(z_t,t)dt + g(z_t,t)dW_t^s\right], \\ dv_t &= \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^v, \end{aligned} \tag{2.17}$$

where

$$\mu(z,t) = r + A\sqrt{v}g(z,t),\ v = G(z)$$

and G *is a strictly monotone* C^2 *function and* θ, κ, ξ, A *are positive constants. If* $g(z,t) = \sqrt{z}$ *and* $G(z) = z$, *one recovers the Heston model; if* $g(z,t) = \sqrt{z}$ *and* $G(z) = z^{-1}$, *one recovers the* 3/2 *model. Zeng and Taksar (2013) derive the explicit solutions to the utility maximization under the model* (2.17) *for power and exponential utilities and prove a verification theorem. The weak duality relation in (2.14) and the strong duality relation in Theorem 2.1 can be extended to the above general stochastic volatility model and other one factor models without necessarily displaying the affine property, provided the underlying asset price process* S *follows a geometric Brownian motion type process. The lower and upper bound approach described in Remark 2.2 can be applied.*

For power utility $U(x) = x^p/p$, we can indeed solve the dual problem and find a closed-form solution to (2.13) and therefore solve the primal problem with the dual method. This is explained in the next result.

Corollary 2.5. *For power utility* $U(x) = (1/p)x^p$, $0 < p < 1$, *the primal value function is given by*

$$\mathcal{W}(t,x,v) = \frac{x^p}{p}\exp\Big[(1-p)(C(t)+D(t)v)\Big], \tag{2.18}$$

where $C(t)$ and $D(t)$ satisfy the following systems of equations:

$$C'(t) = -\kappa\theta D(t) - \frac{rp}{1-p}, \ C(T) = 0,$$

and

$$D'(t) = \frac{1}{2}D^2(t)\xi^2[p(1-\rho^2)-1] + D(t)(\kappa - A\xi\rho\frac{p}{1-p}) - \frac{p}{2(1-p)^2}A^2, \ D(T) = 0.$$

The optimal control at time t is given by

$$\pi_t = \frac{A}{1-p} + \frac{\xi\rho D(t)}{1-p}.$$

Proof. The dual function of U is given by $\widetilde{U}(y) = -(1/q)y^q$, where $q = p/(p-1)$. We may set $\widetilde{\mathcal{W}}(t,y,v) = \widetilde{U}(y)\tilde{f}(t,v)$ and substitute it into the equation (2.13) to get a simplified equation for $\tilde{f}$:

$$\frac{\partial \tilde{f}}{\partial t} - rq\tilde{f} + \kappa(\theta - v)\tilde{f}_v + \frac{1}{2}q(q-1)\tilde{f}A^2 v - \frac{q}{2(q-1)}\xi^2 v\frac{\tilde{f}_v^2}{\tilde{f}}(1-\rho^2) - qAv\xi\rho\tilde{f}_v + \frac{1}{2}\tilde{f}_{vv}\xi^2 v = 0 \quad (2.19)$$

with the terminal condition $\tilde{f}(T,v) = 1$. We can solve equation (2.19) by setting $\tilde{f}(t,v) = \exp(C(t) + D(t)v)$ and substituting $\tilde{f}$ into (2.19) to get two ODEs for C and D in the statement of Corollary 2.5. We can easily find $C(t)$ once $D(t)$ is known and solve the Riccati equation to get a closed-form solution $D(t)$, see Appendix A. Next we solve the equation $\widetilde{\mathcal{W}}_y + x = 0$ to get

$$y^* = [x\exp(-C(t) - D(t)v)]^{p-1}.$$

Using Theorem 2.1, we obtain the primal value function by the relation

$$\mathcal{W}(t,x,v) = \widetilde{W}(t,y^*,v) + xy^*,$$

which gives (2.18). The control π is thus obtained from formula (2.5). □

Corollary 2.5 shows that the dual control method of Theorem 2.1 gives the closed-form formula for the primal value function with the power utility. After communicating the notations, we see that formula (2.18) is the same as that in Proposition 5.2 of Kraft (2005).

3 Monte-Carlo lower and upper bounds

For general utility functions, it seems impossible we can solve the primal problem by using Theorem 2.1, see the discussions after Theorem 2.1. From the weak duality relation (2.14) we have

$$\mathcal{W}(t,x,v) \le \inf_{y>0}[\inf_{\gamma} E_{t,y,v}[\widetilde{U}(Y_T)] + xy] \le \inf_{y>0}[E_{t,y,v}[\widetilde{U}(Y_T)] + xy]. \quad (3.1)$$

for all dual controls γ. For every fixed γ, define

$$\mathcal{Z}(t,y,v) = E_{t,y,v}[\widetilde{U}(Y_T)]. \quad (3.2)$$

Since the dual function $\widetilde{U}$ is nonnegative, $\widetilde{U}(Y_T)$ is a nonnegative random variable, $\mathcal{Z}(t,y,v)$ is well-defined, but may take the value $+\infty$. In that case, the upper bound in (3.1) does not give us any information. We may simply discard γ and choose another one as we are only interested in an upper bound with finite value. Then $\mathcal{Z}$ is an upper bound and can be easily computed with simulation. Note that $\mathcal{Z}(t,y,v)$ depends on the choice of dual control γ. Denote the conjugate function of $\mathcal{Z}(t,y,v)$ for fixed t and v by

$$\overline{\mathcal{W}}(t,x,v)=\inf_{y>0}[\mathcal{Z}(t,y,v)+xy]. \tag{3.3}$$

The following theorem presents the tight lower and upper bounds on the primal value function.

Theorem 3.1. *Let $\mathcal{S}$ be a set of admissible dual controls and $\overline{\mathcal{W}}(t,x,v)$ be given by* (3.3)*. Then the optimal value function $\mathcal{W}(t,x,v)$ defined in* (2.3) *satisfies*

$$\mathcal{W}(t,x,v)\leq \inf_{\gamma\in\mathcal{S}}\overline{\mathcal{W}}(t,x,v). \tag{3.4}$$

Furthermore, assume that $\mathcal{Z}(t,y,v)$ given by (3.2) *is twice continuously differentiable and strictly convex for $y>0$ with fixed t and v, $y^*=y(t,x,v;\gamma)$ is the solution to the equation*

$$\mathcal{Z}_y(t,y,v)+x=0, \tag{3.5}$$

the feedback control $\bar{\pi}(t,x,v)$, defined by

$$\bar{\pi}(t,x,v):=A\frac{y^*}{x}\mathcal{Z}_{yy}(t,y^*,v)-\frac{\xi\rho}{x}\mathcal{Z}_{yv}(t,y^*,v), \tag{3.6}$$

is admissible, and $\bar{X}$ is the unique strong solution to SDE (2.1) *with the feedback control $\pi_t=\bar{\pi}(t,\bar{X}_t,v_t)$ for $t\in[0,T]$. Define*

$$\underline{\mathcal{W}}(t,x,v):=E_{t,x,v}[U(\bar{X}_T)]. \tag{3.7}$$

Then the optimal value function $\mathcal{W}(t,x,v)$ satisfies

$$\mathcal{W}(t,x,v)\geq \sup_{\gamma\in\mathcal{S}}\underline{\mathcal{W}}(t,x,v). \tag{3.8}$$

Proof. It is obvious from (3.1) and the definitions of $\overline{\mathcal{W}}(t,x,v)$ and $\underline{\mathcal{W}}(t,x,v)$. □

Remark 3.2. *It is clear from Theorem 3.1 that it is straightforward to find the upper bound $\overline{\mathcal{W}}(t,x,v)$ as one only needs to compute the value $\mathcal{Z}(t,y,v)$ and to solve a scalar convex minimization problem. It is much more involved to find the lower bound $\underline{\mathcal{W}}(t,x,v)$. One has to assume some conditions on $\mathcal{Z}$, which is unavoidable if one wants to construct a "good" feedback control $\bar{\pi}(t,x,v)$ in (3.6) for the lower bound using the information of the upper bound from the dual problem, see Remark 3.4 for an example of computing $\mathcal{Z}(t,y,v)$ and $\bar{\pi}(t,x,v)$ beyond the power utility. The requirement of existence of a unique strong solution $\bar{X}$ to SDE* (2.1) *is a standard one in stochastic control theory, especially for the verification theorem, see Pham (2009). On the other hand, one can easily find a lower bound without using $\mathcal{Z}(t,y,v)$ at all by choosing any feasible control π for the primal problem and $E_{t,x,v}[U(X_T)]$ is a lower bound for the primal value function. The shortcoming of this approach is that the derived lower bound may be distant from the true value function $\mathcal{W}(t,x,v)$ and does not provide much useful information.*

Remark 3.3. *Clearly, if $\mathcal{S}\subset\widetilde{\mathcal{S}}$, then*

$$\mathcal{W}(t,x,v)\leq \inf_{\gamma\in\widetilde{\mathcal{S}}}\overline{\mathcal{W}}(t,x,v)\leq \inf_{\gamma\in\mathcal{S}}\overline{\mathcal{W}}(t,x,v) \tag{3.9}$$

Using $\widetilde{\mathcal{S}}$ instead of $\mathcal{S}$ gives a tighter upper bound but is more time-consuming in computation. The same applies to the lower bound. For numerical tests in Section 5, we choose the set $\mathcal{S}$ to contain the following dual controls: $\gamma_t = c(t)$, $c(t)\sqrt{v_t}$, $c(t)v_t$, where c is a piecewise constant function

$$c(t) = \sum_{j=1}^{n} c_j 1_{(t_{j-1},t_j]}(t), \tag{3.10}$$

with $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ for $n \geq 1$ and c_j, $j = 1, \ldots, n$ being arbitrary constants.

Remark 3.4. *If the dual function $\widetilde{U}$ has the form $\widetilde{U}(y) = \sum_{i=1}^{K} \widetilde{U}_i(y)$, where $\widetilde{U}_i(y) = -(1/q_i)y^{q_i}$ for $q_i < 0$ and $i = 1, \ldots, K$, then*

$$\mathcal{Z}(t, y, v) = \sum_{i=1}^{K} \widetilde{U}_i(y) F_i(t, v),$$

where $F_i(t, v) = E_{t,v}[\exp(q_i M_{t,T})]$. The upper bound is given by

$$\overline{\mathcal{W}}(t, x, v) = \mathcal{Z}(t, y^*, v) + xy^*$$

and y^* is the unique solution to equation (3.5), that is, $-\sum_{i=1}^{K} y^{q_i-1} F_i(t, v) + x = 0$. The feedback control for the lower bound is given by (3.6):

$$x\bar{\pi}(t, x, v) = \sum_{i=1}^{K} (y^*)^{q_i-1} \left(A(1 - q_i)F_i(t, v) + \xi\rho \frac{\partial}{\partial v} F_i(t, v) \right).$$

For fixed dual control γ_t, $0 \leq t \leq T$, we can use the Monte-Carlo method to compute $F_i(t, v)$ and approximate $\frac{\partial}{\partial v} F_i(t, v)$ with the finite difference $(F_i(t, v + h) - F_i(t, v - h))/(2h)$ for sufficiently small $h > 0$. If $K = 1$, we have a closed-form solution y^*. If $K > 1$, we can use the Newton-Raphson method to find y^*.

Remark 3.5. *If the dual function $\widetilde{U}$ is Lipschitz continuous, then we may use the pathwise differentiation method to compute $\mathcal{Z}_y(t, y, v)$, that is,*

$$\mathcal{Z}_y(t, y, v) = E_{t,y,v}[\widetilde{U}'(y \exp(M_{t,T})) \exp(M_{t,T})].$$

For example, the dual function of Yaari utility (see (4.6)) is given by $\widetilde{U}(y) = L(1 - y)^+$, we have $\widetilde{U}'(y) = -L1_{\{y<1\}}$, where 1_S is an indicator which equals 1 if S happens and 0 otherwise. We can then approximate $\mathcal{Z}_{yy}(t, y, v)$ and $\mathcal{Z}_{yv}(t, y, v)$ with finite differences $(\mathcal{Z}_y(t, y + h, v) - \mathcal{Z}_y(t, y - h, v))/(2h)$ and $(\mathcal{Z}_y(t, y, v + h) - \mathcal{Z}_y(t, y, v - h))/(2h)$, respectively, for sufficiently small $h > 0$.

The Monte-Carlo methods can be used to find the tight lower and upper bounds, analogous to the algorithm developed by Ma et al. (2017). To implement the method, we need to discretize the variance process v, the dual process Y and the wealth process X. We discretize these processes by the full-truncation Euler method (see Lord et al. (2010)):

$$\begin{aligned}
v_{t+\Delta t} &= v_t + \kappa(\theta - v_t^+)\Delta t + \xi\sqrt{v_t^+}\sqrt{\Delta t}Z_1 \\
Y_{t+\Delta t} &= Y_t - rY_t\Delta t - (\rho\gamma_t + A\sqrt{v_t^+})Y_t\sqrt{\Delta t}Z_2 + Y_t\gamma_t\sqrt{\Delta t}Z_1 \\
\bar{X}_{t+\Delta t} &= \bar{X}_t + r\bar{X}_t\Delta t + \bar{\pi}_t\bar{X}_t\sqrt{v_t^+}\left(A\sqrt{v_t^+}\Delta t + \sqrt{\Delta t}Z_2\right),
\end{aligned}$$

where Z_1 and Z_2 are two standard normal variables with correlation ρ and $v_t^+ = \max(0, v_t)$.

For wealth process $\bar{X}$ driven by $\bar{\pi}$, it is possible that an investor may lose all his wealth during the investment period. Thus if $\bar{X}_t \leq 0$ for some $t < T$, we stop generating paths and set $\bar{X}_T = 0$.

Remark 3.6. *There is a well-known Feller condition $2\kappa\theta \geq \xi^2$ for the Heston model, which ensures that v_t is strictly positive (see Feller (1951)). When it is violated, 0 becomes attainable for v_t process with some strongly reflecting properties (see Andersen and Piterbarg (2011)), which may cause problems for numerical simulation and applications. For example, in the stock and foreign exchange market, the calibration of the Heston model often yields high variance parameter ξ which violates the Feller condition (see Duffie et al. (2000)) and may cause numerical errors in getting negative v_t. Among many numerical algorithms for the Heston model, it is known that the full-truncation Euler method can handle the violation case with strong L^1 convergence and outperforms (in terms of bias and root-mean-squared error) the other algorithms in the literature (see Lord et al. (2010)). In this paper, we use the full-truncation Euler method and do not rely on the Feller condition for simulation and analysis, which makes the algorithm more flexible for practical uses.*

Next we describe the Monte-Carlo methods for computing the tight lower and upper bounds at time 0. The tight lower and upper bounds at other time t can be computed similarly. Assume $X_0 = x$, $v_0 = v$ and the dual utility function $\widetilde{U}$ in (2.8) are known. The dual control $\gamma_t = c(t)$ or $c(t)\sqrt{v_t}$ or $c(t)v_t$, where c is a piecewise constant function given by (3.10). Denote by $\mathcal{S}$ the set of vectors $\mathbf{C} := (c_1, \ldots, c_n)$ which form the coefficients of the function c.

Monte-Carlo method for computing tight lower and upper bounds:

Step 1: Fix a vector $\mathbf{C} \in \mathcal{S}$ and a form of dual control γ_t.

Step 2: Generate M sample paths of Brownian motion W^s and W^v, discretize SDE (2.10), compute Y_T with $Y_0 = y$ and the average derivative:

$$\frac{\partial \mathcal{Z}(0, y, v)}{\partial y} \approx \frac{1}{y} \frac{1}{M} \sum_{\ell=1}^{M} Y_T \widetilde{U}'(Y_T).$$

Step 3: Use the bisection method to solve equation (3.5) and obtain the solution $y \approx y^*$.

Step 4: Compute the upper bound

$$\overline{\mathcal{W}}(0, x, v) \approx \mathcal{Z}(0, y^*, v) + xy^*.$$

Step 5: Find the feedback control process $\bar{\pi}$ in (3.6) and generate the wealth process $\bar{X}$ in (2.1).

Step 6: Compute the lower bound

$$\underline{\mathcal{W}}(t, x, v) \approx \frac{1}{M} \sum_{\ell=1}^{M} U(\bar{X}_T).$$

Step 7: Repeat Steps 1 to 6 with different $\mathbf{C} \in \mathcal{S}$ to derive the tight lower bound $\sup_{\mathbf{C}\in\mathcal{S}} \underline{\mathcal{W}}(0, x, v)$ and the tight upper bound $\inf_{\mathbf{C}\in\mathcal{S}} \overline{\mathcal{W}}(0, x, v)$.

Remark 3.7. *It is much more expensive to compute the tight lower bound due to the requirement that one has to solve equation* (3.5) *and obtains the control process $\bar{\pi}$ at each time period while generating sample paths of the wealth process $\bar{X}$, not just at $t = 0$ as in the computation of the tight upper bound. One technique to speed up is to use a four-dimensional matrix $\bar{\pi}_{i_t \times j_x \times k_v \times l_C}$ to pre-save the values of $\bar{\pi}$ on a lattice, and then apply linear interpolation to approximate the exact values we need while generating sample paths of $\bar{X}$.*

4 Closed-form upper bounds

For general dual controls γ_t, $0 \le t \le T$, we have to use the Monte-Carlo method to compute the upper bound $\mathcal{Z}(t, y, v)$ in (3.2). However, for a class of special dual controls and utility functions, we can find the upper bound in closed-form. Since Y satisfies a linear SDE (2.10) and $\widetilde{U}$ is a decreasing and convex function, $\mathcal{Z}(t, y, v)$ is a decreasing and convex function for $y > 0$ with fixed t and v. Moreover, the Feynman-Kac theorem implies that $\mathcal{Z}$ satisfies the following linear PDE:

$$\mathcal{Z}_t - ry\mathcal{Z}_y + \kappa(\theta - v)\mathcal{Z}_v + \frac{1}{2}\mathcal{Z}_{yy}y^2[A^2 v + \gamma^2(1-\rho^2)] + \mathcal{Z}_{yv}y[\gamma\xi\sqrt{v}(1-\rho^2) - Av\xi\rho] + \frac{1}{2}\mathcal{Z}_{vv}\xi^2 v = 0 \quad (4.1)$$

with terminal condition $\mathcal{Z}(T, y, v) = \widetilde{U}(y)$, provided that $\mathcal{Z}(t, y, v)$ is the unique $C^{1,2,2}$ solution to (4.1) satisfying some growth conditions. The choice $\gamma_t = c(t)\sqrt{v_t}$, where c is a piecewise constant function, is particularly interesting as we can get the closed-form solution to the equation (4.1) if $\widetilde{U}$ is a linear combination of power functions. Specifically, if

$$\gamma_t = c(t)\sqrt{v_t},$$

where c is a piecewise constant function defined by (3.10), and

$$\widetilde{U}(y) = \sum_{i=1}^{K} \widetilde{U}_i(y), \quad (4.2)$$

where $\widetilde{U}_i(y) = -(1/q_i)y^{q_i}$ with $q_i < 0$ for $i = 1, \ldots, K$. The solution to (4.1) is given by

$$\mathcal{Z}(t, y, v) = \sum_{i=1}^{K} \widetilde{U}_i(y) \exp(C_i(t) + D_i(t)v), \quad (4.3)$$

where C_i and D_i satisfy the following ODEs:

$$C_i'(t) = -\kappa\theta D_i(t) + rq_i, \ C_i(T) = 0$$

and

$$D_i'(t) = a_i D_i^2(t) + b_i(t)D_i(t) + \eta_i(t), \ D_i(T) = 0$$

with coefficients given by $a_i = -(1/2)\xi^2$, $b_i(t) = \kappa - q_i\xi(c(t)(1-\rho^2) - A\rho)$, and $\eta_i(t) = -(1/2)q_i(q_i - 1)(A^2 + c^2(t)(1-\rho^2))$. Furthermore, since $c(t)$ is a piecewise constant function, D_i is given by

$$D_i(t) = \sum_{j=1}^{n} D_{ij}(t) 1_{(t_{j-1}, t_j]}(t),$$

where D_{ij}, $j = 1, \ldots, n$, are computed recursively as follows: for $j = n$,

$$D_{in}'(t) = a_i D_{in}^2(t) + b_i(t_n)D_{in}(t) + \eta_i(t_n), \ t \in [t_{n-1}, t_n]$$

with terminal condition $D_{in}(t_n) = 0$ and, for $j = n-1, \dots, 1$,

$$D'_{ij}(t) = a_i D^2_{ij}(t) + b_i(t_j) D_{ij}(t) + \eta_i(t_j),\ t \in [t_{j-1}, t_j]$$

with terminal condition $D_{ij}(t_j) = D_{i,j+1}(t_j)$. The closed-form solutions to $C_{ij}(t)$ and $D_{ij}(t)$ are given by (A.3) and (A.2), respectively, in Appendix A. Comparing $\mathcal{Z}$ in Remark 3.4 and (4.3), we see that

$$F_i(t, v) = \exp(C_i(t) + D_i(t)v)$$

and the upper bound $\overline{\mathcal{W}}$ and the feedback control $\bar{\pi}$ are given by

$$\overline{\mathcal{W}}(t, x, v) = \sum_{i=1}^{K} \widetilde{U}_i(y^*) F_i(t, v) + xy^* \tag{4.4}$$

and

$$x\bar{\pi}(t) = \sum_{i=1}^{K} [A(1 - q_i) + \xi\rho D_i(t)](y^*)^{q_i - 1} F_i(t, v), \tag{4.5}$$

where $y^* = y(t, x, v)$ is the unique solution to the equation $\sum_{i=1}^{K} y^{q_i - 1} F_i(t, v) = x$.

Since PDE (4.1) has a closed-form solution, this makes the computation of the upper bound very fast. Even if the dual utility is not in the form of (4.2), but has some simple structure such as call/put option payoff function, one can still compute the upper bound efficiently by using the fast Fourier transform method. We next discuss several examples to illustrate these points.

Example 4.1. *(power utility). For $U(x) = x^p/p$, its dual function is given by $\widetilde{U}(y) = -(1/q)y^q$, where $q = p/(p-1)$. Let $\gamma_t = c\sqrt{v_t}$. This is a special case of (4.2) with $K = 1$ and $q_1 = q$. The dual value function $\mathcal{Z}$, defined by (3.2), is given by (4.3). For power utility, the upper bound $\overline{\mathcal{W}}$ and the feedback control $\bar{\pi}$ can be written out explicitly as*

$$\overline{\mathcal{W}}(t, x, v) = U(x)\exp((1 - p)(C(t) + D(t)v)) \text{ and } \bar{\pi}(t) = (1 - q)[A + \xi\rho D(t)],$$

where $C(t)$ and $D(t)$ are given by (A.3) and (A.2), respectively, with $\underline{t} = 0$, $\bar{t} = T$ and $f_1 = f_2 = 0$. Note that $\bar{\pi}$ is a deterministic function of time t. We can then use the Monte-Carlo method to generate sample paths of the wealth process to compute the lower bound, see Remark 3.4. However, for power utility, there is a fast approximation method to compute the lower bound as shown next. By the Feynman-Kac theorem, the lower bound $\underline{W}$, defined by (3.7), satisfies the following PDE:

$$\frac{\partial \underline{\mathcal{W}}}{\partial t} + (r + A\bar{\pi}(t)v)x\underline{\mathcal{W}}_x + \kappa(\theta - v)\underline{\mathcal{W}}_v + \frac{1}{2}\bar{\pi}(t)^2 x^2 v \underline{\mathcal{W}}_{xx} + \xi\bar{\pi}(t)xv\rho\underline{\mathcal{W}}_{xv} + \frac{1}{2}\xi^2 v \underline{\mathcal{W}}_{vv} = 0$$

with the terminal condition $\underline{\mathcal{W}}(T, x, v) = (1/p)x^p$. Thanks to the power utility, the solution of the above equation is given by

$$\underline{\mathcal{W}}(t, x, v) = \frac{x^p}{p} \exp\left[\bar{C}(t) + \bar{D}(t)v\right],$$

where $\bar{C}(t)$ and $\bar{D}(t)$ satisfy the following ODEs

$$\bar{C}'(t) = -\kappa\theta\bar{D}(t) - rp,\ \bar{C}(T) = 0$$

and

$$\bar{D}'(t) = -\frac{1}{2}\xi^2\bar{D}^2(t) - (\xi\bar{\pi}(t)\rho p - \kappa)\bar{D}(t) - Ap\bar{\pi}(t) - \frac{1}{2}\bar{\pi}(t)^2 p(p - 1),\ \bar{D}(T) = 0.$$

Even though $\bar{D}$ satisfies a Riccati equation, there is no closed form solution for $\bar{D}$ as $\bar{\pi}$ is a continuous function, not a constant. We can nevertheless approximate $\bar{\pi}$ with a piecewise constant function and then get a closed-form approximate solution to $\bar{D}$ with a recursive method. Specifically, we may divide interval $[0,T]$ by grid points $0=\tilde{t}_0<\tilde{t}_1<\dots\tilde{t}_m=T$ and approximate $\bar{\pi}$ by a piecewise constant function

$$\tilde{\pi}(t)=\sum_{k=1}^{m}\bar{\pi}(\tilde{t}_k)1_{(t_{k-1},t_k]}(t).$$

$\tilde{\pi}$ can be made arbitrarily close to $\bar{\pi}$. If we replace $\bar{\pi}$ by $\tilde{\pi}$ in the Riccati equation for $\bar{D}$, the solution to the resulting equation can be written as

$$\tilde{D}(t)=\sum_{k=1}^{m}\tilde{D}_k(t)1_{(t_{k-1},t_k]}(t),$$

where $\tilde{D}_k$, $k=m,\dots,1$, satisfy some Riccati equations with constant coefficients on intervals $(\tilde{t}_{k-1},\tilde{t}_k]$ and can be computed recursively in a closed form with terminal conditions $\tilde{D}_k(\tilde{t}_k)=\tilde{D}_{k+1}(\tilde{t}_k)$, see Appendix A for details. The function $\tilde{D}$ is a good approximation of $\bar{D}$.

Example 4.2. *(non-HARA utility). Assume that*

$$U(x)=\frac{1}{3}H(x)^{-3}+H(x)^{-1}+xH(x),$$

for $x>0$, where

$$H(x)=\left(\frac{2}{-1+\sqrt{1+4x}}\right)^{1/2}.$$

It can be easily checked that U is continuously differentiable, strictly increasing and strictly concave, satisfying $U(0)=0$, $U(\infty)=\infty$, $U'(0)=\infty$ and $U'(\infty)=0$, see Bian and Zheng (2015) for details. The dual function of U is given by

$$\widetilde{U}(y)=\frac{1}{3}y^{-3}+y^{-1}.$$

Let $\gamma_t=c\sqrt{v_t}$. This is a special case of (4.2) with $K=2$ and $q_1=-3$, $q_2=-1$. The dual value function $\mathcal{Z}$ is given by (4.3), the upper bound $\overline{\mathcal{W}}$ by (4.4) and the feedback control $\bar{\pi}$ by (4.5), in the case here, y^* can be computed explicitly as

$$y^*=\left(\frac{F_2(t,v)+\sqrt{F_2(t,v)^2+4xF_1(t,v)}}{2x}\right)^{\frac{1}{2}},$$

where $C_i(t)$ and $D_i(t)$ are given by (A.3) and (A.2), respectively, with $\underline{t}=0$, $\bar{t}=T$ and $f_1=f_2=0$, see Appendix A.

Note that, unlike the case for power utility, there is no closed form formula for the lower bound $\underline{W}$. One has to use the Monte-Carlo method to generate sample paths of the wealth process in order to find its value. We can nevertheless find a reasonable lower bound at more expensive computational cost.

Example 4.3. *(Yaari utility). Assume that*

$$U(x)=x\wedge L, \tag{4.6}$$

where L is a positive constant. U is a continuous, increasing and concave function, but not differentiable at $x = L$ and not strictly concave. Also note that $U'(0) = 1$, so Inada's condition is not satisfied. This utility is called Yaari utility and is used in behavioural finance. The dual function is given by

$$\widetilde{U}(y) = L(1-y)^+.$$

For the dual process (2.10) *with $\gamma_t = c\sqrt{v_t}$, where $c > 0$ is an arbitrarily fixed constant, we evaluate the dual value function*

$$\mathcal{Z}(t,y,v) = E_{t,y,v}[\widetilde{U}(Y_T)] = E_{t,y,v}[L(1-Y_T)^+].$$

This is a European put option pricing problem with the Heston model, see Remark 4.4.

Let $Z_T = \ln Y_T$ and $z = \ln y$. Then

$$\mathcal{Z}(t,y,v) = \widetilde{\mathcal{Z}}(t,z,v) = E_{t,z,v}[L(1-e^{Z_T})^+], \tag{4.7}$$

with terminal condition $\widetilde{\mathcal{Z}}(T,z,v) = L(1-e^z)^+$. Although the conditional probability density function of Z_T is unknown, its conditional characteristic function (namely, the Fourier transform of the density function) can be derived. Therefore, analogous to the well-known Heston method in Heston (1993), function $\mathcal{Z}$ in (4.7) *can be written as an integral formula which can be evaluated by efficient numerical integration (Fourier-cosine expansion, called COS method), see Fang and Oosterlee (2008) for details. For the convenience of the reader, we explain the main ideas of the COS method in Appendix B.*

We now give some details. Define the conditional characteristic function of Z_T by

$$\phi(t,z,v;\omega) = E_{t,z,v}\left[e^{i\omega Z_T}\right].$$

By the Feynman-Kac theorem, ϕ satisfies the following PDE

$$\begin{aligned}\frac{\partial\phi}{\partial t} &- \left\{r + \frac{1}{2}v[A^2 + c^2(1-\rho^2)]\right\}\phi_z + \kappa(\theta - v)\phi_v + \frac{1}{2}v[A^2 + c^2(1-\rho^2)]\phi_{zz} \\ &+ v\xi[c(1-\rho^2) - A\rho]\phi_{zv} + \frac{1}{2}\xi^2 v\phi_{vv} = 0.\end{aligned} \tag{4.8}$$

Assume that ϕ takes the following form

$$\phi(t,z,v;\omega) = \exp(C(t;\omega) + D(t;\omega)v + i\omega z). \tag{4.9}$$

with $C(T;\omega) = 0$ and $D(T;\omega) = 0$. Inserting (4.9) *into* (4.8) *gives that C and D satisfy the Riccati equations in Appendix A with coefficients $d_1 = -\kappa\theta$, $d_2 = ri\omega$ and*

$$a = -\frac{1}{2}\xi^2,\ b = -\{\xi[c(1-\rho^2) - A\rho]i\omega - \kappa\},\ \eta = \frac{1}{2}[A^2 + c^2(1-\rho^2)](\omega^2 + i\omega).$$

The closed form solutions C and D are given by (A.3) *and* (A.2)*, respectively, with $\underline{t} = 0$, $\bar{t} = T$, $f_1 = 0$, $f_2 = 0$. Define $\varphi(t,v;\omega) = e^{-i\omega z}\phi(t,z,v;\omega)$. This is the conditional characteristic function of $Z_T - Z_t = \ln Y_T - \ln Y_t$.*

Following Fang and Oosterlee (2008), we can easily find that the upper bound is given by

$$\overline{\mathcal{W}}(t,x,v) = \mathcal{Z}(t,y^*,v) + xy^*, \tag{4.10}$$

where

$$\mathcal{Z}(t,y^*,v) := \widetilde{\mathcal{Z}}(t,z^*,v) \approx {\sum_{k=0}^{N-1}}' \mathrm{Re}\left\{\varphi\left(t,v;\frac{k\pi}{\zeta_2 - \zeta_1}\right)e^{ik\pi\frac{z^* - \zeta_1}{\zeta_2 - \zeta_1}}\right\}\widetilde{\mathcal{Z}}_k, \tag{4.11}$$

and $\widetilde{\mathcal{Z}}_k$, y^* and other constants are given in Appendix B. The feedback control for computing the lower bound is given by (3.6).

Remark 4.4. *For utility maximization with discretionary stopping (see e.g., Karatzas and Wang (2000)), the corresponding dual problem is a finite-maturity American put option pricing problem with the Heston model, which is much more complex and is not investigated in this paper.*

5 Numerical tests

In the following numerical examples we use the dual-control Monte-Carlo method to solve the optimal control problem (2.3) with power, non-HARA and Yaari utilities. We compute the upper bounds using the closed form formulas for power and non-HARA utilities and the Fourier-cosine method for Yaari utility when $\gamma = c\sqrt{v}$ and everything else (the lower bounds for all γ and the upper bounds for $\gamma = c$ and $\gamma = cv$) using the Monte-Carlo method with path number 100,000 and time steps 100 for discretizing SDEs with the Euler method, see Remarks 3.4 and 3.5.

5.1 Power utility

Example 5.1. *This example is aimed to apply the lower and upper bound method to the power utility when v_t following mean-reversion square-root process. The following parameters*

$$r = 0.05,\ \rho = -0.5,\ \kappa = 10,\ \theta = 0.05,\ \xi = 0.5,\ A = 0.5,\ x_0 = 1,\ v_0 = 0.5,\ T = 1, \tag{5.1}$$

are taken from Zhang and Ge (2016). The comparisons are implemented for the cases of sampling control c for $1, 5, 80$ *times uniformly distributed in* $[-0.5, 0.5]$ *for both the lower and upper bounds. The benchmark value is the primal value explicitly given by Kraft (2005). The parameter p in utility function equals* $1/2$*, and other parameters follow values in* (5.1)*. The numerical results are listed in Table 1 in which the shorthand notations used are diff = UB− LB and rel-diff (%) =* $(UB - LB)/LB \times 100$*. The same notations are used for all subsequent numerical tests.*

Table 1: Lower bound (LB) and upper bound (UB) for power utility (Example 5.1). The benchmark from Example 5.1 equals 2.074842. diff = UB − LB and rel-diff (%) = (UB − LB)/LB × 100

			$\gamma_t = c$			
Num c	LB	UB	diff	rel-diff (%)	LB time (secs)	UB time (secs)
1	2.074824	2.074894	7.01e−5	3.38e−3	2.62e+4	2.90e+0
5	2.074824	2.074894	7.01e−5	3.38e−3	2.63e+4	1.46e+1
80	2.074824	2.074893	6.92e−5	3.34e−3	2.76e+4	2.24e+2
			$\gamma_t = c\sqrt{v_t}$			
Num c	LB	UB	diff	rel-diff (%)	LB time (secs)	UB time (secs)
1	2.074823	2.074845	2.12e−5	1.02e−3	4.65e+0	9.87e−4
5	2.074823	2.074845	2.12e−5	1.02e−3	2.33e+1	3.30e−3
80	2.074823	2.074842	1.88e−5	9.06e−4	3.91e+2	8.18e−3
			$\gamma_t = cv_t$			
Num c	LB	UB	diff	rel-diff (%)	LB time (secs)	UB time (secs)
1	2.074824	2.074894	7.01e−5	3.38e−3	2.64e+4	2.83e+0
5	2.074824	2.074894	6.97e−5	3.36e−3	2.65e+4	1.38e+1
80	2.074824	2.074893	6.88e−5	3.31e−3	2.79e+4	2.15e+2

Example 5.2. *From Example 5.1, we see $\gamma_t = c\sqrt{v_t}$ outperforms other choices of γ_t. In this example we further test the robustness of the dual control Monte-Carlo methods for $\gamma_t = c\sqrt{v_t}$. The comparisons are implemented for the cases of sampling control c for $1, 5, 80$ times uniformly distributed in $[-0.5, 0.5]$ both for the lower and upper bounds. In Table 2, we give the mean and standard deviation of the absolute and relative difference between the lower and upper bounds, which are respectively denoted by mean diff, std diff, mean rel-diff (%), and std rel-diff (%), for the power utility with randomly sampled parameters-sets: 10 samples of r from the uniform distribution on interval $[0.01, 0.08]$, ρ on $[-1, 1]$, κ on $[1, 10]$, θ on $[0.01, 1]$, ξ on $[0.1, 1]$, A on $[0.1, 1.5]$, $x_0 = 1$, $v_0 = 0.5$, and $T = 1$. The gap between the tight lower and upper bounds is narrow and can be improved by adjusting the dual control variable c. This shows the accuracy and reliability of the algorithm. The numerical results are listed in Table 2.*

Table 2: Mean and std of the absolute and relative difference between the lower and upper bounds for power utility (Example 5.2) with many randomly sampled parameters-sets.

Num c	mean diff	std diff	mean rel-diff (%)	std rel-diff (%)	mean time (secs)
1	2.2695e−3	2.7658e−3	9.8159e−2	1.1543e−1	4.29e+1
5	2.2660e−3	2.7632e−3	9.8010e−2	1.1532e−1	2.16e+2
80	1.8253e−3	1.9986e−3	7.9391e−2	8.4384e−2	3.48e+3

5.2 Non-HARA utility

Example 5.3. *This example is aimed to check the correctness of the lower and upper bounds when process v_t always constant through the time, in which case there is explicit solution to the primal value function. Let $v_0 = \theta$, $\xi = 0$, and the other parameters be the same as (5.1). Denote $\bar{W}_1 = \exp[(3r + 6A^2\theta)(T - t)]$ and $\bar{W}_2 = \exp[(r + A^2\theta)(T - t)]$. Then the primal value function has the following explicit form (see Bian and Zheng (2015)):*

$$\mathcal{W}(t, x) = \frac{2}{3}\left(\frac{\bar{W}_2}{y^*} + 2xy^*\right),$$

with

$$y^* = \sqrt{\frac{1}{2x}\left(\bar{W}_2 + \sqrt{\bar{W}_2^2 + 4x\bar{W}_1}\right)}.$$

The lower and upper bounds are computed by the Monte-Carlo method with path number $100,000$ and time steps 100. The numerical results are listed in Table 3, in which the numerics show that the benchmark is between the lower and upper bound, and the difference between these is proportional to 10^{-4} and relative difference 10^{-5}. Therefore, the lower and upper bound methods are reliable and accurate.

Table 3: Lower bound (LB) and upper bound (UB) for Example 5.3 (non-HARA utility).

Benchmark	LB	UB	diff	rel-diff (%)
2.307810	2.307691	2.307843	1.52e−4	6.60e−3

Example 5.4. *This example is aimed to apply the lower and upper bound methods to the non-HARA utility when v_t following mean-reversion square-root process. The comparisons are implemented for the cases of sampling control c for 20 times uniformly distributed in $[-0.5, 0.5]$*

Table 4: Lower bound (LB) and upper bound (UB) for non-HARA utility (Example 5.4).

γ	LB	UB	diff	rel-diff(%)	LB time (secs)	UB time (secs)
c	2.327407	2.327834	4.27e−4	1.83e−2	2.43e+4	5.37e+1
$c\sqrt{v_t}$	2.327573	2.327858	2.84e−4	1.22e−2	1.36e+2	4.35e−3
cv_t	2.327411	2.327833	4.21e−4	1.81e−2	2.41e+4	5.58e+1

both for the lower and upper bounds. The other parameters values are the same as in (5.1)*. The numerical results in Table 4 show that the choice* $\gamma = c\sqrt{v_t}$ *outperforms the others.*

Using the optimal control $c^* = 0.491037386$ *for computing the tight lower bound for* $\gamma_t = c\sqrt{v_t}$ *in Table 4, we draw the 3D figures for the feasible control strategy* $\bar{\pi}$ *and the distribution of the terminal wealth (see Figure 1). The left figure shows the percentage invested in the stock reaches lowest when variance process is large at initial time and reaches highest when variance process is low near the terminal time, which coincides the behavior of risk-aversion investor. The middle figure shows the percentage invested in the stock increases as the wealth increases, which coincides with the relative risk aversion coefficient's property of non-HARA utility (see Bian and Zheng (2015)). The right figure shows the distribution of the terminal wealth which is positively skewed.*

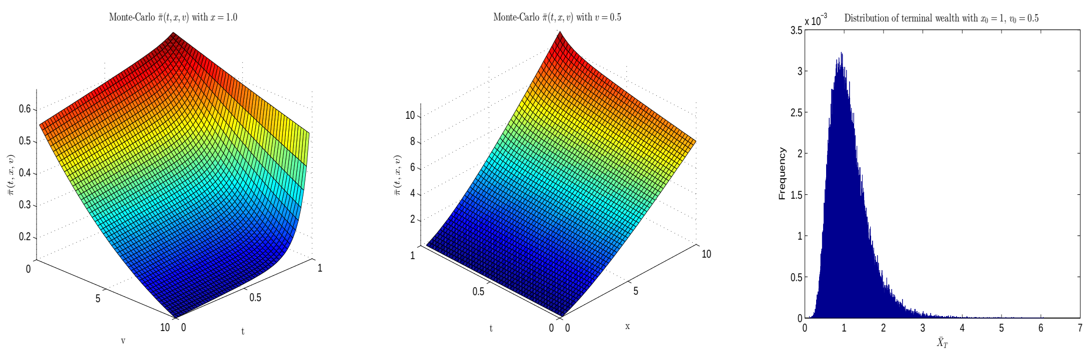

Figure 1: 3D and 2D figures for non-HARA utility (Example 5.4) under the dual control $\gamma_t = c^*\sqrt{v_t}$ with $c^* = 0.491037386$. The left and middle figures are the corresponding control strategies $\bar{\pi}(t, x, v)$ with initial wealth $x_0 = 1$ and initial variance $v_0 = 0.5$, respectively, and the right figure is the distribution of the terminal wealth.

Example 5.5. *In this example, we further examine the robustness of the lower and upper bound methods with* $\gamma_t = c\sqrt{v_t}$*. The comparisons are implemented for the cases of sampling control* c *for* $1, 5, 20$ *times, uniformly distributed in* $[-0.5, 0.5]$*, for both the lower and upper bounds. In Table 5, we give the mean and standard deviations of the absolute and relative differences between the lower and upper bounds, denoted respectively by mean diff, std diff, mean rel-diff (%), and std rel-diff (%), for non-HARA utility with randomly sampled parameters-sets:* 10 *samples of* r *from the uniform distribution on interval* $[0.01, 0.08]$*,* ρ *on* $[-1, 1]$*,* κ *on* $[1, 10]$*,* θ *on* $[0.01, 1]$*,* ξ *on* $[0.1, 1]$*,* A *on* $[0.1, 1.5]$*,* $x_0 = 1$*,* $v_0 = 0.5$*, and* $T = 1$*. The difference between the tight lower and upper bounds is small and can be improved by adjusting the dual control variable* c*. This shows the accuracy and reliability of the algorithm.*

5.3 Yaari utility

Example 5.6. *This example is aimed to check the lower and upper bound methods when process* v_t *always constant through the time. Let* $v_0 = \theta$*,* $\xi = 0$*, and the other parameters be the*

Table 5: Mean and std of the absolute and relative difference between the lower and upper bounds for non-HARA utility (Example 5.5) with many randomly sampled parameters-sets.

Num c	mean diff	std diff	mean rel-diff (%)	std rel-diff (%)	mean time (secs)
1	1.1434e−2	1.7040e−2	4.2995e−1	6.3311e−1	1.33e+2
5	9.3788e−3	1.2686e−2	3.5362e−1	4.6995e−1	6.64e+2
20	8.5262e−3	1.2124e−2	3.1998e−1	4.4861e−1	2.66e+3

same as (5.1)*. Then the analytical solution to the primal value is given by*

$$\mathcal{W}(t,x) = \begin{cases} L\Phi\{\Phi^{-1}[\frac{x}{L}e^{r(T-t)}] + A\sqrt{\theta(T-t)}\}, & 0 \leq x < Le^{-r(T-t)}, \\ L, & x \geq Le^{-r(T-t)}. \end{cases}$$

The upper bound is computed by the Monte-Carlo methods with path number 10,000 *and time steps* 100*, and the lower bound with path number* 100,000 *and time steps* 100*. The threshold L is taken as $L = 2$. The numerical results are listed in Table 6, which confirm that the lower and upper bound methods are reliable and accurate.*

Table 6: Lower bound (LB) and upper bound (UB) for Yaari utility (Example 5.6).

Benchmark	LB	UB	diff	rel-diff (%)
1.139790	1.136091	1.139889	3.80e−3	3.34e−1

Example 5.7. *This example is aimed to apply the lower and upper bound methods to the Yaari utility when v_t following mean-reversion square-root process. The comparisons are implemented for sampling control c for* 20 *times uniformly distributed in* $[-0.5, 0.5]$ *both for the lower and upper bounds. The values of other parameters are the same as Example 5.6. For the Fourier-cosine methods, we set the truncation number as $N = 64$. The numerical results are listed in Table 7. It is shown that the choice $\gamma_t = c\sqrt{v_t}$ outperforms the others.*

Table 7: Lower bound (LB) and upper bound (UB) for Yaari utility (Example 5.7).

γ	LB	UB	diff	rel-diff (%)	LB time (secs)	UB time (secs)
c	1.113889	1.174928	6.10e−2	5.48e+0	1.08e+4	6.08e+0
$c\sqrt{v_t}$	1.172057	1.173366	1.31e−3	1.12e−1	2.37e+3	2.48e−1
cv_t	1.137594	1.174928	3.73e−2	3.28e+0	1.09e+4	5.69e+0

Using the optimal control $c^* = 0.2615142622$ for computing the tight lower bound with $\gamma_t = c\sqrt{v_t}$ in Table 7, we draw the 3D figures for the corresponding control strategies $\bar{\pi}(t, x, v)$ and the distribution of the terminal wealth (see Figure 2). The left and middle figures show that as time t tends to maturity T and variance v or wealth x tends to 0, the percentage invested in the stock tends to infinite, which is consistent with the behavior of a Yaari utility investor who invests heavily by borrowing near the end of an investment period in the hope of reaching the ultimate target. The right figure displays the terminal wealth distribution which is a Bernoulli-like distribution (total loss or total win), similar to those in the GBM and regime-switching cases (see Ma et al. (2017)).

Example 5.8. *In this example, we further test the robustness of the lower and upper bound methods for $\gamma_t = c\sqrt{v_t}$. The comparisons are implemented for the cases of sampling control*

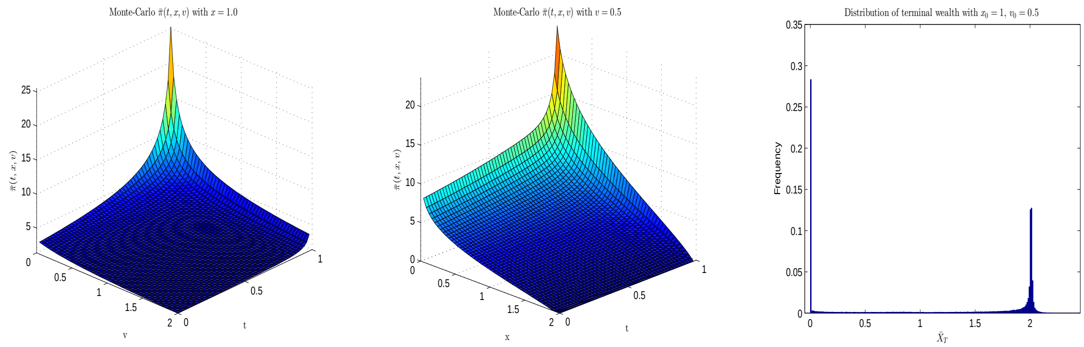

Figure 2: 3D and 2D figures for Yaari utility (Example 5.7) under the dual control $\gamma_t = c^* \sqrt{v_t}$ with $c^* = 0.2615142622$. The left and middle figures are the corresponding control strategies $\bar{\pi}(t, x, v)$ with initial wealth $x_0 = 1$ and initial variance $v_0 = 0.5$, respectively. The right figure is the distribution of the terminal wealth.

c for $1, 5, 20$ times uniformly distributed in $[-0.5, 0.5]$ both for the lower and upper bounds. In Table 8, we list the mean and standard deviations of the absolute and relative differences between the lower and upper bounds with randomly sampled parameters-sets: 10 samples of r from the uniform distribution on interval $[0.01, 0.08]$, ρ on $[-1, 1]$, κ on $[1, 10]$, θ on $[0.01, 1]$, ξ on $[0.1, 1]$, A on $[0.1, 1.5]$, $x_0 = 1$, $v_0 = 0.5$, and $T = 1$. The difference between the tight lower and upper bounds is small and can be improved by adjusting the dual control variable c. This shows the accuracy and reliability of the algorithm.

Table 8: Mean and std of the absolute and relative difference between the lower and upper bounds for Yaari utility (Example 5.8) with many randomly sampled parameters-sets.

Num c	mean diff	std diff	mean rel-diff (%)	std rel-diff (%)	mean time
1	6.9575e−3	5.0882e−3	5.2082e−1	3.4214e−1	1.22e+3
5	6.3116e−3	4.7772e−3	4.7119e−1	3.2324e−1	1.73e+3
20	5.8369e−3	4.5691e−3	4.3418e−1	3.1094e−1	3.65e+3

5.4 Regime-switching model

In this example, we aim to test the robustness of the lower and upper bound methods for the regime-switching Heston model. The coefficients of the dynamics are driven by a continuous time finite state observable Markov chain process (MCP) $\boldsymbol{\alpha}$ which is independent of Brownian motions W^s, W^v and has the following semi-martingale representation:

$$\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_0 + \int_0^t \mathbf{Q}^{tr} \boldsymbol{\alpha}_v dv + \mathbf{M}_t, \ 0 \le t \le T, \tag{5.2}$$

where $\boldsymbol{\alpha}_t$ is a unit vector in the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$ with $\mathbf{e}_i \in \mathbb{R}^d$ being a column vector with 1 in the ith position and 0 elsewhere, $\boldsymbol{\alpha}_0$ is the initial Markov chain state, $\mathbf{Q} = (q_{ij})_{d\times d}$ is the generator of MCP $\boldsymbol{\alpha}$ with $q_{ij} \ge 0$ for $i \ne j$ and $\sum_{j=1}^d q_{ij} = 0$ for each $i \in \mathbb{D} := \{1, \ldots, d\}$, and $\mathbf{M}$ is a purely discontinuous square-integrable martingale with initial value zero, see Elliott et al. (1994).

The dynamics of the bond and the stock price processes B and S are given by the following regime-switching Heston model for $t \in [0, T]$,

$$\begin{aligned} dB_t &= r_t B_t dt, \\ dS_t &= S_t[(r_t + A_t v_t)dt + \sqrt{v_t} dW_t^s], \\ dv_t &= \kappa_t(\theta_t - v_t)dt + \xi_t \sqrt{v_t} dW_t^v, \end{aligned}$$

where $r_t = \boldsymbol{r}\boldsymbol{\alpha}_t$, $A_t = \boldsymbol{A}\boldsymbol{\alpha}_t$, $\kappa_t = \boldsymbol{\kappa}\boldsymbol{\alpha}_t$, $\theta_t = \boldsymbol{\theta}\boldsymbol{\alpha}_t$, $\xi_t = \boldsymbol{\xi}\boldsymbol{\alpha}_t$, $dW_t^s dW_t^v = \rho_t dt = \boldsymbol{\rho}\boldsymbol{\alpha}_t dt$. $\boldsymbol{r} = (r_1, \dots, r_d)$ is a vector of risk-free interest rates with $r_i > 0$ being the rate in regime i. The market price of risk $\boldsymbol{A}$, the mean reverting rate $\boldsymbol{\kappa}$, the long-run average volatility $\boldsymbol{\theta}$, the variance $\boldsymbol{\xi}$, the correlation $\boldsymbol{\rho}$ are defined similarly.

Let X be the wealth process. At time $t \in [0, T]$ the investor allocates a proportion π_t of wealth X in risky asset S and the remaining wealth in savings account B. Then the wealth process X satisfies the SDE:

$$dX_t = X_t[(r_t + \pi_t A_t v_t)dt + \pi_t \sqrt{v_t} dW_t^s], \ X_0 = x_0, \tag{5.3}$$

where π is a progressively measurable control process.

The regime-switching GBM case for utility maximization has been studied in Ma et al. (2017). In the following, we will extend the dual control method to the regime-switching Heston model. Suppose that a dual process has the following form

$$dY_t = Y_t[-r_t dt - (\rho_t \gamma_t + A_t \sqrt{v_t})dW_t^s + \gamma_t dW_t^v + \mathbf{C}_1 d\mathbf{M}_t], \ Y_0 = y, \tag{5.4}$$

where γ is a dual control process and $\mathbf{C}_1$ a constant row vector in R^d with components in $(-0.5, 0.5)$. We take dual controls γ_t in the form of $\mathbf{C}_2\boldsymbol{\alpha}_t$, $\mathbf{C}_2\boldsymbol{\alpha}_t\sqrt{v_t}$, and $\mathbf{C}_2\boldsymbol{\alpha}_t v_t$, where $\mathbf{C}_2$ is a constant row vector in R^d. Using Itô's formula, we can check that XY is a supermartingale for any control process π, which leads to the following weak duality relation

$$\sup_{\pi} E[U(X_T)] \leq \inf_{y} \big(\inf_{\mathbf{C}_1, \mathbf{C}_2} E[\widetilde{U}(Y_T)] + x_0 y \big), \tag{5.5}$$

We can now use the algorithm developed in Ma et al. (2017) to generate MCP $\boldsymbol{\alpha}$ and complete the numerical simulation by the dual control Monte-Carlo method discussed in Section 3.

Consider a 2-state Markov chain process with a generating matrix

$$\mathbf{Q} = \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}, \tag{5.6}$$

where a, b are positive constants. To show the robustness of the algorithm, we have chosen 5 samples of a, b from the uniform distribution on interval $[0.1, 2.0]$, which means the transition of one state to another can be slow (average once every 10 years) or fast (average twice a year) or anything in between. The states can be "growth economy" (state 1) and "recession economy" (state 2). For other parameters, we sample r_1, r_2 from the uniform distribution on interval $[0.01, 0.08]$, ρ_1, ρ_2 on $[-1, 1]$, κ_1, κ_2 on $[1, 10]$, θ_1, θ_2 on $[0.01, 1]$, ξ_1, ξ_2 on $[0.1, 1]$, A_1, A_2 on $[0.1, 1.5]$, $x_0 = 1$, $v_0 = 0.5$, and $T = 1$. The lower and upper bounds are both computed by the Monte-Carlo methods with path number $10,000$ and time steps 50. The control vector $(\mathbf{C}_1, \mathbf{C}_2)$ are sampled in $[-0.4, 0.4]^4$ for 10,000 times for computing the upper bound and 20 times for the lower bound. Table 9 shows that the differences between the lower and upper bounds are very small and the suggested dual control Monte-Carlo method is robust and stable.

Remark 5.9. *In all numerical tests above, we have listed the lower and upper bounds in terms of utilities, that is, the optimal value function is in between LB and UB. Rather than*

Table 9: The difference between the lower bound and upper bound for power utility in regime-switching Heston model.

γ_t	$\mathbf{C}_2\boldsymbol{\alpha}_t$	$\mathbf{C}_2\boldsymbol{\alpha}_t\sqrt{v_t}$	$\mathbf{C}_2\boldsymbol{\alpha}_t v_t$
mean diff	2.3977e−3	2.2781e−3	2.2638e−3
std diff	2.3325e−3	2.2543e−3	2.2135e−3
mean rel-diff (%)	1.1307e−1	1.0743e−1	1.0678e−1
std rel-diff (%)	1.0570e−1	1.0220e−1	1.0035e−1

listing values in units of utility at maturity T, we may express the lower and upper bounds in units of initial wealth at time 0. This is possible in the domain of the utility function U where it is strictly increasing. Once having the utility value z at time T, we can solve the equation $U(e^{rT}x) = z$, where r is the riskless interest rate, and get the initial wealth value x at time 0, either analytically if there is a closed-form formula or numerically with the Newton method. For example, in Table 1 for power utility $U(x) = (1/2)\sqrt{x}$ and dual control $\gamma_t = c\sqrt{v_t}$ with $T = 1$ and $r = 0.05$, we have LB is 2.074823 and UB 2.074845, the corresponding initial wealth values for these two bounds are $x_{LB} = 1.023736$ and $x_{UB} = 1.023756$, the difference of the two is 0.000020, or 0.0020% of the wealth, which may be explained as the additional initial wealth that would make an investor indifferent between the two bounds. Similarly, in Table 4 for non-HARA utility and dual control $\gamma_t = c\sqrt{v_t}$, we have LB is 2.327573 and UB 2.327858, the corresponding initial wealth values for these two bounds are $x_{LB} = 1.032793$ and $x_{UB} = 1.033012$, the difference of the two is 0.000219, or 0.0212% of the wealth. In Table 7 for Yaari utility with the threshold level $L = 2$ and dual control $\gamma_t = c\sqrt{v_t}$, we have LB is 1.172057 and UB 1.173366, the corresponding initial wealth values for these two bounds are $x_{LB} = 1.114895$ and $x_{UB} = 1.116140$, the difference of the two is 0.001245, or 0.1117% of the wealth. We thank the anonymous reviewer for suggesting this way of explaining the lower and upper bounds.

6 Conclusions

In this paper we use the weak duality relation to construct the lower and upper bounds on the primal value function for utility maximization under the Heston stochastic volatility model with general utilities. We propose a dual control Monte-Carlo method to compute the bounds and suggest some simple forms of the dual control γ_t which makes the bounds tighter and computation easier. In particular, if γ is taken as $\gamma_t = c(t)\sqrt{v_t}$ with c being a piecewise constant function, the closed form upper bound can be obtained for a broad class of utilities (including power and non-HARA utilities), and the Fourier-Cosine formula can be used for the Yaari utility. The gap between the lower and upper bounds can be reduced if the number of sampling or the number of time pieces increases. Numerical examples show that the tight bounds can be derived with little computational cost. The applications to the regime-switching Heston model are also studied in the numerical examples. Based on the numerical results, it seems that using the Monte-Carlo method with path numbers 100,000 and time steps 100 would provide tight lower and upper bounds with absolute differences less than 0.01 in most cases.

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A Closed-form solutions to Riccati equations

In the paper we need to solve a number of times the following system of equations:

$$C'(t) = d_1 D(t) + d_2, \quad \underline{t} \le t \le \bar{t}$$

and

$$D'(t) = aD^2(t) + bD(t) + \eta, \quad \underline{t} \le t \le \bar{t},$$

with the terminal conditions $C(\bar{t}) = f_1$ and $D(\bar{t}) = f_2$, where all coefficients are constants. Assume $b^2 - 4a\eta > 0$ and $\frac{m_1}{m_2} \notin [e^{-k_1(\bar{t}-\underline{t})}, 1]$, where

$$k_1 = \sqrt{b^2 - 4a\eta}, \quad m_1 = \frac{-b-k_1}{2a}, \quad m_2 = \frac{-b+k_1}{2a}.$$

The assumption $b^2 - 4a\eta > 0$ ensures m_1 and m_2 are distinct real numbers. We can first find D by writing the equation as

$$\frac{1}{a}\left(\frac{1}{D-m_1} - \frac{1}{D-m_2}\right)\frac{dD}{m_1 - m_2} = dt. \tag{A.1}$$

Using the terminal condition and denote $k_2 = \frac{f_2 - m_1}{f_2 - m_2}$ we obtain the solution to (A.1) as

$$D(t) = \frac{m_1 - m_2}{1 - k_2 \exp[k_1(\bar{t} - t)]} + m_2, \tag{A.2}$$

which leads to a closed-form formula for $D(t)$ on interval $[\underline{t}, \bar{t}]$. As for $C(t)$, we have the following form

$$C(t) = -\frac{d_1(m_1 - m_2)}{k_1} \ln\left(\frac{k_2 - 1}{k_2 - \exp[-k_1(\bar{t} - t)]}\right) - d_1 m_2(\bar{t} - t) - d_2(\bar{t} - t) + f_1. \tag{A.3}$$

The assumption $\frac{m_1}{m_2} \notin [e^{-k_1(\bar{t}-\underline{t})}, 1]$ is to exclude the case of $\int_{\underline{t}}^{\bar{t}} D(s)ds$ being hypersingular integral.

B COS method for the Heston model

Following the main idea of COS method in Fang and Oosterlee (2008), we explain how to derive the upper bound for Yaari utility. For an option pricing problem expressed in (4.7), we rewrite it into the following form

$$\widetilde{\mathcal{Z}}(t,z,v) = E_{t,z,v}[L(1-e^{Z_T})^+] = \int_{\mathbb{R}} \widetilde{\mathcal{Z}}(T,y)g(y|z,v)dy. \tag{B.1}$$

Since the density rapidly decays to zero as $y \to \pm\infty$ in (B.1), we truncate the infinite integration range without loosing significant accuracy to $[\zeta_1, \zeta_2] \subset \mathbb{R}$, and obtain approximation $\widetilde{\mathcal{Z}}^{(1)}$:

$$\widetilde{\mathcal{Z}}^{(1)}(t,z,v) = \int_{\zeta_1}^{\zeta_2} \widetilde{\mathcal{Z}}(T,y)g(y|z,v)dy.$$

In the second step, since $g(y|z,v)$ is usually unknown whereas the characteristic function is, we approximate the density function g by the first N terms of its cosine expansion in y, that is,

$$g(y|z,v) \approx {\sum_{k=0}^{N-1}}' \mathcal{A}_k(z,v)\cos\left(k\pi\frac{y-\zeta_1}{\zeta_2-\zeta_1}\right),$$

where $\sum'$ indicates that the first term in the summation is weighted by one-half and

$$\mathcal{A}_k(z,v) := \frac{2}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} g(y|z,v)\cos\left(k\pi\frac{y-\zeta_1}{\zeta_2-\zeta_1}\right)dy.$$

Interchanging the summation and integration, and inserting the definition

$$\widetilde{\mathcal{Z}}_k := \frac{2}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} \widetilde{\mathcal{Z}}(T,y)\cos\left(k\pi\frac{y-\zeta_1}{\zeta_2-\zeta_1}\right)dy = \frac{2}{\zeta_2-\zeta_1}L[\psi_k(\zeta_1,0)-\chi_k(\zeta_1,0)],$$

where

$$\begin{aligned}
\psi_k(x_1,x_2) &= \begin{cases} \left[\sin\left(k\pi\frac{x_2-\zeta_1}{\zeta_2-\zeta_1}\right) - \sin\left(k\pi\frac{x_1-\zeta_1}{\zeta_2-\zeta_1}\right)\right]\frac{\zeta_2-\zeta_1}{k\pi}, & k \neq 0, \\ x_2 - x_1, & k = 0, \end{cases} \\
\chi_k(x_1,x_2) &= \frac{1}{1+\left(\frac{k\pi}{\zeta_2-\zeta_1}\right)^2}\Bigg[\cos\left(k\pi\frac{x_2-\zeta_1}{\zeta_2-\zeta_1}\right)e^{x_2} - \cos\left(k\pi\frac{x_1-\zeta_1}{\zeta_2-\zeta_1}\right)e^{x_1} \\
&\quad + \frac{k\pi}{\zeta_2-\zeta_1}\sin\left(k\pi\frac{x_2-\zeta_1}{\zeta_2-\zeta_1}\right)e^{x_2} - \frac{k\pi}{\zeta_2-\zeta_1}\sin\left(k\pi\frac{x_1-\zeta_1}{\zeta_2-\zeta_1}\right)e^{x_1}\Bigg],
\end{aligned}$$

we obtain approximation $\widetilde{\mathcal{Z}}^{(2)}$:

$$\widetilde{\mathcal{Z}}^{(2)}(t,z,v) = \frac{1}{2}(\zeta_2-\zeta_1){\sum_{k=0}^{N-1}}' \mathcal{A}_k(z,v)\widetilde{\mathcal{Z}}_k.$$

Noting that

$$\begin{aligned}
\frac{1}{2}(\zeta_2-\zeta_1)\mathcal{A}_k(z,v) &= \int_{\zeta_1}^{\zeta_2} g(y|z,v)\cos\left(k\pi\frac{y-\zeta_1}{\zeta_2-\zeta_1}\right)dy \\
&\approx \int_{\mathbb{R}} g(y|z,v)\cos\left(k\pi\frac{y-\zeta_1}{\zeta_2-\zeta_1}\right)dy
\end{aligned}$$

$$= \quad \mathrm{Re}\left\{\phi\left(t, z, v; \frac{k\pi}{\zeta_2-\zeta_1}\right) e^{-ik\pi\frac{\zeta_1}{\zeta_2-\zeta_1}}\right\},$$

we obtain

$$\widetilde{\mathcal{Z}}(t,z,v) \approx \widetilde{\mathcal{Z}}^{(3)}(t,z,v) = {\sum_{k=0}^{N-1}}' \mathrm{Re}\left\{\phi\left(t, z, v; \frac{k\pi}{\zeta_2-\zeta_1}\right) e^{-ik\pi\frac{\zeta_1}{\zeta_2-\zeta_1}}\right\} \widetilde{\mathcal{Z}}_k.$$

Define $\varphi(t,v;\omega) = e^{-i\omega z}\phi(t,z,v;\omega)$. We have derived (4.11). To find y^* in (4.10) and the feedback control $\bar{\pi}$ in (3.6), we need to compute derivatives $\mathcal{Z}_y$, $\mathcal{Z}_{yy}$ and $\mathcal{Z}_{yv}$, which is easy and straightforward from the above approximate formula for $\mathcal{Z}(t,y,v)$. Finally, according to Fang and Oosterlee (2008), we can choose the boundary of integral as

$$[\zeta_1, \zeta_2] := \left[c_1 - L_1\sqrt{|c_2|},\ c_1 + L_1\sqrt{|c_2|}\right],$$

where

$$c_n = \frac{1}{i^n}\frac{\partial^n \ln(\varphi(t,v;\omega))}{\partial \omega^n}|_{\omega=0},$$

and L_1 is a constant chosen large enough to guarantee $\zeta_1 < 0 < \zeta_2$. Cumulant c_2 may become negative for sets of Heston parameters that do not satisfy the Feller condition, i.e., $2\kappa\theta \geq \xi^2$. We therefore use the absolute value of c_2.