

Convergence analysis and optimal strike choice for static hedges of general path-independent payoffs

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Abstract

In this paper we propose a new algorithm to find the optimal static replicating portfolios for general path-independent nonlinear payoff functions and give an estimate for the rate of convergence that is absent in the literature. We choose the static replication by designing an adaptation function arising in the error bound between the nonlinear payoff function and the linear spline approximation and derive the equidistribution equation for selecting the optimal strikes. The numerical tests for variance swaps, swaptions, static quadratic hedges, and also for a jump diffusion process allowing for the default of the underlying asset, show that the proposed iterative equidistribution equation algorithm is simple, fast and accurate. The paper generalizes and improves the results of the static replication and approximation in the literature.

JEL classification. C, C6, C63, G, G1, G12

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1 Introduction

It is well known that hedging a derivative is in general much more difficult than pricing it, as hedging requires the determination of the feasible trading strategy, whereas pricing only involves the computation of the expected payoff, which may be found with the numerical integration or simulation. Dynamic replication can be used for hedging with the help of the martingale representation theorem if the market is complete, but it is often difficult to implement as the market is in fact incomplete. Static replication is a viable alternative.

The idea of using a portfolio of options to replicate complex payoffs dates back to Ross (1976) and Breeden and Litzenberger (1978). If options with strikes from zero to infinity are all available, then a nonlinear path-independent payoff function at maturity with certain regularity conditions can be replicated exactly with static hedging. Under the assumption of no arbitrage, the price of the derivative being replicated is then the total premium of the replicating options. Compared to dynamic replication, which may incur prohibitively high transaction costs, static replication has many advantages, see e.g., Derman et al. (1995), Carr and Chou (1997), Carr et al. (1998),

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Demeterfi et al. (1999). Static replication with a portfolio of European calls and puts is easy to implement and does not incur running transaction costs. Carr and Wu (2014) discuss and compare static hedging with delta hedging when the underlying asset price is exposed to the possibility of jumps of random sizes. They conclude that static hedging strongly outperforms delta hedging.

To find a static replication, one needs first to have a good approximation to the payoff function. Linear spline approximation is a simple yet effective method. The key benefit of using a linear spline is that the resulting static replicating portfolio consists of simple European calls, puts, and digital options, and the weights of these options can be easily computed however complex the payoff function. In theory the approximation error can be made arbitrarily small if the maximal distance between adjacent grid points is sufficiently small. In practice one has to strike a balance between accuracy and cost, which means one needs to choose grid points carefully to minimize the error if the number of grid points is fixed.

Demeterfi et al. (1999) use European calls and puts with equally-spaced strikes to replicate the log payoff, which is not optimal because of the equal spacing. Broadie and Jain (2008) propose a simulation method to obtain the optimal approximation of a static replication; this minimizes the approximation error, but is computationally expensive. Liu (2010) discusses three optimal approximations of nonlinear payoffs. The first two approaches are to minimize the expected area (simple average and weighted average) enclosed by the payoff curve and the chords, which implicitly assume the payoff function is convex (or concave) and cannot be applied to general payoffs. The third is to minimize the expected sum of squared differences of the payoff and the replicating portfolio, which is computationally expensive in solving the optimization equation for complex nonlinear payoffs. These papers do not discuss convergence rates.

In this paper, motivated by the idea from de Boor (1973), we propose a new algorithm for finding the optimal approximation for general path-independent nonlinear payoffs satisfying suitable regularity conditions and provide the convergence theory for the algorithm. We first give an estimate of the error bound between the nonlinear payoff function and the linear spline approximation. We then choose the strikes of the static replication so that the error bound achieves an optimal order of convergence. The reason for working on the error bound is that we can derive a tractable equidistribution equation for selecting the optimal strikes, which would be difficult by directly working on the error itself. This approach of static replication works well if options with strikes from zero to infinity are all available. In practice, we use options with given strikes traded in the market and we may have no choice over the strikes. In that case we use static quadratic hedging to find the optimal weights of the options. The equidistribution equation algorithm is again useful in computing the optimal weights with some modified payoff functions.

The main contribution of the paper is that a simple, fast and accurate iterative equidistribution equation algorithm is proposed to find the optimal static replicating portfolio for general nonlinear payoff functions, and convergence rates are given. The results of the paper improve and generalize those of Liu (2010) and others in the literature.

The paper is organized as follows. In section 2 we discuss the approximation of a nonlinear payoff function by a linear spline and a portfolio of calls and puts and estimate the error bound (Theorem 2.1). In section 3 we propose an iterative equidistribution equation algorithm to find the optimal strikes in static replication and estimate of convergence rate (Theorem 3.1). We also apply quadratic hedging to find the optimal weights of the static replicating portfolio when the number of traded calls and puts in the market are fixed and finite. In section 4 we perform some numerical tests and compare the results with those from analytic formulas or simulations for different payoffs and asset price distributions. We also compare the static hedging performances for variance swaps with the equidistribution equation method and with the spanning relation method of Carr and Chou (1997). In section 5 we conclude. In the appendix we give the proofs of Theorems 2.1 and

3.1 and the derivations of formulas (2) and (26).

2 Static replication and error bound on approximation

In this section, we give a formula as in Liu (2010) for replicating the nonlinear payoff with a basket of European options, and derive the error bound on the approximation.

Let S be a nonnegative random variable, representing the asset price at maturity, and $f(S)$ the derivative value, with f a continuous payoff function defined on the positive real line. Let $[0, +\infty)$ be partitioned by X_0, X_1, \dots, X_n , with $0 < X_{min} \equiv X_0 < X_1 < \dots < X_n \equiv X_{max} < +\infty$ and X_{min}, X_{max} being fixed so that the probabilities $P(S < X_{min})$ and $P(S > X_{max})$ are extremely small. Then f can be approximated by the following piecewise linear functions:

$$L_i(S) = \frac{X_{i+1} - S}{h_i} f(X_i) + \frac{S - X_i}{h_i} f(X_{i+1}), \quad S \in [X_i, X_{i+1}], \quad (1)$$

where $h_i \equiv X_{i+1} - X_i$, $i = 0, 1, \dots, n-1$. We can represent the payoff curve approximately by

$$\begin{aligned} f(S) &\approx \sum_{i=0}^{n-1} L_i(S) \mathbf{1}_{X_i \leq S < X_{i+1}} + f(S) \mathbf{1}_{S < X_0} + f(S) \mathbf{1}_{S \geq X_n} \\ &= [f(S) - (L_0(X_0) - b_0(X_0 - S))] \mathbf{1}_{S < X_0} \\ &\quad + [f(S) - (L_{n-1}(X_n) + b_{n-1}(S - X_n))] \mathbf{1}_{S \geq X_n} + L_k(X_k) \\ &\quad + \sum_{i=1}^{k-1} (b_i - b_{i-1})(X_i - S) \mathbf{1}_{S < X_i} - b_{k-1}(X_k - S) \mathbf{1}_{S < X_k} \\ &\quad + b_k(S - X_k) \mathbf{1}_{S \geq X_k} + \sum_{i=k+1}^{n-1} (b_i - b_{i-1})(S - X_i) \mathbf{1}_{S \geq X_i}, \end{aligned} \quad (2)$$

with $\mathbf{1}_A$ the indicator function ($\mathbf{1}_A = 1$ if $x \in A$ and 0 otherwise.) and $b_i = (f(X_{i+1}) - f(X_i))/h_i$, see Appendix A or Liu (2010) for its proof. In (2), the first two terms of the last equality can be regarded as the error of linear polynomial approximations of $f(S) \mathbf{1}_{S < X_0}$ and $f(S) \mathbf{1}_{S \geq X_n}$, respectively, the third term is a cash amount, and the remaining terms are European puts and calls with strikes X_i , $i = 0, 1, \dots, n-1$. Both puts and calls are likely to be out-of-the-money options.

Since the probabilities $P(S < X_0)$ and $P(S > X_n)$ are extremely small, the approximation errors involving these events in the formula (2) have little impact on the valuation, so may be replaced. In such a case, the following formula from Demeterfi et al. (1999) can be used for static replication:

$$\begin{aligned} f(S) &\approx L_k(X_k) + \sum_{i=1}^{k-1} (b_i - b_{i-1})(X_i - S)^+ - b_{k-1}(X_k - S)^+ \\ &\quad + b_k(S - X_k)^+ + \sum_{i=k+1}^{n-1} (b_i - b_{i-1})(S - X_i)^+. \end{aligned} \quad (3)$$

Liu (2010) chooses the strikes X_0, X_1, \dots, X_n so that the total area enclosed by the payoff curve and the chords

$$\sum_{i=0}^{n-1} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)] dS$$

is minimized. Liu (2010) also finds that the performance can be improved if the weighted total area

$$\sum_{i=0}^{n-1} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)]g(S) dS$$

is used, where g is the density function of S (conditional on today's price of the underlying). It is clear that f needs to be convex to ensure all integrands are nonnegative. For a general payoff function f we measure the error with the weighted squared norm

$$\text{Error} \equiv \sqrt{\sum_{i=0}^{n-1} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)]^2 g(S) dS}.$$

The error bound on the linear spline approximation (1) is given by the following theorem.

Theorem 2.1 *Assume that f is continuous on $[X_0, X_n]$ and twice continuously differentiable on (X_i, X_{i+1}) , $i = 0, 1, \dots, n-1$, with finite second-order left and right directional derivatives at X_i , $i = 0, 1, \dots, n$. Then the error of the linear spline approximation (1) to the nonlinear payoff function f is bounded by*

$$\sqrt{\sum_{i=0}^{n-1} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)]^2 g(S) dS} \leq \sqrt{2 \sum_{i=0}^{n-1} h_i^4 \int_{X_i}^{X_{i+1}} G(S) (f''(S))^2 dS},$$

where

$$\begin{aligned} G(S) &\equiv \widehat{G}\left(\frac{S - X_i}{h_i}\right), \\ \widehat{G}(t) &\equiv \int_0^t \widehat{g}_i(\xi) \frac{\xi^2(1-\xi)^3}{3} d\xi + \int_t^1 \widehat{g}_i(\xi) \frac{(1-\xi)^2 \xi^3}{3} d\xi, \\ \widehat{g}_i(\xi) &\equiv g(X_i + h_i \xi). \end{aligned}$$

Proof See Appendix B.

3 Equidistribution equation methods and convergence analysis

In this section we propose two algorithms for static replication of nonlinear payoff functions with European call and put options. The first algorithm selects the optimal strikes when the strikes of options from zero to infinity are all available. The second algorithm finds the optimal weights of the options when there are only a limited number of strikes are available.

We develop an equidistribution equation to determine the values of strikes X_i , $i = 1, \dots, n-1$, (with fixed boundary values X_0 and X_n) so that the error estimation in Theorem 2.1 achieves an optimal order of convergence. (From numerical analysis the second-order convergence is optimal for the linear spline approximation.) The idea of the equidistribution equation is that for approximation of a function without good smoothness, equidistributing a suitable quantum monitored by an adaptation function can lead to an optimal order of convergence for the approximation. In this paper we define an adaptation function that purely stems from the approximation error bound given by Theorem 2.1 and equidistribute the area enclosed by the adaptation function curve (see Fig. 1 (right) in Section 4.1) to give the equidistribution equation for selecting the strikes. The

equidistribution equation can be implemented by an efficient and reliable iteration algorithm. This approach for selecting the strikes for the static replication is called the *equidistribution equation method* in this paper.

Liu (2010) proposes a minimum expected area method (the area enclosed by the payoff function and the approximating linear spline function) that leads to a system of highly nonlinear algebraic equations for the strikes. Liu (2010) provides an iteration algorithm to solve the system without convergence analysis, which leaves unclear whether the algorithm actually works. Our iterative equidistribution equation method here is simple to implement and has theoretical convergence rate.

In the following we design the adaptation function ρ_i . To better control the strike concentration without damaging the convergence rate of the approximation, using a similar idea of Huang (2005), we introduce an intensity parameter¹,

$$\alpha_h \equiv \left[\frac{1}{X_n - X_0} \sum_{i=0}^{n-1} h_i \left(\frac{1}{h_i} \int_{X_i}^{X_{i+1}} G(S)(f''(S))^2 dS \right)^{\gamma/2} \right]^{2/\gamma}, \quad (4)$$

for some $\gamma \in (0, 2]$, in the error bound provided in Theorem 2.1 and get the following inequality:

$$\sqrt{\sum_{i=0}^{n-1} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)]^2 g(S) dS} \leq \sqrt{2\alpha_h \sum_{i=0}^{n-1} h_i^5 \left(1 + \frac{1}{\alpha_h h_i} \int_{X_i}^{X_{i+1}} G(S)(f''(S))^2 dS \right)}.$$

Referring to the above inequality, we define the adaptation function

$$\rho_i \equiv \left(1 + \frac{1}{\alpha_h h_i} \int_{X_i}^{X_{i+1}} G(S)(f''(S))^2 dS \right)^{\gamma/2}, \quad (5)$$

and the equidistribution equation for selecting strikes X_1, \dots, X_{n-1} ,

$$h_i \rho_i = \frac{\sum_{j=0}^{n-1} h_j \rho_j}{n}, \quad i = 0, \dots, n-1. \quad (6)$$

To solve equation (6), we rewrite it as

$$\sum_{\ell=0}^{i-1} h_\ell \rho_\ell = \frac{i}{n} \sum_{j=0}^{n-1} h_j \rho_j, \quad i = 1, \dots, n. \quad (7)$$

Define a piecewise constant function

$$\bar{\rho}_X(x) = \rho_i, \quad \text{when } x \in [X_i, X_{i+1}], \quad i = 0, \dots, n-1.$$

Then equation (7) can be rewritten as

$$\int_{X_0}^{X_i} \bar{\rho}_X(x) dx = \frac{i}{n} \int_{X_0}^{X_n} \bar{\rho}_X(x) dx. \quad (8)$$

Noting that equation (8) cannot be solved exactly, we propose the following iterative algorithm to solve it.

¹Note that α_h is uniformly bounded, see Appendix C. As mentioned in Huang (2005), the optimal choice of γ for the smallest error bound is $\gamma = 2/5$.

Algorithm 3.1 (*Iterative Equidistribution Equation Algorithm*) Set initial values

$$X_i^{(0)} = X_0 + i \frac{X_n - X_0}{n}, \quad i = 0, 1, \dots, n.$$

Then the $(k+1)$ th-step values for $k = 0, 1, \dots$, are calculated by the following iteration

$$\int_{X_0^{(k+1)}}^{X_i^{(k+1)}} \bar{\rho}_{X^{(k)}}(x) dx = \frac{i}{n} \int_{X_0^{(k)}}^{X_n^{(k)}} \bar{\rho}_{X^{(k)}}(x) dx, \quad (9)$$

where $X_0^{(k+1)} \equiv X_0$, $X_n^{(k+1)} \equiv X_n$ and $\bar{\rho}_{X^{(k)}}$ is a piecewise constant function defined by

$$\bar{\rho}_{X^{(k)}}(x) = \rho_i^{(k)}, \quad \text{when } x \in [X_i^{(k)}, X_{i+1}^{(k)}], \quad i = 0, \dots, n-1,$$

and $\rho_i^{(k)}$ is given by (5) with X_i being replaced by $X_i^{(k)}$.

The piecewise constant function $\bar{\rho}_{X^{(k)}}$ in iteration equation (9) is defined on the (k) -th step mesh $X^{(k)}$. To evaluate the integrals we need to determine an interval of the (k) -th step in which the point $X_i^{(k+1)}$ of the $(k+1)$ -th step locates. Let j index an interval $(X_j^{(k)}, X_{j+1}^{(k)})$ of the (k) -th step in which the point $X_i^{(k+1)}$ of the $(k+1)$ -th step falls, i.e., $X_i^{(k+1)} \in (X_j^{(k)}, X_{j+1}^{(k)})$. Then using (9), we get the following form:

$$\sum_{\ell=0}^{j-1} h_\ell^{(k)} \rho_\ell^{(k)} < \frac{i}{n} \sum_{\ell=0}^{n-1} h_\ell^{(k)} \rho_\ell^{(k)} \leq \sum_{\ell=0}^j h_\ell^{(k)} \rho_\ell^{(k)},$$

which can be used to determine the index j in the implementation, since all the information is known in the (k) -th step. So the iteration equation (9) explicitly determines

$$X_i^{(k+1)} = X_j^{(k)} + \frac{\frac{i}{n} \sum_{\ell=0}^{n-1} h_\ell^{(k)} \rho_\ell^{(k)} - \sum_{\ell=0}^{j-1} h_\ell^{(k)} \rho_\ell^{(k)}}{\rho_j^{(k)}}, \quad i = 1, \dots, n-1, \quad (10)$$

where $h_\ell^{(k)} \equiv X_{\ell+1}^{(k)} - X_\ell^{(k)}$.

In the implementation of Algorithm 3.1, we need to find $\rho_j^{(k)}$, $j = 0, \dots, n-1$ in expression (10). $\rho_j^{(k)}$ with $\gamma = 2/5$ can be calculated approximately by some quadrature rule, e.g., the rectangle rule:

$$\rho_j^{(k)} \approx \left(1 + \frac{G(X_{j+1}^{(k)}) \left(f''(X_{j+1}^{(k)}) \right)^2}{\left(\frac{1}{X_n - X_0} \sum_{\ell=0}^{n-1} h_\ell^{(k)} \left(G(X_{\ell+1}^{(k)}) \right)^{1/5} \left(f''(X_{\ell+1}^{(k)}) \right)^{2/5} \right)^5} \right)^{1/5},$$

with

$$G(X_{\ell+1}^{(k)}) = \hat{G}(1) = \int_0^1 \hat{g}_\ell(\xi) \frac{\xi^2(1-\xi)^3}{3} d\xi,$$

$$\hat{g}_\ell(\xi) \equiv g(X_\ell^{(k)} + h_\ell^{(k)} \xi).$$

The sequences $X_i^{(k)}$, $i = 0, 1, \dots, n$, generated by the iteration equation (9) (or equivalently from (10)) converge to X_i , $i = 0, 1, \dots, n$, generated by the equidistribution equation (6) (or the equivalent forms (7), (8)) as the iteration number $k \rightarrow +\infty$. Since the proof falls into the mathematical

framework of Xu et al. (2011), the details are omitted. We only need to show the convergence rate of the approximation to the nonlinear payoff with the equidistribution equation (6) for selecting the strikes.

Since the strikes are not uniformly distributed, the error estimation (2.1) cannot reveal the convergence rate of the approximation. In the following theorem, we derive the convergence rate with respect to the number of the strikes which are generated by the equidistribution equation (6).

Theorem 3.1 *Assume that f is continuous on $[X_0, X_n]$ and twice continuously differentiable on (X_i, X_{i+1}) , $i = 0, 1, \dots, n-1$, with finite second-order left and right directional derivatives at X_i , $i = 0, 1, \dots, n$. Then the convergence rate of the static replication of the nonlinear payoff f using the linear spline approximation (1) with the equidistribution equation (6) for selecting the strikes X_i , $i = 0, \dots, n$, is given by*

$$\sqrt{\sum_{i=0}^{n-1} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)]^2 g(S) dS} \leq Cn^{-2},$$

with C a positive constant independent of the strikes X_i , $i = 1, \dots, n-1$, and n the number of the strikes used in the replication.

Proof See Appendix C.

In the options market only a limited number of options with fixed strikes are traded. Suppose that the fixed strikes are \bar{X}_j , $j = 1, \dots, n$, in increasing order. We form a portfolio of call options at maturity to replicate the nonlinear payoff f ,

$$f(S) \approx \Pi \equiv \sum_{j=1}^n w_j (S - \bar{X}_j)^+,$$

with w_i , $i = 1, \dots, n$, chosen to minimize the approximation error

$$V(w_1, \dots, w_n) \equiv \int_0^\infty [f(S) - \Pi]^2 g(S) dS.$$

The first-order optimality conditions lead to a system of equations

$$\mathbf{Q}\mathbf{w} = \mathbf{u},$$

with $\mathbf{Q} = (q_{ij})_{i,j=1,\dots,n}$, $\mathbf{w} = (w_1, \dots, w_n)^T$, $\mathbf{u} = (u_1, \dots, u_n)^T$ and

$$q_{ij} = \int_{\max\{\bar{X}_i, \bar{X}_j\}}^\infty (S - \bar{X}_i)(S - \bar{X}_j)g(S) dS, \quad i, j = 1, \dots, n, \quad (11)$$

$$u_i = \int_{\bar{X}_i}^\infty (S - \bar{X}_i)f(S)g(S) dS, \quad i = 1, \dots, n. \quad (12)$$

In the literature, there are many discussions on this quadratic hedging approach, see, e.g., Carr and Mayo (2007), Broadie and Jain (2008), Liu (2010). In general, for complex nonlinear payoff f , there is no closed-form formula for u_i , which have to be computed numerically. The fast algorithm developed in this paper can be readily used to compute u_i . More precisely, write $\tilde{f}_i(S) \equiv (S - \bar{X}_i)^+ f(S)$. Then

$$u_i = \int_0^\infty \tilde{f}_i(S)g(S) dS = E[\tilde{f}_i(S_T)],$$

from which we can apply Algorithm 3.1 to the new nonlinear payoff function \tilde{f}_i with $X_0 \equiv \bar{X}_i$.

In many cases, q_{ij} in (11) can be computed explicitly. For the lognormal distribution, for example, (see models in Section 4.1), q_{ij} can be calculated by the following formula (see Liu (2010)):

$$q_{ij} = S_0^2 \Phi(d_0) e^{2r + \sigma^2} T - (\bar{X}_i + \bar{X}_j) S_0 \Phi(d_1) e^{rT} + \bar{X}_i \bar{X}_j \Phi(d_2),$$

with Φ the standard normal distribution function and

$$d_1 = \frac{\ln [S_0 / \max\{\bar{X}_i, \bar{X}_j\}] + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}, \quad d_0 = d_1 + \sigma\sqrt{T}.$$

4 Numerical implementations and applications

4.1 Static replication under lognormal process

Let S be a lognormal variable under a risk-neutral measure. Then $\ln S$ is a normal variable with mean $\ln S_0 + (r - \sigma^2/2)T$ and variance $\sigma^2 T$, where r is the constant risk-free interest rate, σ the constant volatility, and T the maturity. The European call and put prices can be computed explicitly by the Black-Scholes formula.

Example 4.1 A variance swap: Consider the following nonlinear payoff

$$f(S) = \frac{2}{T} \left(\frac{S - S_0}{S_0} - \ln \frac{S}{S_0} \right), \quad (13)$$

studied in Liu (2010) and Demeterfi et al. (1999). This gives a \$1 exposure for one volatility point squared.

In Table 1 we list the replication values for different maturities and volatilities. The data used are $r = 5\%$, $S_0 = 100$, $T = 0.25, 0.5, 1$ and $\sigma = 20\%, 30\%, 60\%$. When asset volatility σ increases, one has to decrease X_{min} and increase X_{max} to ensure the probabilities $P(S < X_{min})$ and $P(S > X_{max})$ remain extremely small. Since the interval $[X_{min}, X_{max}]$ becomes wider, the number of grid points n needs to be increased to keep the accuracy. We implement Algorithm 3.1 by selecting 18 strikes (i.e., 20 grid points, including two fixed boundary points) between 45 and 140 for volatility $\sigma = 20\%$, 78 strikes between 25 and 200 for volatility $\sigma = 30\%$, and 158 strikes between 15 and 300 for volatility $\sigma = 60\%$. The computational results are shown for a notional exposure of \$100 per volatility point squared. We see that the replication values using formula (3) are very close to the true values of the nonlinear payoffs.

Table 1: Replications by Algorithm 3.1 (The numbers outside the brackets are the values of replications by Algorithm 3.1 and those inside the brackets are the exact values of the nonlinear payoffs. The computational results are shown for a notional exposure of \$100 per volatility point squared.)

$T \backslash \sigma$	20%	30%	60%
0.25	4.1122 (4.0123)	8.9664 (8.9502)	35.6283 (35.6148)
0.5	4.0729 (4.0242)	8.9114 (8.9007)	35.2220 (35.2341)
1	3.9718 (4.0467)	8.7864 (8.8029)	34.2173 (34.4861)

In Table 2, we test the convergence rate of Algorithm 3.1 for the case $\sigma = 20\%$ and $T = 0.25$. By increasing the total number of strikes between 45 and 200, we calculate the total value of

the replication. We see that the replication values converge to the true value (4.0123) of the given nonlinear payoff (13) as the total number of strikes n goes to infinity. In addition, we test the convergence rate as follows. Let $\text{TRV}(n)$ denote the total replication values using n points. Assume that the convergence rate of Algorithm 3.1 is p , i.e.,

$$|\text{Error}(n)| \equiv |\text{TRV}(n) - (\text{True Value})| = O(n^{-p}), \quad (14)$$

where $O(n^{-p})$ means that there exists a positive constant C such that $O(n^{-p}) \approx Cn^{-p}$. Equation (14) gives the formula for testing the convergence rate

$$p \approx \frac{\log(|\text{Error}(n)|/|\text{Error}(2n)|)}{\log 2}. \quad (15)$$

In Table 2 we calculate the value p using formula (15) and find $p \approx 2$. So the convergence rate of Algorithm 3.1 is 2, consistent with the theoretical result of Theorem 3.1.

Table 2: Convergence rates of Algorithm 3.1 for replications (The computational results are shown for a notional exposure of \$100 per volatility point squared.)

Number of strikes (n)	Total replication values (TRV)	Error(n)	Convergence rates (p)
20	4.1651	0.1528	–
40	4.0484	0.0361	2.1
80	4.0211	0.0088	2.0
160	4.0145	0.0022	2.0
320	4.0128	0.0005	2.1
640	4.0124	0.0001	2.3

In Table 3 we test for the static quadratic replication algorithm for nonlinear payoff (13). A set of strikes {50, 70, 90, 100, 110, 130} are used for the replications². The optimal weights and the replication values are computed by the static quadratic replication algorithms in Section 3. The numerical results in Table 3 show that the total replication value is 4.0224, close to the true value 4.0123.

Table 3: Numerical results for the static quadratic replication algorithms (The computational results are shown for a notional exposure of \$100 per volatility point squared.)

Strikes	Weight	Value per option	Cost today
50	1.7393	50.6211	88.0450
70	–3.3196	30.8698	–102.4741
90	1.2107	11.6701	14.1288
100	0.7073	4.6150	3.2642
110	0.8639	1.1911	1.0290
130	1.2978	0.0228	0.0296
Total			4.0224

Carr and Chou (1997), Carr et al. (1998), Takahashi and Yamazaki (2009a,b), Carr and Wu (2014) use a spanning relation in the static hedging. At maturity time T , for any twice continuously

²In real applications, the strikes can be picked up from the traded option market.

differentiable f , performing a Taylor expansion at point X_{min} gives a spanning relation:

$$f(S) = f(X_{min}) + f'(X_{min})(S - X_{min}) + \int_0^{X_{min}} f''(\mathcal{K})(\mathcal{K} - S)^+ d\mathcal{K} + \int_{X_{min}}^{\infty} f''(\mathcal{K})(S - \mathcal{K})^+ d\mathcal{K}.$$

Choose a very small X_{min} and large X_{max} so that the probabilities $P(S < X_{min})$ and $P(S > X_{max})$ are extremely small. Then the above spanning relation gives an approximation

$$f(S) \approx f(X_{min}) + f'(X_{min})(S - X_{min}) + \int_{X_{min}}^{X_{max}} f''(\mathcal{K})(S - \mathcal{K})^+ d\mathcal{K}.$$

To implement the hedging in practice one needs to discretize the integral and may use the following Gauss-Legendre quadrature:

$$\int_a^b f''(\mathcal{K})(S - \mathcal{K})^+ d\mathcal{K} \approx \sum_{i=1}^n W_i f''(\mathcal{K}_i)(S - \mathcal{K}_i)^+, \quad (16)$$

with

$$W_i = \frac{b-a}{2} w_i, \quad \mathcal{K}_i = \frac{b-a}{2} z_i + \frac{a+b}{2}, \quad i = 1, \dots, n,$$

where the Gauss points z_i , $i = 1, \dots, n$, are the roots of the n -th order normalized Legendre polynomials $P_n(x)$ and the weights w_i , $i = 1, \dots, n$, are given by the formula

$$w_i = \frac{2}{(1 - z_i)^2 [P'_n(z_i)]^2},$$

see Abramowitz and Stegun (1965) for the lists of Gauss points z_i and weights w_i . Writing the integral as

$$\int_{X_{min}}^{X_{max}} f''(\mathcal{K})(S - \mathcal{K})^+ d\mathcal{K} = \sum_{j=0}^{m-1} \int_{\mathcal{C}_j}^{\mathcal{C}_{j+1}} f''(\mathcal{K})(S - \mathcal{K})^+ d\mathcal{K},$$

with

$$\mathcal{C}_j = X_{min} + \frac{X_{max} - X_{min}}{m} j, \quad j = 0, \dots, m,$$

we may then apply the Gauss-Legendre quadrature (16) to each sub-integral $\int_{\mathcal{C}_j}^{\mathcal{C}_{j+1}} f''(\mathcal{K})(S - \mathcal{K})^+ d\mathcal{K}$ with $a \equiv \mathcal{C}_j$ and $b = \mathcal{C}_{j+1}$. We name the above described approach the *spanning relation method*.

Taking the variance swap as an example, with parameters $r = 5\%$, $S_0 = 100$, $T = 0.25$, $\sigma = 20\%$, we compare the hedging performance of the equidistribution equation method with that of the spanning relation method. The hedging error is defined as $\hat{f}(S) - f(S)$, where $\hat{f}(S)$ is obtained by either method for the static hedging portfolio setup at time 0. The statistics performance of the hedging error is calculated by Monte-Carlo methods. For the spanning relation method, we take $X_{min} = 45$ and $X_{max} = 140$ and use five-point Gauss-Legendre quadrature for each sub-integral with the number of sub-integrals $m = 2, 4, 8, 16$ (i.e., the number of strikes are 10, 20, 40, 80). The numerical results are summarized in Table 4. The exact value for the variance swap is 4.0123.

For the equidistribution equation method the hedging errors (mean, deviation, maximum, minimum) decrease to zero as the number of grid points increases. The minimum hedging errors are always positive, which means our static hedging portfolio is actually a super-hedging portfolio. This is reasonable as the payoff for the variance swap is a convex function and the linear spline

approximation is above the payoff function. The static hedging portfolio value at time 0 is greater than the exact value and the excess may be interpreted as the premium for superhedging.

For the spanning relation method the hedging errors (mean, deviation, maximum) also tend to zero as the number of grid points increases, although at slower rates. In contrast to our method, the minimum hedging errors are always negative and do not tend to zero as the number of grid points increases, which means the static hedging portfolio has not fully hedged the variance swap and the shortfall (-7.6547) is significant under some scenarios, even though the number of grid points (80) is large and the initial static hedging portfolio value (4.0262) is higher than the exact value (4.0123). For this numerical example it is clear that the equidistribution equation method outperforms the spanning relation method.

Table 4: Simulated statistics performance of hedging error (The computational results are shown for a notional exposure of \$100 per volatility point squared.)

Static hedging	Equidistribution equation method				Spanning relation method			
	20	40	80	160	10	20	40	80
Grid points	20	40	80	160	10	20	40	80
Mean	0.1542	0.0365	0.0089	0.0022	-0.0026	-0.0004	-0.0003	-0.0003
Std deviation	0.1015	0.0232	0.0057	0.0014	0.3406	0.1093	0.0353	0.0322
Maximum	3.0594	0.7589	0.1591	0.0432	0.9884	0.2446	0.0683	0.0180
Minimum	1.1e-05	7.4e-07	9.8e-08	1.1e-08	-4.0413	-6.9229	-4.1317	-7.6547
Skewness	3.7674	3.6587	3.7182	3.5360	-0.7892	-5.8809	-36.54	-168.61
Kurtosis	45.3871	56.4423	50.5866	48.0058	3.7934	277.59	3383	35419
Option value	4.1651	4.0484	4.0211	4.0145	3.9936	4.0097	3.9866	4.0262

In Figure 1 (left), we plot the variance swap payoff function and its linear spline approximation by the equidistribution equation method (with $T = 0.25$, $\sigma = 20\%$, $S_0 = 100$, $r = 5\%$, $K = 0.01$, and 20 strikes.). As seen from the figure, the variance swap payoff is well approximated by the linear splines with the selected strikes. In Figure 1 (right), we plot the adaptation function (5) used in the equidistribution equation method for selecting the strikes and the area enclosed by the adaptation function curve. It can be shown by the figure that the equidistribution equation (6) reads as equidistributing the total area enclosed by the adaptation function curve.

Example 4.2 A swaption: Consider the following nonlinear payoff

$$f^c(S) = \left(\frac{2}{T} \left(\frac{S - S_0}{S_0} - \ln \frac{S}{S_0} \right) - K \right)^+. \quad (17)$$

We compute the replication value of nonlinear payoff (17) using Algorithm 3.1. It is more convenient to replicate the put swaption with

$$f^p(S) = \left(K - \frac{2}{T} \left(\frac{S - S_0}{S_0} - \ln \frac{S}{S_0} \right) \right)^+, \quad (18)$$

and then the call swaption can be computed easily by put-call parity. Let

$$h(S) = K - \frac{2}{T} \left(\frac{S - S_0}{S_0} - \ln \frac{S}{S_0} \right), \quad S > 0.$$

A simple check shows that h is strictly concave, has the maximum value at $S = S_0$ and has only two solutions S_L and S_R ($S_L < S_R$) to the nonlinear equation $h(S) = 0$.

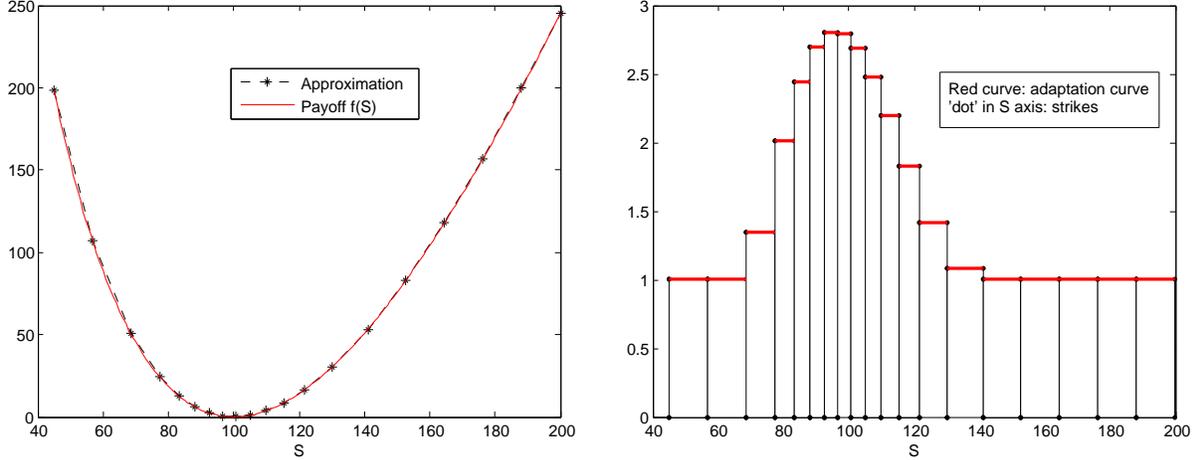


Figure 1: The left figure is for variance swap payoff function and its linear spline approximation. The marks ‘*’ on the approximation curves indicate the positions of the strikes obtained by the equidistribution equation method. The right figure is for the adaptation function and the area enclosed by the adaptation function curves.

The strikes used in the replication can be selected between S_L and S_R . The values of S_L and S_R are calculated by Newton’s method:

$$S^{(k)} = S^{(k-1)} - \frac{h(S^{(k-1)})}{h'(S^{(k-1)})}, \quad k = 1, \dots \quad (19)$$

with the initial point $S^{(0)}$ to be chosen sufficiently small (and large) such that $h(S^{(0)}) < 0$. Then Newton’s iteration (19) converges to S_L (and S_R) quadratically.

In Table 5 we list the replication values of the call swaption for different maturities and volatilities with $S_0 = 100$, $r = 5\%$ and $K = 0.01$. Using Newton’s iteration (19), we obtain the values $S_L \approx 95.0840$; $S_R \approx 105.0827$ for $T = 0.25$, $S_L \approx 93.0956$; $S_R \approx 107.2377$ for $T = 0.5$, $S_L \approx 90.3315$; $S_R \approx 110.3351$ for $T = 1$. 18 strikes between S_L and S_R are selected using Algorithm 3.1.

Table 5: Replications by Algorithm 3.1 for swaption (17) (The numbers outside the brackets are the values of replications by Algorithm 3.1 and those inside the brackets are the values of Monte-Carlo simulation. The computational results are shown for a notional exposure of \$100 per volatility point squared.)

$T \backslash \sigma$	20%	30%	60%
0.25	3.2796 (3.2791)	8.1353 (8.1469)	34.7138 (34.6813)
0.5	3.2998 (3.3019)	8.0960 (8.0907)	34.3438 (34.3599)
1	3.3389 (3.3457)	8.0180 (8.0034)	33.6168 (33.5928)

The put swaption payoff function and its linear spline approximation by the equidistribution equation method are drawn in Figure 2 (left). The figure shows that the payoff function is well approximated by the linear spline with the selected strikes.

The replication methods may also be applied to nonlinear path-dependent payoffs. Consider, for example, $f(M_T)$ with $M_T = \max_{0 \leq t \leq T} S_t$. It is known from Shreve (2004) that the pdf of M_T has an explicit form. So Algorithm 3.1 can be used and the value of $f(M_T)$ may be replicated by a portfolio of barrier options with payoff $\mathbf{1}_{M_T \geq K}$ and lookback options with payoffs $(M_T - K)^+$ or $(K - M_T)^+$. Note that this is a very special case and hedging for nonlinear path-dependent payoffs is in general much more involved; see Carr and Chou (1997), Carr et al. (1998), Carr and Wu (2014).

4.2 Static replication for a jump-diffusion model

In this section we consider a financial market model with a risky asset subject to counterparty risk: the dynamics of the risky asset is affected by the possibility of the counterparty defaulting. However, this stock still exists and can be traded after such default.

Let $W = (W_t)_{t \in [0, T]}$ be a Brownian motion with horizon $T < \infty$ on the probability space (Ω, \mathcal{G}, P) and denote by $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ the natural filtration of W . Let τ , a nonnegative and finite random variable on (Ω, \mathcal{G}, P) , represent the default time. Before τ , the filtration \mathbb{F} represents the information accessible to the investors. After τ , the investors add this new information τ to the reference filtration \mathbb{F} .

Write the risky asset price S_t as

$$S_t = S_t^{\mathbb{F}} \mathbf{1}_{t < \tau} + S_t^d(\tau) \mathbf{1}_{t \geq \tau}, \quad 0 \leq t \leq T,$$

with $S_t^{\mathbb{F}}$ \mathbb{F} -adapted and $S_t^d(\theta)$ θ -measurable and \mathbb{F} -adapted. Then we assume that the asset price follows the following dynamics under physical measure:

$$dS_t^{\mathbb{F}} = S_t^{\mathbb{F}} \left(\mu^{\mathbb{F}} dt + \sigma^{\mathbb{F}} dW_t \right), \quad 0 \leq t < \tau, \quad (20)$$

$$dS_t^d(\tau) = S_t^d(\tau) \left(\mu_t^d(\tau) dt + \sigma_t^d(\tau) dW_t \right), \quad \tau < t \leq T, \quad (21)$$

$$S_\tau^d(\tau) = S_{\tau-}^{\mathbb{F}} (1 - \gamma_{\tau}^{\mathbb{F}}). \quad (22)$$

Here for simplicity we assume that

$$\mu^{\mathbb{F}} = \mu_1, \quad \sigma^{\mathbb{F}} = \sigma_1, \quad \mu_t^d(\tau) = \mu_2, \quad \sigma_t^d(\tau) = \sigma_2, \quad \gamma_{\tau}^{\mathbb{F}} = \gamma,$$

with $\mu_1, \sigma_1, \mu_2, \sigma_2$ nonnegative constants and the distribution of γ ($\gamma \leq 1$) fixed. Moreover γ, τ, W_t are independent and τ is an exponential variable with parameter λ . For more general set-ups, see Jiao and Pham (2011).

Assume that r is a riskless interest rate. Changing measure by Girsanov's theorem, the dynamics (20)–(22) for asset price S_t under physical measure are transformed into the following form under the equivalent martingale measure:

$$dS_t^{\mathbb{F}} = S_t^{\mathbb{F}} \left((r + \lambda m) dt + \sigma_1 dW_t \right), \quad 0 \leq t < \tau, \quad (23)$$

$$dS_t^d(\tau) = S_t^d(\tau) \left(r dt + \sigma_2 dW_t \right), \quad \tau < t \leq T, \quad (24)$$

$$S_\tau^d(\tau) = S_{\tau-}^{\mathbb{F}} (1 - \gamma), \quad (25)$$

where $m = E(\gamma)$. From (23)–(25), we see that if $\gamma = 0$ then there is no jump of asset price at time τ , and this is a simple regime switching model. In practice, we may assume γ is a discrete random variable to simplify the computation, e.g., we may assume that γ takes value γ_i with probability

p_i for $i = 1, 2, 3$, where $0 < \gamma_1 \leq 1$ (loss), $\gamma_2 = 0$ (no change) and $\gamma_3 < 0$ (gain). The distribution function of random variable $S = S_T$ is given by

$$F(S) = e^{-\lambda T} \Phi \left(\frac{\ln(S/S_0) - a(T)}{b(T)} \right) + \sum_{i=1}^3 p_i \int_0^T \lambda e^{-\lambda t} \Phi \left(\frac{1}{b(t)} \left(\ln \left(\frac{S}{S_0(1-\gamma_i)} \right) - a(t) \right) \right) dt, \quad (26)$$

with $a(t) = (r + \lambda m - \sigma_1^2/2)t + (r - \sigma_2^2/2)(T - t)$, $b(t) = \sqrt{\sigma_1^2 t + \sigma_2^2 (T - t)}$ and Φ the standard normal distribution function. The proof of formula (26) is given in Appendix D.

Combining the distribution function F in (26) and the formula

$$\int_0^\infty (S - K)^+ d\Phi \left(\frac{1}{B} \left(\ln \left(\frac{S}{C} \right) - A \right) \right) = C e^{A + \frac{B^2}{2}} \Phi(x_0 + B) - K \Phi(x_0),$$

with A a constant, B, C, K positive constants and $x_0 = \frac{1}{B} (A - \ln(\frac{K}{C}))$, we can easily compute the value of a call option at time 0 with counterparty risk as

$$e^{-rT} E [(S - K)^+] = S_0 e^{-(1-m)\lambda T} \Phi(\tilde{d}_0 + b(T)) - K e^{-(r+\lambda)T} \Phi(\tilde{d}_0) + e^{-rT} \sum_{i=1}^3 p_i \int_0^T \lambda e^{-\lambda t} \left[S_0 (1 - \gamma_i) e^{a(t)+b^2(t)/2} \Phi(\tilde{d}_i(t) + b(t)) - K \Phi(\tilde{d}_i(t)) \right] dt, \quad (27)$$

with $\tilde{d}_0 = \frac{1}{b(T)} (a(T) - \ln(\frac{K}{S_0}))$ and $\tilde{d}_i(t) = \frac{1}{b(t)} (a(t) - \ln(\frac{K}{S_0(1-\gamma_i)}))$ for $i = 1, 2, 3$. The value of a put option can be computed by put-call parity:

$$e^{-rT} E [(K - S)^+] - e^{-rT} E [(S - K)^+] = K e^{-rT} - S_0 e^{-(1-m)\lambda T} - \lambda S_0 e^{-rT} \left(\int_0^T e^{a(t)+b^2(t)/2-\lambda t} dt \right) \sum_{i=1}^3 p_i (1 - \gamma_i). \quad (28)$$

Now we can use Algorithm 3.1 to replicate a variance swap with the payoff f in (13) and with the asset price S having the distribution function F in (26). In Table 6 we list the replication values and the true values for different jump sizes and probabilities. The data used are $S_0 = 100$, $T = 1$, $r = 5\%$, $\sigma_1 = 40\%$, $\sigma_2 = 20\%$, $\lambda = 0.5$; others are in Table 6. Since f is simple we have a closed-form formula for the valuation that can be used to compare the accuracy of the algorithm. We choose $X_0 = 5$ and $X_n = 400$ and $n = 80$. It is clear that the static replication values are very close to the true values.

Table 6: Numerical results for the static replication of nonlinear payoff under counterparty risk (The computational results are shown for a notional exposure of \$100 per volatility point squared.)

γ_1	γ_2	γ_3	p_1	p_2	p_3	Replication value	True value
0.5	0	-0.2	0.3	0.5	0.2	17.6584	17.6316
0.9	0	-0.2	1	0	0	118.0538	118.0215
0.9	0	-0.2	0.9	0	0.1	107.6932	107.6547

We plot the approximation curve of the nonlinear variance swap payoff function for the jump-diffusion process in Figure 2 (right), which shows that the linear splines with the selected strikes produce a good approximation.

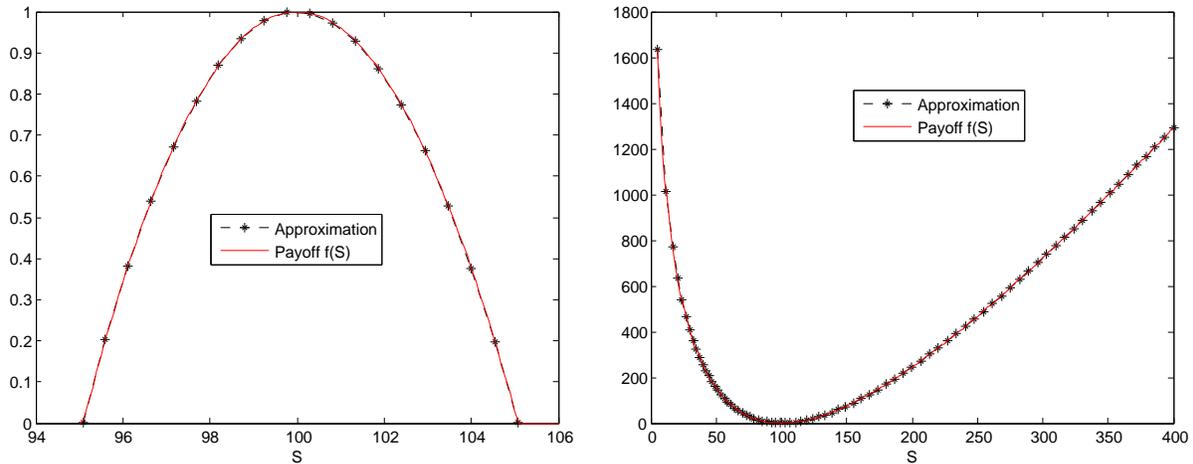


Figure 2: The left figure is for the linear spline approximation of the put swaption payoff function in Example 4.2. The right figure is for the linear spline approximation of the variance swap payoff function under jump-diffusion model. Marks ‘*’ on the curves indicate the positions of the strikes obtained by the equidistribution equation method.

5 Conclusions

In this paper we propose a new algorithm for optimally approximating the nonlinear path-independent payoff and give rigorous convergence theory. We define an equidistribution equation for selecting the strikes and construct a simple, fast and accurate iterative algorithm for implementation. In addition we perform some numerical tests, including examples of the static quadratic replication with the options traded in the market and the asset-price model with default. The results of the paper generalize and improve those of the static replication and approximation in the literature.

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A Proof of formula (2)

Mathematically we can write

$$\begin{aligned}
\sum_{i=0}^{n-1} L_i(S) \mathbf{1}_{X_i \leq S < X_{i+1}} &= L_k(X_k) - L_0(X_0) \mathbf{1}_{S < X_0} + b_0(X_0 - S) \mathbf{1}_{S < X_0} \\
&\quad - L_{n-1}(X_{n-1}) \mathbf{1}_{S \geq X_n} - b_{n-1}(S - X_{n-1}) \mathbf{1}_{S \geq X_n} \\
&\quad + \sum_{i=1}^{k-1} (b_i - b_{i-1})(X_i - S) \mathbf{1}_{S < X_i} - b_{k-1}(X_k - S) \mathbf{1}_{S < X_k} \\
&\quad + b_k(S - X_k) \mathbf{1}_{S \geq X_k} + \sum_{i=k+1}^{n-1} (b_i - b_{i-1})(S - X_i) \mathbf{1}_{S \geq X_i}. \tag{29}
\end{aligned}$$

Formula (29) is essentially proved by Liu (2010). Here we re-prove it with added details for the convenience of the reader. Since

$$\mathbf{1}_{X_i \leq S < X_{i+1}} = \mathbf{1}_{S < X_{i+1}} - \mathbf{1}_{S < X_i}, \quad \mathbf{1}_{X_i \leq S < X_{i+1}} = \mathbf{1}_{S \geq X_i} - \mathbf{1}_{S \geq X_{i+1}},$$

we can write

$$\begin{aligned}
\sum_{i=0}^{n-1} L_i(S) \mathbf{1}_{X_i \leq S < X_{i+1}} &= \sum_{i=0}^{k-1} L_i(S) \mathbf{1}_{X_i \leq S < X_{i+1}} + \sum_{i=k}^{n-1} L_i(S) \mathbf{1}_{X_i \leq S < X_{i+1}} \\
&= \sum_{i=0}^{k-1} [L_i(S) \mathbf{1}_{S < X_{i+1}} - L_i(S) \mathbf{1}_{S < X_i}] + \sum_{i=k}^{n-1} [L_i(S) \mathbf{1}_{S \geq X_i} - L_i(S) \mathbf{1}_{S \geq X_{i+1}}] \\
&= -L_0(S) \mathbf{1}_{S < X_0} + \sum_{i=1}^{k-1} [L_{i-1}(S) - L_i(S)] \mathbf{1}_{S < X_i} \\
&\quad + L_{k-1}(S) \mathbf{1}_{S < X_k} + L_k(S) \mathbf{1}_{S \geq X_k} \\
&\quad + \sum_{i=k+1}^{n-1} [L_i(S) - L_{i-1}(S)] \mathbf{1}_{S \geq X_i} - L_{n-1}(S) \mathbf{1}_{S \geq X_n}. \tag{30}
\end{aligned}$$

We may re-write $L_i(S)$ as

$$L_i(S) = L_i(X_i) + b_i(S - X_i) \quad \text{or} \quad L_i(S) = L_i(X_{i+1}) + b_i(S - X_{i+1}). \tag{31}$$

Therefore, we have

$$L_{i-1}(S) - L_i(S) = (b_i - b_{i-1})(X_i - S), \tag{32}$$

and

$$\begin{aligned}
L_{k-1}(S) \mathbf{1}_{S < X_k} + L_k(S) \mathbf{1}_{S \geq X_k} &= [L_{k-1}(X_k) + b_{k-1}(S - X_k)] \mathbf{1}_{S < X_k} \\
&\quad + [L_k(X_k) + b_k(S - X_k)] \mathbf{1}_{S \geq X_k} \\
&= L_k(X_k) [\mathbf{1}_{S < X_k} + \mathbf{1}_{S \geq X_k}] \\
&\quad + b_{k-1}(S - X_k) \mathbf{1}_{S < X_k} + b_k(S - X_k) \mathbf{1}_{S \geq X_k} \\
&= L_k(X_k) + b_{k-1}(S - X_k) \mathbf{1}_{S < X_k} + b_k(S - X_k) \mathbf{1}_{S \geq X_k}, \tag{33}
\end{aligned}$$

where in the second equality we have used the fact $L_{k-1}(X_k) = L_k(X_k)$. Combining formulas (30) to (33) leads to formula (29). Formula (2) clearly follows from (29).

B Proof of Theorem 2.1

Without the density function g involved, this is the standard error bound on the linear spline interpolation (see e.g., Atkinson and Han (2005)). Following the proof of Atkinson and Han (2005) with some additional analysis on g , we obtain the desired error bound as follows.

Transform $S \in [X_i, X_{i+1}]$ into $\xi \in [0, 1]$ by mapping $S = X_i + h_i\xi$ and denote by $\widehat{f}_i(\xi) \equiv f(X_i + h_i\xi)$. Then, from (1), we have

$$\widehat{L}_i(\xi) = L_i(X_i + h_i\xi) = \widehat{f}_i(0)(1 - \xi) + \widehat{f}_i(1)\xi, \quad \xi \in [0, 1]. \quad (34)$$

Taylor's theorem gives that

$$\widehat{f}_i(0) = \widehat{f}_i(\xi) - \xi \widehat{f}'_i(\xi) - \int_{\xi}^0 t \widehat{f}''_i(t) dt, \quad (35)$$

$$\widehat{f}_i(1) = \widehat{f}_i(\xi) + (1 - \xi) \widehat{f}'_i(\xi) + \int_{\xi}^1 (1 - t) \widehat{f}''_i(t) dt. \quad (36)$$

Using (34), (35) and (36), we have

$$\widehat{f}_i(\xi) - \widehat{L}_i(\xi) = -\xi \int_{\xi}^1 (1 - t) \widehat{f}''_i(t) dt - (1 - \xi) \int_0^{\xi} t \widehat{f}''_i(t) dt.$$

Therefore, using $(a + b)^2 \leq 2(a^2 + b^2)$ and the Cauchy-Schwartz inequality, we derive that

$$\begin{aligned} & \int_0^1 [\widehat{f}_i(\xi) - \widehat{L}_i(\xi)]^2 \widehat{g}_i(\xi) d\xi \\ & \leq 2 \int_0^1 \left(\xi^2 \left(\int_{\xi}^1 (1 - t) \widehat{f}''_i(t) dt \right)^2 + (1 - \xi)^2 \left(\int_0^{\xi} t \widehat{f}''_i(t) dt \right)^2 \right) \widehat{g}_i(\xi) d\xi \\ & \leq 2 \int_0^1 \left(\xi^2 \frac{(1 - \xi)^3}{3} \int_{\xi}^1 (\widehat{f}''_i(t))^2 dt + (1 - \xi)^2 \frac{\xi^3}{3} \int_0^{\xi} (\widehat{f}''_i(t))^2 dt \right) \widehat{g}_i(\xi) d\xi \\ & = 2 \int_0^1 \widehat{G}(t) (\widehat{f}''_i(t))^2 dt, \end{aligned} \quad (37)$$

where

$$\widehat{G}(t) \equiv \int_0^t \widehat{g}_i(\xi) \frac{\xi^2(1 - \xi)^3}{3} d\xi + \int_t^1 \widehat{g}_i(\xi) \frac{(1 - \xi)^2 \xi^3}{3} d\xi. \quad (38)$$

We can now estimate the weighted squared error

$$\begin{aligned} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)]^2 g(S) dS &= h_i \int_0^1 [\widehat{f}_i(\xi) - \widehat{L}_i(\xi)]^2 \widehat{g}_i(\xi) d\xi \\ &\leq 2h_i \int_0^1 \widehat{G}(\xi) \left[\frac{d^2 f(X_i + h_i\xi)}{d\xi^2} \right]^2 d\xi \\ &= 2h_i^4 \int_{X_i}^{X_{i+1}} G(S) (f''(S))^2 dS, \end{aligned} \quad (39)$$

where

$$G(S) \equiv \widehat{G}\left(\frac{S - X_i}{h_i}\right), \quad \text{for } S \in [X_i, X_{i+1}].$$

□

C Proof of Theorem 3.1

Following the idea of Huang (2005) using a different measure, we prove this theorem. First we prove that $\sum_{j=0}^{n-1} h_j \rho_j$ is bounded. Note that $X_0 \equiv X_{min}$ and $X_n \equiv X_{max}$ being fixed. Then it follows from Jensen's inequality and the definition (4) for α_h that

$$\begin{aligned} \sum_{j=0}^{n-1} h_j \rho_j &= \sum_{j=0}^{n-1} h_j \left(1 + \alpha_h^{-1} \left(\frac{1}{h_j} \int_{X_j}^{X_{j+1}} G(S) (f''(S))^2 dS \right) \right)^{\gamma/2} \\ &\leq \sum_{j=0}^{n-1} h_j \left(1 + \alpha_h^{-\gamma/2} \left(\frac{1}{h_j} \int_{X_j}^{X_{j+1}} G(S) (f''(S))^2 dS \right)^{\gamma/2} \right) \\ &= 2(X_n - X_0) = 2(X_{max} - X_{min}). \end{aligned} \quad (40)$$

From the error bound in Theorem 2.1, the definition (4) for α_h , and the definition (5) for ρ_i , we derive that

$$\begin{aligned} &\sqrt{\sum_{i=0}^{n-1} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)]^2 g(S) dS} \\ &\leq \sqrt{2 \sum_{i=0}^{n-1} h_i^5 \left(\alpha_h + \frac{1}{h_i} \int_{X_i}^{X_{i+1}} G(S) (f''(S))^2 dS \right)} \\ &= \sqrt{2\alpha_h \sum_{i=0}^{n-1} h_i^5 \rho_i^{2/\gamma}}. \end{aligned} \quad (41)$$

Now take $\gamma = 2/5$. Then combining (40) with (41) and using the definition of (6), we derive that

$$\begin{aligned} \sqrt{\sum_{i=0}^{n-1} \int_{X_i}^{X_{i+1}} [L_i(S) - f(S)]^2 g(S) dS} &\leq \sqrt{2\alpha_h \sum_{i=0}^{n-1} h_i \rho_i (h_i \rho_i)^4} \\ &= \sqrt{2\alpha_h \sum_{i=0}^{n-1} h_i \rho_i \left(\frac{\sum_{j=0}^{n-1} h_j \rho_j}{n} \right)^4} \\ &\leq \sqrt{2^6 \alpha_h (X_n - X_0)^5 n^{-2}} \\ &\leq C n^{-2}, \end{aligned}$$

where in the last inequality, we have used the fact that α_h is uniformly bounded. This fact can be proved as follows. From the definition of $G(S)$ (see Theorem 2.1), we can see that $G(S)$ has a uniform upper bound in $[X_0, X_n]$ with $X_0 \equiv X_{min}$ and $X_n \equiv X_{max}$ being fixed. Therefore, we can derive that

$$\alpha_h \leq \left[\frac{C}{X_n - X_0} \right]^{2/\gamma} \left[\sum_{i=0}^{n-1} h_i \left(\frac{1}{h_i} \int_{X_i}^{X_{i+1}} (f''(S))^2 dS \right)^{\gamma/2} \right]^{2/\gamma} = \left[\frac{C}{X_{max} - X_{min}} \right]^{2/\gamma} \tilde{\alpha}_h,$$

where we denote

$$\tilde{\alpha}_h \equiv \left[\sum_{i=0}^{n-1} h_i \left(\frac{1}{h_i} \int_{X_i}^{X_{i+1}} (f''(S))^2 dS \right)^{\gamma/2} \right]^{2/\gamma}.$$

Note that $\tilde{\alpha}_h$ can be bounded by the following lower and upper Riemann sums:

$$\left[\sum_{i=0}^{n-1} h_i \min_{S \in [X_i, X_{i+1}]} (f''(S))^\gamma \right]^{2/\gamma} \leq \tilde{\alpha}_h \leq \left[\sum_{i=0}^{n-1} h_i \max_{S \in [X_i, X_{i+1}]} (f''(S))^\gamma \right]^{2/\gamma}.$$

Then, as $n \rightarrow +\infty$, $\tilde{\alpha}_h$ converges to its continuous form

$$\tilde{\alpha}_h \rightarrow \left[\int_{X_0}^{X_n} (f''(S))^\gamma dS \right]^{2/\gamma} = \left[\int_{X_{min}}^{X_{max}} (f''(S))^\gamma dS \right]^{2/\gamma}.$$

Therefore, α_h is uniformly bounded. □

D Proof of formula (26)

The solutions to (23)–(25) are given by

$$S_t^{\mathbb{F}} = S_0 e^{(r + \lambda m - \sigma_1^2/2)t + \sigma_1 W_t}, \quad 0 \leq t < \tau, \quad (42)$$

$$S_t^d(\tau) = S_\tau^d(\tau) e^{(r - \sigma_2^2/2)(t - \tau) + \sigma_2(W_t - W_\tau)}, \quad \tau < t \leq T, \quad (43)$$

$$S_\tau^d(\tau) = S_{\tau-}^{\mathbb{F}}(1 - \gamma). \quad (44)$$

We now compute the distribution of S_T .

$$\begin{aligned} F(S) &= P(S_T \leq S) = E[\mathbf{1}_{S_T \leq S}] \\ &= E[\mathbf{1}_{S_T \leq S} \mathbf{1}_{\tau \geq T}] + E[\mathbf{1}_{S_T \leq S} \mathbf{1}_{\tau < T}] \\ &\equiv A + B. \end{aligned} \quad (45)$$

A and B can be computed as follows:

$$\begin{aligned} A &= E[E[\mathbf{1}_{S_T \leq S} \mathbf{1}_{\tau \geq T} | \tau]] \\ &= \int_T^\infty \lambda e^{-\lambda t} E[\mathbf{1}_{S_T^{\mathbb{F}} \leq S} | \tau = t] dt \\ &= \int_T^\infty \lambda e^{-\lambda t} P(S_T^{\mathbb{F}} \leq S) dt \\ &= e^{-\lambda T} \Phi\left(\frac{\ln(S/S_0) - (r + \lambda m - \sigma_1^2/2)T}{\sigma_1 \sqrt{T}}\right) \end{aligned} \quad (46)$$

and

$$\begin{aligned} B &= E[E[\mathbf{1}_{S_T \leq S} \mathbf{1}_{\tau < T} | \tau]] \\ &= \int_0^T \lambda e^{-\lambda t} E[\mathbf{1}_{S_T^d(\tau) \leq S} | \tau = t] dt \\ &= \int_0^T \lambda e^{-\lambda t} E\left[P(S_T^d(t) \leq S | \gamma)\right] dt \\ &= \int_0^T \lambda e^{-\lambda t} E\left[\Phi\left(\frac{1}{b(t)} \left(\ln\left(\frac{S}{S_0(1-\gamma)}\right) - a(t)\right)\right)\right] dt \\ &= \sum_{i=1}^3 p_i \int_0^T \lambda e^{-\lambda t} \Phi\left(\frac{1}{b(t)} \left(\ln\left(\frac{S}{S_0(1-\gamma_i)}\right) - a(t)\right)\right) dt, \end{aligned} \quad (47)$$

where $a(t) \equiv (r + \lambda m - \sigma_1^2/2)t + (r - \sigma_2^2/2)(T - t)$ and $b(t) = \sqrt{\sigma_1^2 t + \sigma_2^2 (T - t)}$. We have used (42)–(44) in computing $P(S_T^d(t) \leq S | \gamma)$. □