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On Modeling Economic Default Time: A Reduced-Form Model Approach

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Abstract In the aftermath of the global financial crisis, much attention has been paid to investigating the appropriateness of the current practice of default risk modeling in banking, finance and insurance industries. A recent empirical study by Guo et al. (Rev Deriv Res 11(3): 171–204, 2008) shows that the time difference between the economic and recorded default dates has a significant impact on recovery rate estimates. Guo et al. (http://arxiv.org/abs/1012.0843, 2011) develop a theoretical structural firm asset value model for a firm default process that embeds the distinction of these two default times. In this paper, we assume the market participants cannot observe the firm asset value directly and we develop reduced-form models for characterizing the economic and recorded default times. We derive the probability distributions of these two default times. Numerical experiments with empirical data are given to demonstrate the proposed models. Our approach helps researchers to gain a new perspective for economic and recorded defaults and is more feasible in general practice compared with current

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method. Our results can also contribute to the understanding of the impacts of various parameters on the economic and recorded default times.

Keywords Economic default time · Reduced-form model · Affine jump diffusion model

1 Introduction

Modeling default risk has long been an important problem in both theory and practice of banking and finance. Popular credit risk models currently used have their origins in two major classes of models. The first class of models was pioneered by Black and Scholes (1973) and Merton (1974) and is called the structural firm value model. The basic idea of the model is to describe explicitly the relationship between the asset value and the default of a firm. More specifically, the default of the firm is triggered by the event that the asset value of the firm falls below a certain threshold level related to the liabilities of the firm. The structural firm value model provides the theoretical basis for the commercial KMV model which has been widely used for default risk model in the financial industry. The second class of models was developed by Jarrow and Turnbull (1995) and Madan and Unal (1998) and is called the reduced-form credit risk model. The basic idea of the model is to consider defaults as exogenous events and to model their occurrences by using Poisson processes and their variants. Other models such as Hidden Markov Models (HMMs) have also been employed for modeling default data Ching et al. (2009).

A recent empirical study by Guo et al. (2008) on the time-series behavior of market debt prices around the recorded default date reveals the fact that the market anticipates the default event well before default is recorded. Their statistical analysis shows that the time span between the economic and recorded default dates has a significant impact on recovery rate estimates. Guo et al. (2011) develop a theoretical structural firm asset value model for a firm default process that embeds a distinction between an economic and a recorded default time and study the probability distributions of the economic and recorded default times.

In this paper, to be more consistent with the market practice, we assume that the market participants cannot observe the firm asset value directly, instead, they are aware of the firm's operation state. For example, the firm may have two states: normal and default (under stress). The firm's state process is characterized by a continuous-time Markov chain with stochastic transition rates. By this assumption, our proposed model, different from the one proposed by Guo et al. (2011), is a "reduced-form" model. Under this framework, the economic and recorded default time is defined in a similar way as the one in Guo et al. (2011). We derive the probability law of the economic and recorded default time. Numerical experiments with empirical data are given to demonstrate the proposed models.

Our work contributes to the literature in two aspects. Firstly, we introduce the reduced-form model to study the distribution of economic and recorded default times. The current method is based on structural value model and requires full information on the asset value of the firm, which is not consistent with the market practice.

Our approach helps researchers to gain a new perspective for economic and recorded defaults and is more realistic in general. Secondly, in addition to studying the distribution of economic and recorded default times, we also investigate the calibration and implementation of our reduced-form model. Our results can contribute to the understanding of the impacts of various parameters on the economic and recorded default times.

The rest of the paper is organized as follows. Section 2 provides a review on Guo et al.'s structural firm asset value model Guo et al. (2011). Section 3 gives the construction of our proposed reduced-form model. Section 4 presents the main results of this paper concerning the distribution of economic and recorded default time. Section 5 provides the numerical illustrations on the computation of economic and recorded default time distribution. Section 6 then concludes the paper.

2 Literature Review

Guo et al. (2008) show that identifying the "economic" default date, as distinct from the recorded default date, is crucial for obtaining unbiased recovery estimates. For most debt issues, the economic default date occurs far in advance of the reported default date. An implication is that the standard industry practice of using 30-day post default prices to compute recovery rate yields biased estimates. This result, unfortunately, reveals that the empirical studies investigating the economic characteristics of industry based recovery rates are using biased data. Hence, the study of the economic default date is essential.

To be more specific, Guo et al. (2008) proposed a recovery rate model which fits the stressed bond prices well with an average pricing error of less than one basis point. In their model, the "modified recovery rate" process is defined to price the stressed bonds as follows:

$$R_t = \delta_t e^{-\int_{\tau_e}^{t} r_u du}, \quad t > \tau_e$$

where δ_t denotes the recovery rate process and τ_e is the economic default time. We remark that R_t implicitly depends on the economic default time.

In Guo et al.'s model (Guo et al. 2011), for a given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ that satisfies the usual conditions, the value of the firm $S = (S_t)_{t \geq 0}$ follows a geometric Lévy process together with its natural filtration $\mathcal{F}_t$. The firm needs to make debt repayments at a predetermined (deterministic) set of discrete times, denoted by $N_1, N_2, \ldots,$. For simplicity, let $N_k = kN$ for a fixed $N > 0$, at time N_k, the amount of debt in the firm is D_k. For simplicity, we assume that $D_k = D$ is constant over time. To be consistent with a structural model, the recorded default time τ_r is the first time that the firm is unable to make a debt repayment, i.e.,

$$\tau_r = \inf\{N_k : S_k \leq D\}$$

while economic default time to be the last time, before the onset of recorded default, when the firm is able to make a debt repayment, i.e.,

$$\tau_e = \sup\{t \in [\tau_r - N, \tau_r] : S_t \geq D\}.$$

The following proposition characterizes the distribution of the important quantity $(\tau_r - \tau_e)$, the time lap between the recorded default time and the economic default time.

Proposition 1 (Guo et al. (2011)) *Assume that $S = (S_t, t \geq 0)$ is a geometric spectrally positive Lévy process, then*

$$P_x(\tau_r - \tau_e \in ds) = \int_D^\infty \sum_{n=1}^\infty \psi(u,s) u_n(x) P_x(S_{(n-1)N} \in du \mid \tau_r = nN)$$

where $u_n(x) = P_x(\tau_r = nN)$ and

$$\psi(x,s) = \int_0^N P_{(u,D)}(\tau_r - \tau_e \in ds \mid \tau_r = N) P_x(H_D \in du)$$

where $H_D = \inf\{t : S_t \leq D\}$ and $P_{(u,D)}$ denotes the distribution of S starting from D at time $t = u$.

Suppose $(S_t, t \geq 0)$ is a geometric Brownian motion with zero drift, i.e.,

$$S_t = \exp\left(\mu W_t - \frac{\mu^2 t}{2}\right)$$

where μ is the constant volatility. In this case, S_t is an exponential martingale. Under the risk neutral measure with W_t being a standard Brownian motion, then we have

$$P_{(u,D)}(\tau_r - \tau_e \in ds \mid \tau_r = N) = \frac{ds}{\pi\sqrt{s(N-u-s)}} \phi\left(\frac{\mu}{2}\sqrt{N-u-s}\right),$$

with

$$\phi(a) = \int_0^\infty dt e^{-t} \cosh(a\sqrt{2t}).$$

Therefore, the distribution of $(\tau_r - \tau_e)$ is a mixture of Arcsine law. The empirical study in Guo et al. (2008), shows that the density of time difference between the economic and the recorded default has a "U-shape" in the time interval $[0, N]$, while this feature can be well captured by the Arcsine law.

3 The Reduced-Form Models

We present our proposed reduced-form model in this section. The distinction of the economic and recorded default time is also embeded. We begin with a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Under this probability space, we are given a stochastic process $(X_t)_{t\geq 0}$, right-continuous with left limits, representing the macroeconomic environment common factor. We consider a firm with K states, i.e., $1, 2, \ldots, K$, where state K represents the default state. Let stochastic process $(S_t)_{t\geq 0}$ denotes the state process of the given firm and we assume that $(S_t)_{t\geq 0}$ is a continuous-time Markov chain with stochastic transition rates, i.e., $\lambda_{i,j}(X_s)$, where each $\lambda_{i,j}$ is a bounded continuous function defined on $\mathbf{R}$. Heuristically, one can think of, $\lambda_{i,j}(X_s)\Delta t$ as the probability that a firm in state i will jump to state j within the (small) time interval Δt. With these notations, the transition rate depends on the stochastic process $(X_s)_{s\geq 0}$ characterizing the common factor. Let

$$\lambda_i(X_s) = \sum_{k\neq i} \lambda_{i,k}(X_s), \quad i = 1, 2, \ldots, K.$$

Here $\lambda_i(X_s)\Delta t$ is the probability that a firm in state i will jump to different states within the (small) time interval Δt.

We redefine the economic and recored default time under the given framework. First, we assume that the firm has to make certain required payment at some fixed time, i.e., $0 = N_0, N_1, \ldots, N_i, \ldots$. For simplicity, we assume that the $N_i = iN$. If the firm is in the “default” state at the payment date, its payment will be missed. The recorded default time τ_r is defined to be $\tau_r = \inf\{N_i : S_{N_i} = K\}$ while the economic default time is defined to be $\tau_e = \sup\{t \leq \tau_r : S_t \neq K\}$. The information set available to the market participants up to time t is then given by $\mathcal{F}_t = \sigma(X_s, S_s, 0 \leq s \leq t)$. For the ease of discussion, we also define $\mathcal{G}_t = \sigma(X_s : 0 \leq s \leq t)$.

4 The Distribution of the Economic Default Time τ_e

In this section, we focus on finding the distributions of τ_r and τ_e. There are two cases to be discussed: constant transition rates and stochastic transition rates. We begin with the following proposition which gives the probability law of the two random variables.

Proposition 2 *For a non-negative integer i, we have*

$$\begin{aligned} &P(\tau_e \in (N_i, N_i + t] \mid \mathcal{G}_\infty) \\ &= \left(\prod_{j=0}^{i-1} P_X^{**}(N_j, N_{j+1}) \cdot P_X^{*}(N_i, N_i + t) \right)_{S_0, K} \exp\left\{ -\int_{N_i+t}^{N_{i+1}} \lambda_K(X_u) du \right\} \end{aligned} \tag{1}$$

and

$$P(\tau_r = N_{i+1} \mid \mathcal{G}_\infty) = \left(\prod_{j=0}^{i-1} P_X^{**}(N_j, N_{j+1}) \cdot P_X^{*}(N_i, N_{i+1}) \right)_{S_0, K} \tag{2}$$

and

$$P(\tau_r - \tau_e > t \mid \mathcal{G}_\infty) = \sum_{i=0}^{\infty} \left(\prod_{j=0}^{i-1} P_X^{**}(N_j, N_{j+1}) \cdot P_X^{*}(N_i, N_{i+1} - t) \right)_{S_0, K} \exp\left\{ - \int_{N_{i+1}-t}^{N_{i+1}} \lambda_K(X_u) du \right\} \tag{3}$$

where conditioning on the underlying process $(X_t)_{t\geq 0}$, $P_X(s,t)$ denotes the transition probability matrix of the state process $(S_t)_{t\geq 0}$, i.e., the (i, j) entry of $P_X(s,t)$ denotes the probability that the firm stays in state j at time t given that the firm stays in state i at time s. $P_X^*(s,t)$ is the $(K-1)\times K$ matrix that results from deleting the Kth row of $P_X(s,t)$ and $P_X^{**}(s,t)$ is the $(K-1)\times(K-1)$ matrix that results from deleting the Kth column and Kth row of $P_X(s,t)$.

Proof See Appendix 1. □

From Proposition 1, one can see that the probability law of τ_r and τ_e depends on the transition matrix $P_X(s,t)$. In the following, we discuss the issue of calculating $P_X(s,t)$ in different cases: constant rate model and stochastic rate model.

4.1 Constant Transition Rates

In this subsection, we assume that the underlying stochastic process is degenerate, which means that $X_u = c$, $u \geq 0$ for some constant c. Let $\lambda_{i,j}(c) = \lambda_{i,j}$ and $\lambda_i(c) = \lambda_i$ for all i, j and $P_X(s,t) = P(s,t)$. Let

$$A = \begin{pmatrix} -\lambda_1 & \lambda_{1,2} & \lambda_{1,3} & \cdots & \cdots & \lambda_{1,K} \\ \lambda_{2,1} & -\lambda_2 & \lambda_{2,3} & \cdots & \cdots & \lambda_{2,K} \\ \lambda_{3,1} & \lambda_{3,2} & -\lambda_3 & \cdots & \cdots & \lambda_{3,K} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & \cdots & -\lambda_{K-1} & \lambda_{K-1,K} \\ \lambda_{K,1} & \lambda_{K,2} & \cdots & \cdots & \lambda_{K,K-1} & -\lambda_K \end{pmatrix}$$

then by Kolmogorov's backward equations, one can obtain

$$\frac{\partial P(s,t)}{\partial s} = -AP(s,t). \tag{4}$$

Solving these equations, we obtain

$$P(s,t) = \exp\left(A(t-s)\right).$$

In the following, we give an example of two states to demonstrate the model.

Example 1 In this example, we assume that the firm's state process follows a two-state continuous-time Markov chain with normal state "1" and default state "2". The transition rates are given, respectively, by λ_1 and λ_2. Therefore we have

$$A = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$$

and

$$P(s,t) = \begin{pmatrix} \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)(t-s)} + \frac{\lambda_2}{\lambda_1+\lambda_2} & -\frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)(t-s)} + \frac{\lambda_1}{\lambda_1+\lambda_2} \\ -\frac{\lambda_2}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)(t-s)} + \frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)(t-s)} + \frac{\lambda_1}{\lambda_1+\lambda_2} \end{pmatrix}.$$

By Proposition 1, one obtains

$$\begin{aligned} P(\tau_e \in (N_i, N_i + t]) &= \left(\frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)N} + \frac{\lambda_2}{\lambda_1+\lambda_2} \right)^i \\ &\quad \times \left(\frac{\lambda_1}{\lambda_1+\lambda_2} - \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t} \right) e^{-\lambda_2(N-t)} \end{aligned} \tag{5}$$

and

$$\begin{aligned} P(\tau_r = N_{i+1}) &= \left(\frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)N} + \frac{\lambda_2}{\lambda_1+\lambda_2} \right)^i \\ &\quad \times \left(-\frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)N} + \frac{\lambda_1}{\lambda_1+\lambda_2} \right) \end{aligned} \tag{6}$$

and

$$P(\tau_r - \tau_e > t) = \frac{e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)N} e^{\lambda_1 t}}{1 - e^{-(\lambda_1+\lambda_2)N}}. \tag{7}$$

4.2 Stochastic Transition Rates

We define the following matrix

$$A_X(s) = \begin{pmatrix} -\lambda_1(X_s) & \lambda_{1,2}(X_s) & \lambda_{1,3}(X_s) & \dots & \dots & \lambda_{1,K}(X_s) \\ \lambda_{2,1}(X_s) & -\lambda_2(X_s) & \lambda_{2,3}(X_s) & \dots & \dots & \lambda_{2,K}(X_s) \\ \lambda_{3,1}(X_s) & \lambda_{3,2}(X_s) & -\lambda_3(X_s) & \dots & \dots & \lambda_{3,K}(X_s) \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1}(X_s) & \lambda_{K-1,2}(X_s) & \dots & \dots & -\lambda_{K-1}(X_s) & \lambda_{K-1,K}(X_s) \\ \lambda_{K,1}(X_s) & \lambda_{K,2}(X_s) & \dots & \dots & \lambda_{K,K-1}(X_s) & -\lambda_K(X_s) \end{pmatrix}$$

and we obtain

$$\frac{\partial P_X(s,t)}{\partial s} = -A_X(s) P_X(s,t). \tag{8}$$

As shown in Lando (1998), in general, we have

$$P_X(s,t) \neq \exp\left[\int_s^t A_X(u)du\right].$$

Hence we adopt the special structure of $A_X(s)$ in Lando (1998) by assuming that

$$A_X(s) = B\mu(X_s)B^{-1},$$

where $\mu(X_s)$ denotes the $K \times K$ diagonal matrix : diag$(\mu_1(X_s), \ldots, \mu_{K-1}(X_s), \mu_K(X_s))$ with $\mu_K(X_s) = 0$, and B is the $K \times K$ matrix whose columns consist of K eigenvectors of $A_X(s)$. Let

$$\begin{aligned} &E_X(s,t) \\ &\quad = \mathrm{diag}\left(\exp\left[\int_s^t \mu_1(X_u)du\right], \ldots, \exp\left[\int_s^t \mu_{K-1}(X_u)du\right], \exp\left[\int_s^t \mu_K(X_u)du\right]\right). \end{aligned}$$

Then one can obtain the following lemma.

Lemma 1 *We have*

$$P_X(s,t) = BE_X(s,t)B^{-1}$$

satisfying Eq. (8) and is the desired transition probability matrix.

Proof By using the similar argument in Lando (1998). □

4.2.1 An Affine Jump Diffusion Model for $(X_s)_{s\geq 0}$

In this subsection, we adopt an affine jump diffusion process to characterize the dynamics of $(X_s)_{s\geq 0}$. As we know, the basic affine process is attractive in modeling credit risk for its tractability, see for instance Duffie and Kan (1996) and Duffie and Garleanu (2001) and Wu and Yang (2013). We assume that

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dB_t + dJ_t \tag{9}$$

where B_t is a standard Brownian motion and

$$J_t = \sum_{i=1}^{N(t)} Z_i$$

with $N(t)$ being counting jumps in Poisson with intensity λ and $\{Z_i\}_{i=1}^{\infty}$ a sequence of i.i.d. exponentials with mean γ. Then the expectation

$$E\left[e^{\int_t^T RX_u du + wX_T}\Big|\mathcal{G}_t\right] = e^{\alpha(T-t;R,w)+\beta(T-t;R,w)X_t}, \tag{10}$$

where R, w are constants and α, β are coefficient functions satisfying the ODEs

$$\begin{cases} \dfrac{d\alpha(s;R,w)}{ds} = \kappa\theta\beta(s;R,w) + \dfrac{\lambda\gamma\beta(s;R,w)}{1-\gamma\beta(s;R,w)} \\ \dfrac{d\beta(s;R,w)}{ds} = -\kappa\beta(s;R,w) + \dfrac{1}{2}\sigma^2\beta(s;R,w)^2(s) + R \end{cases}$$

with $\alpha(0;R,w) = 0$ and $\beta(0;R,w) = w$. The explicit form of $\alpha(s;R,w)$ and $\beta(s;R,w)$ can be found in Duffie and Garleanu (2001). The solution to $\beta(s;R,w)$ is given by

$$\beta(s;R,w) = \frac{1+ae^{bs}}{c+de^{bs}}$$

where the coefficients depend on R and w,

$$\begin{cases} a = (d+c)w - 1 \\ b = \dfrac{d(-\kappa+2Rc) + a(-\kappa c+\sigma^2)}{ac-d} \\ c = \dfrac{\kappa+\sqrt{\kappa^2-2R\sigma^2}}{2R} \\ d = (1-cw)\dfrac{-\kappa+\sigma^2 w+\sqrt{(-\kappa+\sigma^2 w)^2-\sigma^2}}{-2\kappa w+\sigma^2 w^2+2R} \end{cases}$$

and $\alpha(s;R,w)$ follows from solving the ODE by substituting $\beta(s;R,w)$.

In what follows, we implement the calculation of the distributions of τ_e and τ_r given the dynamics of $(X_s)_{s\geq 0}$ in Eq. (9). We assume that $\mu_i(X_s) = \mu_i X_s$ with μ_i being a constant for $i = 1, 2, \ldots, K-1$, and $\mu_K = 0$. Although the computational method works in multi-state case, here for simplicity of discussion, we assume that $K = 2$, i.e., the operation state of a firm is either "normal" or "default". Before we state the main result, we have the following observations:

$$P_X^{**}(s,t) = B^* E_X(s,t) B_*^{-1} \quad \text{and} \quad P_X^{*}(s,t) = B^* E_X(s,t) B^{-1}$$

where B^* denotes the $(K-1)\times K$ matrix that results from deleting the Kth row of B, B_*^{-1} denotes the $K\times(K-1)$ matrix that results from deleting the Kth column of B^{-1}. When $K = 2$,

$$P_X^{**}(s,t) = m_1 \exp\left[\int_s^t \mu_1(X_u)du\right] + m_2 \exp\left[\int_s^t \mu_2(X_u)du\right]$$

where $m_1 = b_{11}b_{11}^{(-1)}$ and $m_2 = b_{12}b_{21}^{(-1)}$ with $b_{i,j} = B_{i,j}$ and $b_{ij}^{(-1)} = B_{i,j}^{-1}$. We have

$$
\begin{aligned}
&P_X^*(s,t)\\
&\quad=\left(m_1\exp\left[\int_s^t\mu_1(X_u)du\right]+m_2\exp\left[\int_s^t\mu_2(X_u)du\right],n_1\exp\left[\int_s^t\mu_1(X_u)du\right]\right.\\
&\qquad\left.+n_2\exp[\int_s^t\mu_2(X_u)du]\right)
\end{aligned}
$$

where $n_1=b_{11}b_{12}^{(-1)}$ and $n_2=b_{12}b_{22}^{(-1)}$. And

$$\lambda_2(X_s)=-p_1\mu_1(X_u)-p_2\mu_2(X_u)$$

where $p_1=b_{21}b_{12}^{(-1)}$ and $p_2=b_{22}b_{22}^{(-1)}$. Let

$$\hat{E}_i:=\{\mathbf{e}=(e_0,e_1,\ldots,e_i):e_k\in\{1,2\}\}$$

and for each $\mathbf{e}\in\hat{E}_i$, we let

$$\hat{m}(\mathbf{e})=n_{e_i}\prod_{j=0}^{i-1}m_{e_j}$$

and

$$
\begin{aligned}
\hat{\mu}(\mathbf{e},s)\;=\;&1_{\{s\in[N_i+t,N_{i+1})\}}[p_1\mu_1(X_s)+p_2\mu_2(X_s)]+1_{\{s\in[N_i,N_i+t)\}}\mu_{e_i}(X_s)\\
&+\sum_{j=0}^{i-1}1_{\{s\in[N_j,N_{j+1})\}}\mu_{e_j}(X_s).
\end{aligned}
$$

Proposition 3 *If $K=2$, $\mu_i(X_s)=\mu_iX_s$ with μ_1 being a constant, $\mu_2=0$, the distribution of τ_e is given by*

$$P(\tau_e\in(N_i,N_i+t])=\sum_{\mathbf{e}\in\hat{E}_i}\hat{m}(\mathbf{e})\left(\prod_{j=0}^{i+1}v_j(\mathbf{e})\right)\exp[\beta(N;R_0(\mathbf{e}),w_0(\mathbf{e}))X_0]\tag{11}$$

where R_j, w_j, v_j are defined in Appendix 2.1. The distributions of τ_r and the difference $\tau_r-\tau_e$ are given by,

$$P(\tau_r=N_{i+1})=P(\tau_e\in(N_i,N_{i+1}])\tag{12}$$

and

$$P(\tau_r-\tau_e>t)=\sum_{i=0}^{\infty}P(\tau_e\in(N_i,N_{i+1}-t]).\tag{13}$$

Proof See Appendix 2.1. □

We remark that in conducting our numerical experiments, we have to apply Eq. (13) to approximate $P(\tau_r - \tau_e > t)$, where the error is given by

$$\left| P(\tau_r - \tau_e > t) - \sum_{i=0}^{k} P(\tau_e \in (N_i, N_{i+1} - t]) \right| < P(\tau_r > N_{k+1}) \to 0$$

as $k \to \infty$. For the ease of computing the probability $P(\tau_e \in (N_i, N_i + t])$, we establish the following.

Proposition 4 *Under the assumptions of Proposition 3, the distribution of the economic default time τ_e is given by*

$$P(\tau_e \in (N_i, N_i + t]) = \sum_{j=1}^{2^{i+1}} a_{i,j} \exp(b_{i,j} X_0), \tag{14}$$

where

$$a_{i+1,j} = \begin{cases} m_1 a_{i,j} \exp(\alpha(N, \mu_1, b_{i,j})), & j = 1, 2, \ldots, 2^{i+1} \\ m_2 a_{i,j-2^{i+1}} \exp(\alpha(N, \mu_2, b_{i,j-2^{i+1}})), & j = 2^{i+1} + 1, 2^{i+1} + 2, \ldots, 2^{i+2} \end{cases}$$
$$b_{i+1,j} = \begin{cases} \beta(N, \mu_1, b_{i,j}), & j = 1, 2, \ldots, 2^{i+1} \\ \beta(N, \mu_2, b_{i,j-2^{i+1}}), & j = 2^{i+1} + 1, 2^{i+1} + 2, \ldots, 2^{i+2} \end{cases}$$

and

$$\begin{cases} a_{0,1} = n_1 \exp[\alpha(N - t, p_1\mu_1 + p_2\mu_2, 0)\alpha(t, \mu_1, \beta(N - t, p_1\mu_1 + p_2\mu_2, 0))] \\ a_{0,2} = n_2 \exp[\alpha(N - t, p_1\mu_1 + p_2\mu_2, 0)\alpha(t, \mu_2, \beta(N - t, p_1\mu_1 + p_2\mu_2, 0))] \\ b_{0,1} = \beta(t, \mu_1, \beta(N - t, p_1\mu_1 + p_2\mu_2, 0)) \\ b_{0,2} = \beta(t, \mu_2, \beta(N - t, p_1\mu_1 + p_2\mu_2, 0)). \end{cases}$$

Proof See Appendix 2.2. □

5 Numerical Experiments and Discussions

In this section, we first discuss the constant intensity rate model. The model parameters can be solved by employing the maximum likelihood approach. We state the sufficient conditions for the density function to have the "U-shape" property. Numerical results are then given to demonstrate the model. We also present the numerical results for the stochastic intensity model. It is found by varying the parameters κ, γ and σ, different "U-shape" density functions can be obtained. Thus it is clear that the stochastic intensity rate model can better fit the real data as it includes the constant rate intensity model as its particular case.

In the stochastic intensity rate model, we note that if the mean-reverting rate κ is getting large, the effect of stochastic part will be diminished. Eventually the process will be dominated by deterministic part $dX_t = \kappa(\theta - X_t)dt$. The parameter κ characterized the internal factor of the firm default process. One expects that when κ increases, the distribution seems to converge to certain "U-shape" function and this is consistent with the results in Fig. 2.

The parameter γ, the mean jump size of the jump process J_t, is a positive quantity and can be regarded as the severity of an external event causing the stress. We remark that the sign of the jump is always positive. The larger the value is, the more likely that the time lap between the economic default time and the recorded time is short. Thus we expect that when γ increases, the distribution will have a flatter and flatter tail and this is consistent with the results in Fig. 3.

Finally, the non-negative parameter σ controls the effect of the stochastic part of a Brownian motion σdW which can be positive or negative and it represents the external market risk. We expect that when σ increases, the better capital-structured companies have larger time gap between the economic and the recorded default while worse capital-structured companies have shorter time gap between those. The impact of increasing σ on both type of companies reveals in the time difference of the two default times as in Fig. 4.

In a more economic sense, the parameter σ can be interpreted as a measure of degree the macroeconomic fluctuation or market condition. The larger σ is, the more firms are to default given their original status. As shown by Jacobson et al. (2011), strong evidence for a substantial and stable impact from aggregate fluctuations and business defaults are found in large banking crisis. Moreover, default frequencies tend to increase significantly when the economy fluctuates more. Intuitively speaking, when market conditions or macroeconomy becomes more uncertain or worse, bank or other lenders tend to be less confident and retract their lending to firms, making firms more easily to default. Another interesting facts about our model is that there is a "shift" in the distribution of firms' "default gap classes". Comparing the first and the third graph in Fig. 4, it is not hard to see that the distribution of firms' default gap tends to shift along the parabola rightwards, lifting the right tail up while pressing the left tail down. Moreover, it is obvious that the shifts from the classes with larger default gap are bigger than those from the class with smaller default gap. This interesting phenomenon can be interpreted in a very reasonable way. It is known to all that firms' capital structure and governance manner etc. are very important measure of firms' strength. In particular, these properties tend to be more variable or of larger variance in start-up firms or less matured firms. Baeka et al. (2004) and Ivashina and Scharfstein (2008) Ivashina and Scharfstein (2010) found that firms with better governance manner and capital structure are more likely to survive from defaults during crisis. Start-up firms or less-developed firms (lower class firms) systematically have larger default gaps than those larger and matured firms (higher class firms). Good candidates in the lower class, namely those firms less-matured, but with relatively better governance manner or reasonable capital structures, will have better access to funding or lending during crisis compared with their peers in the same class. We expect the good candidates in each classes that are making the shift. And the shift magnitudes are larger in the lower classes because the variance of capital structure and governance manner are larger in these lower classes.

Table 1 Time lag between the economic and recorded default dates

Day	(0, 18]	(18, 36]	(36, 54]	(54, 72]	(72, 90]
Number of Firms	24	13	6	5	3
Day	(90, 108]	(108, 126]	(126, 144]	(144, 162]	(162, 180]
Number of Firms	1	4	4	2	11

5.1 Constant Intensity

In this section, we first present some estimation method for solving the model parameters. We then compare our proposed model with the real data extracted from Guo et al. (2008). For the real data, Table 1 reports the time difference between the economic and recorded default date with $N = 180$ days, extracted from Guo et al. (2008). From the table, one can easily observe that the density function of the time difference between the economic and the recorded default time has the "U-shape"property.

Regarding our model, we assume the state process follows the two-state continuous time Markov chain as in Example 1. Indeed, from Eq. (7), we observe that the density function of the time difference between the economic and recorded default time is always convex. In fact, it can be shown easily that

Lemma 2 *The density function has the "U-shape" property as long as the following conditions are satisfied:*

(i) $0 \leq \lambda_2 - e^{-(\lambda_1+\lambda_2)N/2}\lambda_1$
(ii) $0 \leq \lambda_1 - \lambda_2$.

We remark that if N is large, then $e^{-(\lambda_1+\lambda_2)N/2} \approx 0$ and therefore essentially the sufficient condition in the above lemma will become $\lambda_2 \leq \lambda_1$. To estimate the model parameters, we adopt the Maximum Log-likelihood method to estimate the desired parameter λ_1 and λ_2 (see Appendix 3), from which we obtain the estimate of the two parameters: $\lambda_1 = 0.3631$ and $\lambda_2 = 0.0238$. We also present the density function of the time difference between the economic and recorded default time with comparison of the proposed model (Fig. 1) and the real data. We note that the two-state constant rate model has the "U-shape property". However, the restriction in the number of states and the lack of dynamics in the intensity result in a thinner tail. We investigate the stochastic intensity in the next subsection.

5.2 Stochastic Intensity

In this example, we assume that the state process of the firm $(S_t)_{t\geq 0}$ follows a two-state continuous-time Markov chain with stochastic transition rates depending on the underlying process $(X_t)_{t\geq 0}$ as described in previous section. We set the parameters as follow:

$$\mu_1 = -0.52, \quad \mu_2 = 0, \quad \theta = 1, \quad \lambda = 0.2, \quad X_0 = 1, \quad N = 180$$

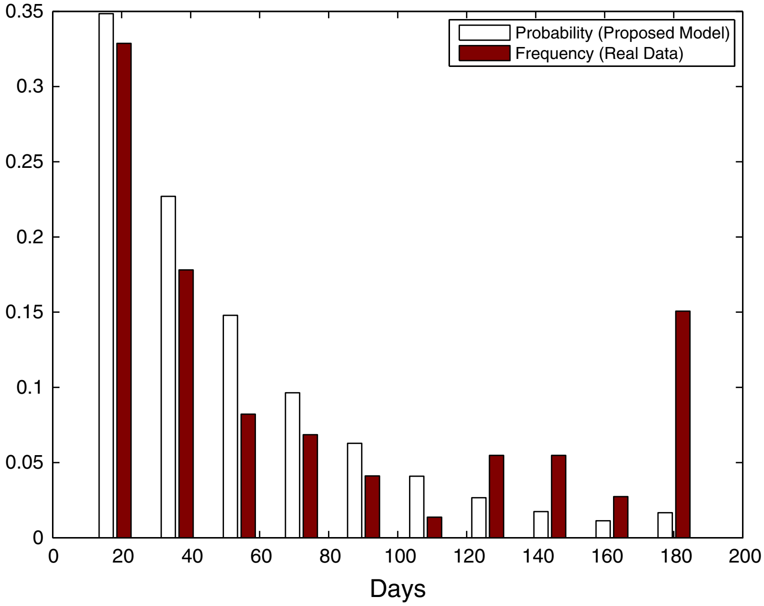

Fig. 1 A comparison of the two-state constant rate model and the real data

and

$$B = \begin{pmatrix} -0.9992 & -0.7071 \\ 0.0400 & -0.7071 \end{pmatrix}$$

We then vary the value of parameters κ, γ and σ, and compute the density function of the time between the recorded and the economic default in Figs. 2, 3 and 4. By setting parameters as above, the initial state is A_X is given by

$$A_X(0) = \begin{pmatrix} -0.5000 & 0.5000 \\ 0.0200 & -0.0200 \end{pmatrix}.$$

Figure 3 shows that as the jump size increase, which means that the common factor suffers from a larger jump, the difference of the two default time tends to decrease. We demonstrate in Fig. 4 that, as the volatility of the common factor decrease, the difference of the default times increases.

For the two-state stochastic transition rate model, again we present the distribution of time difference between economic and recorded default in Fig. 5. We assume the parameters are given by

$$\mu_1 = -0.5120, \quad \mu_2 = 0, \theta = 1, \lambda = 0.2, \kappa = 1, \sigma = 9, \gamma = 3.6, X_0 = 1, N = 180$$

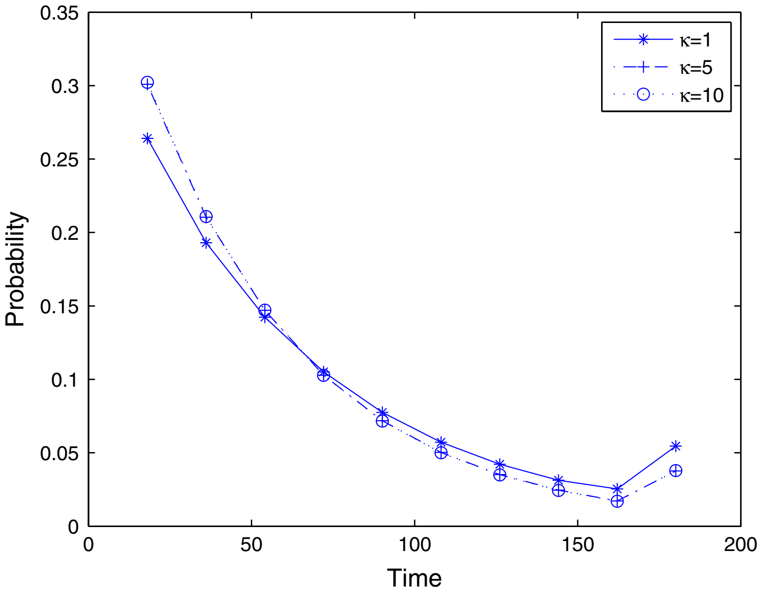

Fig. 2 Distribution of time lag between economic and recorded default with $\sigma = 5$, $\gamma = 0.1$ and different κ

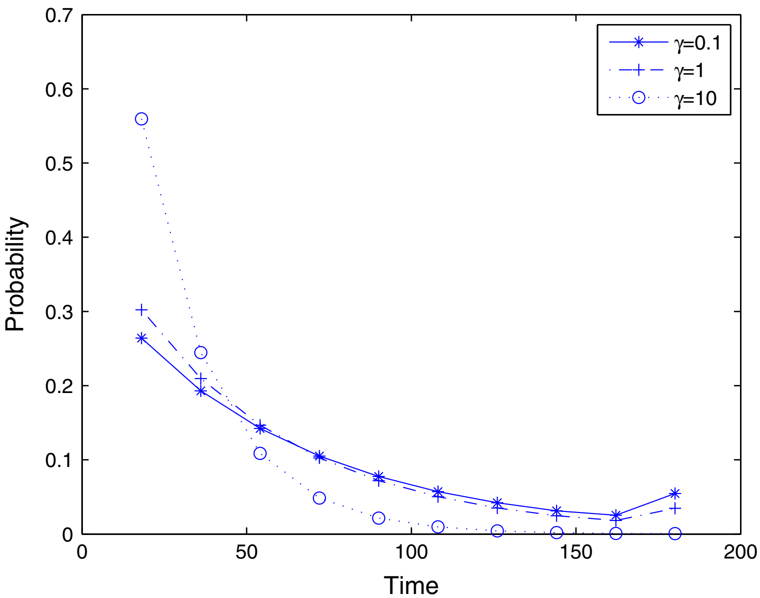

Fig. 3 Distributions of time lag between economic and recorded default with $\kappa = 1$, $\sigma = 5$ and different γ

and

$$B = \begin{pmatrix} -0.9997 & -0.7071 \\ 0.0246 & -0.7071 \end{pmatrix},$$

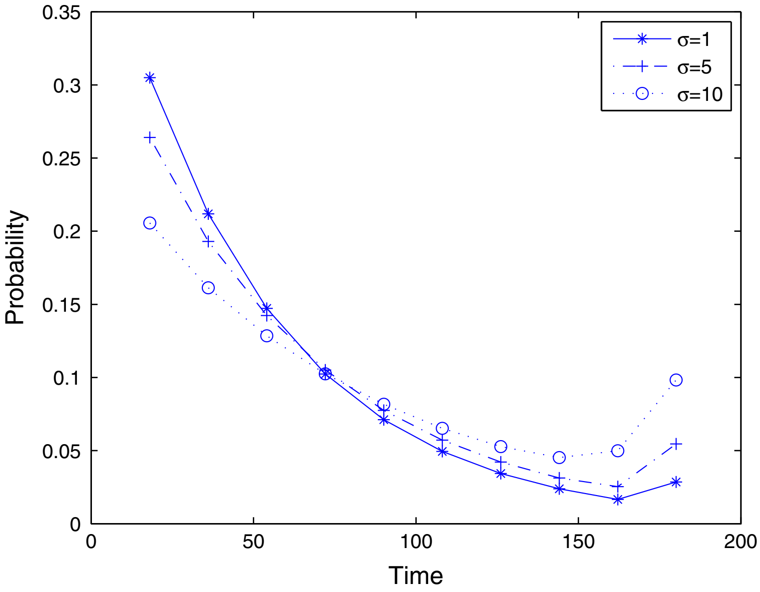

Fig. 4 Distributions of time lag between economic and recorded default with $\kappa = 1, \gamma = 0.1$ and different σ

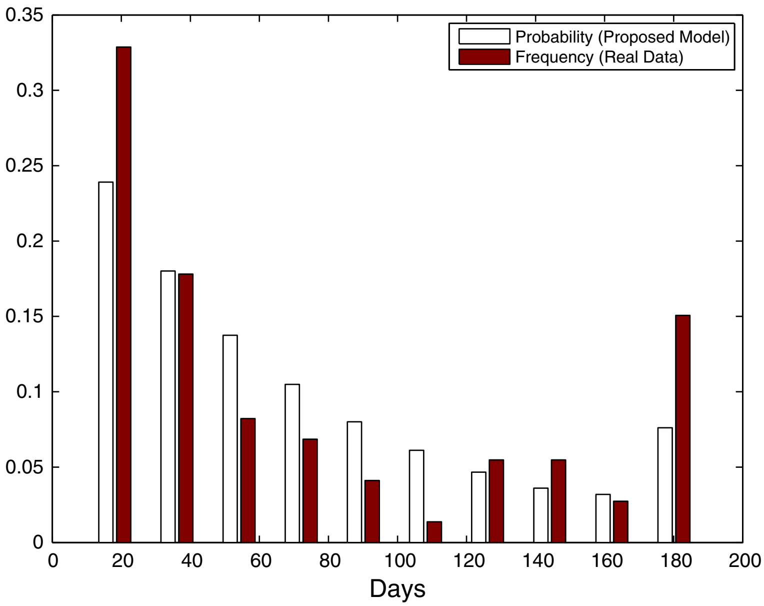

Fig. 5 A comparison of the two-state stochastic rate model and the real data

where the initial state of A_X is given by

$$A_X(0) = \begin{pmatrix} -0.5000 & 0.5000 \\ 0.0120 & -0.0120 \end{pmatrix}.$$

The above set of parameters are obtained by performing a grid search on κ, σ and γ with the object of minimizing the mean squares of errors. Therefore the two-state stochastic rate model fits the real data quite well.

6 Concluding Remarks

In this paper, we develop two reduced-form models, constant rate model and stochastic rate model, for characterizing the economic default time and the recorded default time. For the two-state constant rate model, maximum likelihood approach can be employed to estimate the model parameters easily. It can also capture the "U-shape" property and we have demonstrated this by some real data set. For the stochastic rate model, we assume the state process follows a continuous-time Markov chain with stochastic transition rates depending on the macroeconomic common factor. We derive the probability law of τ_e and τ_r which depend on the stochastic transition matrix $P_X(s,t)$. We also present the evaluation of $P_X(s,t)$ in different cases. The stochastic rate model is a generalization of the constant rate model and therefore it can better fit the real data set. We investigate the probability distribution of the economic and recorded default time with constant transition rates and also with underlying common factor following basic affine jump diffusion. Numerical experiments show that our proposed model can capture the features of empirical data. For our future research, for the constant rate model, we shall consider a multi-state constant rate model. We expect the introduction of extra states can help to improve the model and hence better fit the real data. Regarding two-state stochastic rate model, we applied grid search method to obtain the model parameters. We shall develop estimation method for the model parameters in our future research.

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Appendix

Appendix 1: Proof of Proposition 2

We note that Eq. (2) follows from Eq. (1) by using

$$P(\tau_r = N_{i+1} \mid \mathcal{G}_\infty) = P(\tau_e \in (N_i, N_{i+1}] \mid \mathcal{G}_\infty).$$

Eq. (3) follows from Eq. (1) by using

$$P(\tau_r - \tau_e > t \mid \mathcal{G}_\infty) = \sum_{i=0}^{\infty} P(\tau_e \in (N_i, N_{i+1} - t] \mid \mathcal{G}_\infty).$$

And Eq. (1) follows by

$$
\begin{aligned}
&P(\tau_e \in (N_i, N_i + t] \mid \mathcal{G}_\infty) \\
&\quad = \sum_{n_i=1}^{K-1} P(S_{N_1} \neq K, \ldots, S_{N_{i-1}} \neq K, S_{N_i} = n_i \mid \mathcal{G}_\infty) \\
&\qquad \times P(\tau_e \in (N_i, N_i + t] \mid S_{N_i} = n_i, \mathcal{G}_\infty) \\
&\quad = \sum_{n_i=1}^{K-1} \sum_{n_{i-1}=1}^{K-1} \ldots \sum_{n_1=1}^{K-1} P(S_{N_1} = n_1, \ldots, S_{N_{i-1}} = n_{i-1}, S_{N_i} = n_i \mid \mathcal{G}_\infty) \\
&\qquad \times P_X(N_i, N_i + t)_{n_i, K} \exp\left\{- \int_{N_i+t}^{N_{i+1}} \lambda_K(X_u) du\right\} \\
&\quad = \sum_{n_i=1}^{K-1} \sum_{n_{i-1}=1}^{K-1} \ldots \sum_{n_1=1}^{K-1} P_X(N_0, N_1)_{S_0, n_1} \ldots P_X(N_{i-1}, N_i)_{n_{i-1}, n_i} P_X(N_i, N_i + t)_{n_i, K} \\
&\qquad \times \exp\left\{- \int_{N_i+t}^{N_{i+1}} \lambda_K(X_u) du\right\} \\
&\quad = \left(\prod_{j=0}^{i-1} P_X^{**}(N_j, N_{j+1}) \cdot P_X^{*}(N_i, N_i + t) \right)_{S_0, K} \exp\left\{- \int_{N_i+t}^{N_{i+1}} \lambda_K(X_u) du\right\}.
\end{aligned}
$$

Appendix 2.1: Proof of Proposition 3

Proof Eq.s (12) and (13) are obvious and it suffices to show Eq. (11). Now we have

$$
\begin{aligned}
&P(\tau_e \in (N_i, N_i + t] \mid \mathcal{G}_\infty) \\
&\quad = \prod_{j=0}^{i-1} \left(m_1 \exp\left[\int_{N_j}^{N_{j+1}} \mu_1(X_u) du\right] + m_2 \exp\left[\int_{N_j}^{N_{j+1}} \mu_2(X_u) du\right]\right) \\
&\qquad \times \left(n_1 \exp\left[\int_{N_i}^{N_i+t} \mu_1(X_u) du\right] + n_2 \exp\left[\int_{N_i}^{N_i+t} \mu_2(X_u) du\right]\right) \\
&\qquad \times \exp\left[\int_{N_i+t}^{N_{i+1}} p_1\mu_1(X_u) + p_2\mu_2(X_u) du\right] \\
&\quad = \sum_{\mathbf{e} \in \hat{E}_i} \hat{m}(\mathbf{e}) \exp\left[\int_{N_0}^{N_{i+1}} \hat{\mu}(\mathbf{e}, u) du\right].
\end{aligned}
$$

Hence

$$
P(\tau_e \in (N_i, N_i + t]) = \sum_{\mathbf{e} \in \hat{E}_i} \hat{m}(\mathbf{e}) E\left(\exp\left[\int_{N_0}^{N_{i+1}} \hat{\mu}(\mathbf{e}, u) du\right]\right). \tag{15}
$$

For a fixed $\mathbf{e} \in \hat{E}_i$, let

$$
\begin{aligned}
R_{i+1}(\mathbf{e}) &= p_1\mu_1 + p_2\mu_2 \\
R_j(\mathbf{e}) &= \mu_{e_j},\ j = 0, 1, \ldots, i \\
w_{i+1}(\mathbf{e}) &= 0 \\
w_i(\mathbf{e}) &= \beta(N-t;\ R_{i+1}(\mathbf{e}), w_{i+1}(\mathbf{e})) \\
w_{i-1}(\mathbf{e}) &= \beta(t;\ R_i(\mathbf{e}), w_i(\mathbf{e})) \\
w_j(\mathbf{e}) &= \beta(N;\ R_{j+1}(\mathbf{e}), w_{j+1}(\mathbf{e})),\ j = 0, 1, \ldots, i-2 \\
v_{i+1}(\mathbf{e}) &= \exp[\alpha(N-t;\ R_{i+1}(\mathbf{e}), w_{i+1}(\mathbf{e}))] \\
v_i(\mathbf{e}) &= \exp[\alpha(t;\ R_i(\mathbf{e}), w_i(\mathbf{e}))] \\
v_j(\mathbf{e}) &= \exp[\alpha(N;\ R_j(\mathbf{e}), w_j(\mathbf{e}))],\ j = 0, 1, \ldots, i-1.
\end{aligned}
$$

Then we can rewrite $\hat{\mu}(\mathbf{e}, s)$ as

$$
\begin{aligned}
\hat{\mu}(\mathbf{e}, s) = {} & 1_{\{s\in[N_i+t, N_{i+1})\}}(R_{i+1}(\mathbf{e})X_s) + 1_{\{s\in[N_i, N_i+t)\}}(R_i(\mathbf{e})X_s) \\
& + \sum_{j=0}^{i-1} 1_{\{s\in[N_j, N_{j+1})\}}(R_j(\mathbf{e})X_s).
\end{aligned}
$$

Using the iterated expectation and Eq. (10) we obtain

$$
\begin{aligned}
& E\left(\exp[\textstyle\int_{N_0}^{N_{i+1}} \hat{\mu}(\mathbf{e}, u)du]\right) \\
& = E\left(\exp[\textstyle\int_{N_0}^{N_i+t} \hat{\mu}(\mathbf{e}, u)du] E(\exp[\int_{N_i+t}^{N_{i+1}} R_{i+1}(\mathbf{e})X_u du] \mid \mathcal{G}_{N_i+t})\right) \\
& = v_{i+1}(\mathbf{e}) E\left(\exp[\textstyle\int_{N_0}^{N_i+t} \hat{\mu}(\mathbf{e}, u)du] \exp[w_i(\mathbf{e})X_{N_i+t}]\right) \\
& = v_{i+1}(\mathbf{e}) E\left(\exp[\textstyle\int_{N_0}^{N_i} \hat{\mu}(\mathbf{e}, u)du] E(\exp[\int_{N_i}^{N_i+t} R_i(\mathbf{e})X_u du + w_i(\mathbf{e})X_{N_i+t}] \mid \mathcal{G}_{N_i})\right) \\
& = v_{i+1}(\mathbf{e}) v_i(e) E\left(\exp[\textstyle\int_{N_0}^{N_i} \hat{\mu}(\mathbf{e}, u)du] \exp[w_{i-1}(\mathbf{e})X_{N_i}]\right) \\
& = v_{i+1}(\mathbf{e}) v_i(\mathbf{e}) E\left(\exp[\textstyle\int_{N_0}^{N_{i-1}} \hat{\mu}(\mathbf{e}, u)du] E(\exp[\int_{N_{i-1}}^{N_i} R_{i-1}(\mathbf{e})X_u du + w_{i-1}(\mathbf{e}) \right. \\
& \left. X_{N_i}] \mid \mathcal{G}_{N_{i-1}})\right) \\
& = v_{i+1}(\mathbf{e}) v_i(\mathbf{e}) v_{i-1}(\mathbf{e}) E\left(\exp[\textstyle\int_{N_0}^{N_{i-1}} \hat{\mu}(\mathbf{e}, u)du] \exp[w_{i-2}(\mathbf{e})X_{N_{i-1}}]\right) \\
& = \left(\textstyle\prod_{j=0}^{i+1} v_j(\mathbf{e})\right) \exp[\beta(N;\ R_0(\mathbf{e}), w_0(\mathbf{e}))X_0] \text{ (by iteration)}
\end{aligned}
$$

Hence Eq. (11) follows. □

Appendix 2.2: Proof of Proposition 4

Proof We let

$$
H_i(X_0, t) := P(\tau_e \in (N_i, N_i + t])
$$

then by the proof of Proposition 3, for $i \geq 1$,

$$P(\tau_e \in (N_i, N_i + t] \mid \mathcal{F}_{N_1}) = \left(m_1 \exp\left[\int_{N_0}^{N_1} \mu_1(X_u)du\right] + m_2 \exp\left[\int_{N_0}^{N_1} \mu_2(X_u)du\right]\right) H_{i-1}(X_{N_1}, t)$$

$$H_i(X_0, t) = E\left[\left(m_1 \exp\left[\int_{N_0}^{N_1} \mu_1(X_u)du\right] + m_2 \exp\left[\int_{N_0}^{N_1} \mu_2(X_u)du\right]\right) H_{i-1}(X_{N_1}, t)\right] \quad (16)$$

By Proposition 3, we obtain that

$$H_0(x, t) = a_{0,1} \exp(b_{0,1}x) + a_{0,2} \exp(b_{0,2}x).$$

Combining Eqs. (16) and (10), Proposition 4 follows. □

Appendix 3

Let $\delta = 18$ days, $t_i = \delta i, i = 0, 1, \ldots, 10$. Let N_i denote the number of firms whose time difference of economic and recorded default date is inside the interval $(t_{i-1}, t_i]$. Then the log-likelihood function is given by

$$\mathcal{L}(\lambda_1, \lambda_2) = \sum_{i=1}^{10} N_i \left(\ln\left[(e^{-\lambda_2 t_{i-1}} - e^{-\lambda_2 t_i}) - e^{-(\lambda_1+\lambda_2)N}(e^{\lambda_1 t_{i-1}} - e^{\lambda_1 t_i})\right] - \ln\left[1 - e^{-(\lambda_1+\lambda_2)N}\right]\right)$$

By setting

$$\begin{cases} \dfrac{\partial \mathcal{L}(\lambda_1, \lambda_2)}{\partial \lambda_1} = 0 \\ \dfrac{\partial \mathcal{L}(\lambda_1, \lambda_2)}{\partial \lambda_2} = 0, \end{cases}$$

we have two nonlinear equations for λ_1 and λ_2. Solving these equations numerically yields $\lambda_1 = 0.3631$ and $\lambda_2 = 0.0238$.

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