Optimal investment of DC pension plan under short-selling constraints and portfolio insurance

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Abstract

In this paper we investigate an optimal investment problem under short-selling and portfolio insurance constraints faced by a defined contribution pension fund manager who is loss averse. The financial market consists of a cash bond, an indexed bond and a stock. The manager aims to maximize the expected S-shaped utility of the terminal wealth exceeding a minimum guarantee. We apply the dual control method to solve the problem and derive the representations of the optimal wealth process and trading strategies in terms of the dual controlled process and the dual value function. We also perform some numerical tests and show how the S-shaped utility, the short-selling constraints and the portfolio insurance impact the optimal terminal wealth.

Keywords: short-selling constraints, S-shaped utility, dual control, inflation risk, portfolio insurance

JEL classification: C61; G11; C20.

1 Introduction

Pension funds act as one of the most important institutions in financial markets since they help to ensure personal life after retirement. There are two major categories of pension plans: defined benefit (DB) plans and defined contribution (DC) plans. In a DB plan, the benefits at retirement are fixed in advance and the contributions from the pension plan participation are set and subsequently adjusted to keep the fund in balance. The financial risk associated with a DB plan is borne by the plan sponsor rather than the plan members. In a DC plan, only the contributions, often as a fixed percentage of salary, are defined and the employee's retirement benefits are determined by the size of the accumulation at retirement. The financial risk linked to a DC plan is shifted from the sponsor to the contributor. In recent years, DC plans have become increasingly popular in the pension market due to the demographic evolution and the development of the equity markets.

The retirement benefit of a DC plan is mainly affected by the performance of its fund portfolios before retirement. Asset allocation decisions are important for risk management of DC pension funds during the accumulation phase. The optimal investment strategies of DC pension plans with different objectives have been widely discussed in the literatures, such as the mean-variance criterion (see Sun et al. (2016), Yao et al. (2013), Wu and Zeng (2015)) and the expected utility maximization (see Boulier et al. (2001), Cairns et al. (2006), Blake et al. (2013), Blake et al. (2014), Chen and Delong (2015), Zeng et al. (2018)). As the investment time horizon for a DC pension plan is usually quite long, some studies incorporate the inflation risk into the model. For example, Zhang et al. (2007) and Zhang and Ewald (2010) investigate the optimal asset allocation with inflation risk for DC pension funds. Han and Hung (2012) discuss a continuous-time optimization model for optimal DC plan management with inflation risk under CRRA utility maximization and mean-variance criterion, respectively.

Most studies investigate utility maximization problems of DC pension funds for concave utility functions. Prospect theory, proposed by Kahneman and Tversky (1979), assumes that in behavioral

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finance, the pain associated with a loss is greater than the pleasure associated with an equivalent gain. Instead of focusing on the absolute level of wealth itself, Kahneman and Tversky (1979) propose an S-shaped utility function which is defined over gains and losses relative to a reference point and is commonly used to describe loss aversion utility. For example, Blake et al. (2013) investigate an asset allocation problem of a DC pension plan under a loss-averse preference.

As the purpose of a pension plan is to provide an adequate retirement income for the members, some research incorporates a portfolio insurance (PI) constraint into an investment problem for DC pension funds, see Boulier et al. (2001), Cairns et al. (2006) and Deelstra et al. (2003). The PI constraint can enhance the protection for policyholders or the pension members by imposing a minimum guarantee on the insurance contract. Specifically, the PI constraint requires the manager to keep the portfolio value above a minimum guarantee at retirement, which is essential for the welfare of the members. One popular form of the minimum guarantee is the minimum annual income to its retired members, see Boulier et al. (2001). Guan and Liang (2014) extend the average life expectancy to a random time with a deterministic force of mortality. When the riskless interest rate is a constant, the forms of the minimum guarantee at retirement in Boulier et al. (2001) and Guan and Liang (2014) are equivalent to a deterministic lump sum payment to the members.

The closest to our research is the paper by Chen et al. (2017) who investigate the optimal investment of a DC plan under loss aversion and PI constraints with inflation risk. However, the market in their model is complete, which allows one to adopt the martingale method to solve the portfolio choice problem. The martingale method is often used to solve optimal investment problems in a complete market setting, see Pliska (1986), Cox and Huang (1989) and Karatzas et al. (1986). One may first solve a static optimization problem to find the optimal strategy and then use the martingale representation theorem to find the optimal strategy to replicate it. However, in the presence of market imperfections such as short-selling constraints, the martingale representation theorem cannot be applied. Most of the existing literature on the optimal investment of a DC plan assumes that there is no limitation on short-selling stocks, which is often prohibited in the real world by financial regulations. Therefore, it is reasonable to incorporate short-selling constraints into the optimal allocation problems for a DC plan. Extending Chen et al. (2017), we investigate the optimal investment of a DC plan under loss aversion, short-selling and PI constraints. The considered market becomes incomplete due to short-selling prohibitions and the optimization problem cannot be solved analytically by using the martingale method.

The dual control method is effective in solving the portfolio optimization problem with control constraints since it relates the original stochastic optimal control problem to a dual problem which may be relatively easier to solve than the primal problem. Xu and Shreve (1992) use a duality method to characterize solutions for the optimal wealth and portfolio processes under prohibition of short-selling of stocks. Cvitanić and Karatzas (1992) solve the concave utility maximization problem under convex portfolio constraints, including short-selling or borrowing constraints, and so on. Bian et al. (2011) investigate a constrained portfolio choice problem with utility functions that are not necessarily differentiable or strictly concave. Although the dual control method has been widely used in the constrained portfolio choice problem, few literature focuses on its applications in insurance and actuarial science. In the present paper, we extend the dual control method to investigate the optimal portfolio choice of a DC pension plan under short-selling constraints. Our paper is different from the portfolio selection literature at least in two aspects. Firstly, most literature on the constrained portfolio choice problem focuses on maximizing the expectation of a smooth utility of terminal wealth, whereas we need to solve a non-concave utility maximization problem with control constraints, which is much more complex to deal with. Secondly, the contribution term in a DC pension plan makes the wealth process not a self-financing portfolio, which requires some modification of the procedures of our dual control method from those in the portfolio selection literature.

We apply the dual control method to solve the problem and characterize explicitly the optimal portfolio and wealth processes. Furthermore, we extend Chen et al. (2017) to an S-shaped utility function and general utilities on loss and gain segments, including power and non-HARA utilities, and give the explicit expressions for the optimal allocation strategies. Our theoretical and numerical results show that the short-selling and PI constraints can significantly improve the risk management for investors and regulators due to their preference for less volatility. We also find that the reference point and the minimum guarantee play important roles in asset allocation of a DC pension plan. Numerical tests show that the expected optimal terminal wealth has a V-shaped pattern with a minimum at a particular reference point, which is approximately equal to the difference between the accumulated value at retirement when all the wealth is invested in the cash bond and the protection level. When a reference point in the S-shaped utility moves away from that particular reference point, the pension manager takes more risk by investing more in risky assets to achieve higher expected gains. We also note that the PI constraint well protects the members' benefits by keeping the optimal terminal wealth always above the minimum guarantee. If the protection level is low, then the PI constraint can not provide a significant improvement for risk management. If the protection level is high, then the pension manager may take more prudent investment, which results in a relatively low expected terminal wealth.

The main contribution of this paper to the actuarial/insurance literature is that we extend the application of the dual control method in the portfolio selection literature to a constrained non-concave utility maximization problem of a DC pension plan involving continuous contribution inflows. We convert such an optimization problem into a classic constrained optimal portfolio choice problem and characterize explicitly the optimal wealth process and the optimal investment strategy under the assumption that the amount of money contributed to the pension fund is deterministic. Although our paper is written in terms of DC pension plan, our analysis can also, for example, be applied to the optimization problem under short-selling constraints of life insurance contracts which provide minimum guarantees or the optimal portfolio problem with life insurance purchase under short-selling constraints.

The rest of the paper is organized as follows. In Section 2, we formulate the market model of the investment problem faced by a DC plan manager. In Section 3, we use the concavification technique and the dual control method to solve the optimal investment problem with the S-shaped utility, the PI and short-selling constraints and characterize explicitly the optimal wealth process and the optimal investment strategy in terms of the dual controlled process and the dual value function. In Section 4, we perform and analyze some numerical tests. Section 5 concludes.

2 The financial market and DC pension plan

We consider the investment problem of a DC pension plan from perspective of a pension manager. Consider a continuous-time model with a finite time horizon $\mathcal{T} = [0, T]$. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space where the filtration $\mathbb{F} := \{\mathcal{F}_t | t \in \mathcal{T}\}$ satisfies the usual conditions. Assume that all random variables and stochastic processes in this paper are well defined in this probability space. The pension fund starts at time 0 and the retirement time is T.

Let $W(t) := (W_1(t), W_2(t))^\top$ be a two-dimensional, standard Brownian motion, where $W_1(t)$ and $W_2(t)$ are independent and $a^\top$ is the transpose of a. We consider a continuous-time financial market with inflation risk, which is often represented by CPI (Consumer Price Index) and may be regarded as a price level process. Following Brennan and Xia (2002), Han and Hung (2012) and Chen et al. (2017), we use a price level process, $P(t)$, to capture the inflation risk, and assume that

$P(t)$ satisfies the following stochastic differential equation:

$$\frac{dP(t)}{P(t)} = idt + \sigma_I dW_1(t), P(0) = p_0 > 0,$$

where $i > 0$ is the expected rate of inflation and $\sigma_I > 0$ is the volatility of the price level.

Assume that the financial market consists of three traded assets: a riskless cash bond, a risky stock and an indexed bond which has the same risk source as the price level process.

The indexed bond $I(t)$ offers a constant rate of real return r_I and its price process follows

$$\frac{dI(t)}{I(t)} = r_I dt + \frac{dP(t)}{P(t)} = (r_I + i)dt + \sigma_I dW_1(t).$$

The riskless money market account $B(t)$ evolves as

$$\frac{dB(t)}{B(t)} = rdt,$$

where r is a constant riskless rate of nominal return. The stock price $S(t)$ follows a geometric Brownian motion:

$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma_S(\rho dW_1(t) + \overline{\rho} dW_2(t)),$$

where μ_S is the growth rate, σ_S the volatility, $-1 < \rho < 1$ the correlation between the indexed bond and the stock and $\overline{\rho} = \sqrt{1-\rho^2}$. We assume that $r_I + i > r, \mu_S > r$.

Define the volatility matrix

$$\sigma = \begin{pmatrix} \sigma_I & 0 \\ \sigma_S \rho & \sigma_S \overline{\rho} \end{pmatrix}$$

which is nonsingular. There exists a unique market price of risk, ξ, given by

$$\xi = \sigma^{-1} \begin{pmatrix} r_I + i - r \\ \mu_S - r \end{pmatrix} = \begin{pmatrix} \vartheta_I \\ (\vartheta_S - \rho \vartheta_I)/\overline{\rho} \end{pmatrix},$$

where $\vartheta_I = \frac{r_I + i - r}{\sigma_I} > 0$ and $\vartheta_S = \frac{\mu_S - r}{\sigma_S} > 0$ are the market prices of risk associated with the indexed bond and the stock, respectively.

In a DC plan, the pension members make a continuous contribution of salary to the pension plan before retirement time T and the amount of money contributed to the pension fund at time t is assumed to be $c(t) > 0$. In practice, the contribution rate may follow a stochastic process (see Chen et al. (2017) and Guan and Liang (2016)), which is more realistic but also makes essentially impossible to explicitly solve the optimization problem with trading constraints. To derive closed-form investment strategies, we assume $c(t)$ is a deterministic, non-decreasing function in this paper. This is based on the fact that, in the labor market, the average salary of employees and the contribution rate often steadily increase in the long run and such an assumption is a reasonable approximation to reality, see Boulier et al. (2001) in which $c(t)$ is an exponential function of time t. Furthermore, as mentioned in Chen et al. (2017), it is more realistic to set $c(t)$ to be less than a certain level due to government regulations on DC pension funds, which makes even the unconstrained optimization problem more difficult to solve under the assumption that $c(t)$ follows a diffusion process. However, this more realistic consideration does not bring any difficulty in mathematics in our model.

Assume that there are no transaction costs or taxes in the market and short selling is not allowed. The pension account is endowed with an initial wealth $x_0 \geq 0$. The pension manager can invest in the financial market. Let $\pi_1(t)$ and $\pi_2(t)$ be the amounts of wealth invested in the indexed bond and the stock, respectively. Then, the wealth process $X^\pi(t)$ satisfies

$$dX^\pi(t) = (rX^\pi(t) + \pi^\top(t)\sigma\xi)dt + \pi^\top(t)\sigma dW(t) + c(t)dt, X^\pi(0) = x_0 \geq 0, \tag{2.1}$$

where $\pi(t) = (\pi_1(t), \pi_2(t))^\top$. We next define the set of admissible trading strategies.

Definition 2.1. *A portfolio strategy $\pi = (\pi_1, \pi_2)^\top$ is said to be admissible if it is a progressively measurable, $\mathcal{F}$-adapted process which satisfies $E[\int_0^T \|\pi(t)\|^2 dt] < \infty$, $\pi_1(t) \geq 0, \pi_2(t) \geq 0$, a.s., and there exists a unique strong solution $X^\pi(t)$ to (2.1). The set of all admissible portfolio strategies is denoted by $\mathcal{A}$.*

In Definition 2.1, the condition $\pi_1(t) \geq 0, \pi_2(t) \geq 0$, a.s. rules out short-selling of the indexed bond and stock, but borrowing is still allowed. Note that, when the constraint sets are closed convex cones that include no borrowing or no short-selling constraints, we can use the dual control method to explicitly solve the constrained portfolio choice problem. For the case with both borrowing and short-selling constraints, that is, $\pi_1(t) \geq 0, \pi_2(t) \geq 0, X^\pi(t) - \pi_1(t) - \pi_2(t) \geq 0$, a.s., the constraint set is not a closed convex cone, which leads to a nonlinear dual HJB equation and may not admit a closed-form solution, see Karatzas and Shreve (1998).

The pension manager aims to find the best allocations in the indexed bond and stock to maximize the expected utility of the terminal wealth over a guaranteed threshold value, that is,

$$\begin{cases} \max\limits_{\pi \in \mathcal{A}} E[U(X^\pi(T) - L)], \\ s.t.\ X^\pi(t) \text{ satisfies (2.1)},\ \ X^\pi(T) \geq L, \end{cases} \tag{2.2}$$

where U is a continuous increasing function on $[0, \infty)$ and $U(x) = -\infty$ for $x < 0$, L is a positive constant which represents a lump sum to the members at retirement.

3 Optimal investment strategy

Some earlier research mainly focuses on concave utilities maximization in a DC pension fund, see Boulier et al. (2001) and Deelstra et al. (2003). Kahneman and Tversky (1979) claim that most investors are loss-averse and make decisions relative to some reference levels. Based on experiments, Kahneman and Tversky (1979) propose an S-shaped utility function to characterize different behaviors of people over gains and losses relative to a reference point, that is, risk averse over gains and risk seeking over losses. Mathematically, the S-shaped utility on $[0, \infty)$ is defined by

$$U(x) = \begin{cases} -U_2(\theta - x), & 0 \leq x < \theta, \\ U_1(x - \theta), & x \geq \theta, \end{cases} \tag{3.1}$$

where $\theta > 0$ is a reference point, U_1 and U_2 are strictly increasing, strictly concave, continuously differentiable on $[0, \infty)$ satisfying

$$U_1'(x) < U_2'(x), \tag{3.2}$$

$$U_i(0) = 0, \lim_{x \to +\infty} U_i(x) = +\infty, \lim_{x \to 0^+} U_i'(x) = +\infty, \lim_{x \to +\infty} U_i'(x) = 0, \tag{3.3}$$

and

$$0 \le U_i(x) \le M_i(1 + x^{p_i}), \ x \ge 0, \tag{3.4}$$

for some constants $M_i > 0, 0 < p_i < 1, i = 1, 2$, see Xu and Shreve(1992) and Bian et al. (2011). Define $U_i(x) = -\infty$ for $x < 0$. Note that U is convex when x is less than θ (on the loss domain) and concave when x is greater than θ (on the gain domain), which demonstrates that people tend to be risk averse with respect to gains and risk seeking with respect to losses. Therefore, (3.1) gives an S-shaped graph and is called an S-shaped utility function. The reference point θ is chosen in advance to be associated with the contribution rate and initial wealth. Condition (3.2) holds for loss aversion, which implies that people are more sensitive to a loss than to a gain of the same amount, see Tversky and Kahneman(1992). For example, we can take $U_2(x) = \lambda U_1(x)$, for some loss aversion degree $\lambda > 1$.

Condition (3.3) ensures that the strictly decreasing function U_1' has a strictly decreasing inverse $I_1 : (0, \infty) \to (0, \infty)$, that is

$$U_1'(I_1(y)) = y, \forall y > 0, \ I_1(U_1'(x)) = x, \forall x > 0.$$

Next, we employ the concavification technique from Carpenter (2000) to find the optimal solution of problem (2.2). Denote by f^c the concave envelope of a function f with domain D by:

$$f^c(x) := \inf\{g(x) : D \to R | g(t) \text{ is a concave function}, g(t) \ge f(t), \forall t \in D\}, \forall x \in D.$$

We first derive the concave envelope of U, see Carpenter (2000). Let z be the tangent point of the straight line starting at $(0, -U_2(\theta))$ to the curve $U_1(x), x \ge \theta$. Simple calculus shows the concave

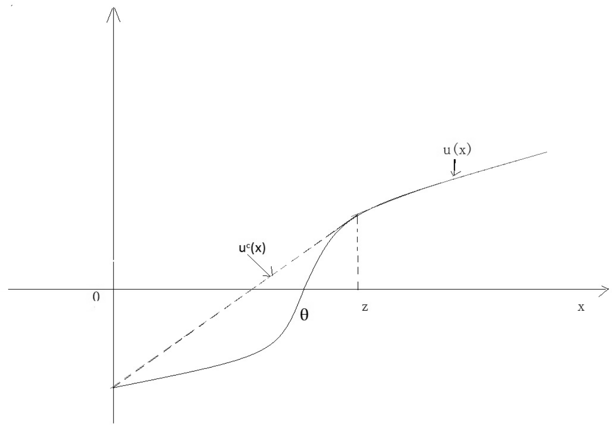

Figure 1: Concave envelope of $U(x)$

envelope of U is given by

$$U^c(x) = \begin{cases} kx - U_2(\theta), & 0 \le x < z, \\ U_1(x - \theta), & x \ge z, \end{cases} \tag{3.5}$$

where

$$k = U_1'(z - \theta) \tag{3.6}$$

is the slope of the tangent line (see Figure 1) and $z > \theta$ is the unique solution of the following equation

$$U_1(x - \theta) + U_2(\theta) - xU_1'(x - \theta) = 0. \tag{3.7}$$

Example 3.1. *For the utility proposed by Kahneman and Tversky (1979)*

$$U(x) = \begin{cases} -A(\theta - x)^{\gamma_1}, & 0 \le x < \theta, \\ B(x-\theta)^{\gamma_2}, & x \ge \theta, \end{cases} \tag{3.8}$$

where $A > B > 0$, $0 < \gamma_1, \gamma_2 < 1$, we have that z is the unique solution of the equation

$$B(x-\theta)^{\gamma_2} - B\gamma_2 x(x-\theta)^{\gamma_2 - 1} + A\theta^{\gamma_1} = 0,$$

and $k = B\gamma_2(z-\theta)^{\gamma_2-1}$.

The utility function (3.8) and the restrictions on the parameters are supported by experiments and statistics in Kahneman and Tversky (1979) and commonly used in optimization problems, see Berkelaar et al. (2004), Chen et al. (2017), Guan and Liang (2016), He and Kou (2018) and Lin et al. (2017). In particular, when there are no short-selling constraints, Chen et al. (2017) and Guan and Liang (2016) use the martingale method to solve the optimization problem (2.2) with utility (3.8). However, when short-selling is not allowed, the martingale representation theorem can not be used.

The dual control method is useful in solving constrained portfolio optimization problems. The dual function plays an important role in establishing the dual relation between the primal and dual problems. The dual function of U is defined by

$$V(y) = \sup_{x \ge 0}\{U(x) - xy\}, y > 0. \tag{3.9}$$

and that of the concave envelope U^c is defined by

$$V^c(y) = \sup_{x \ge 0}\{U^c(x) - xy\}, y > 0. \tag{3.10}$$

It is easy to find that $x^*(y)$ which solves both (3.9) and (3.10) is given by

$$x^*(y) = \begin{cases} \theta + I_1(y), & 0 < y < k, \\ 0, & y \ge k. \end{cases} \tag{3.11}$$

From Lemma 2.9 of Reichlin (2013), we have, for $y > 0$,

$$\begin{aligned} V(y) &= V^c(y) = U^c(x^*(y)) - x^*(y)y \\ &= \begin{cases} U_1(I_1(y)) - (I_1(y) + \theta)y, & 0 < y < k, \\ -U_2(\theta), & y \ge k. \end{cases} \end{aligned} \tag{3.12}$$

We next define the set of admissible dual controls.

Definition 3.2. *A dual control process is a progressively measurable, $\mathcal{F}$-adapted process $\nu = (\nu_1, \nu_2)^\top$ which satisfies $E[\int_0^T \|\nu(t)\|^2 dt] < \infty$ and $\nu_1(t) \ge 0, \nu_2(t) \ge 0$, a.s.. We denote the set of all dual control processes by $\mathcal{A}_0$.*

To use the dual control method to study the utility maximization problem, the choice of dual processes is crucial. For the wealth process (2.1) with $c(t) = 0$, Bian et al. (2011) choose a nonnegative supermartingale process Y^ν such that $X^\pi(t)Y^\nu(t)$ is a supermartingale. As the wealth process (2.1) is not self-financing for $c(t) > 0$, the dual control method in Bian et al. (2011) cannot be directly applied. To convert the optimization problem (2.2) into a classical portfolio optimization problem with control constraints, we follow Zhang and Ewald (2010) to define

$$D(t) = \int_t^T c(s)e^{-r(s-t)}ds, \tag{3.13}$$

the discounted value at time t of total pension contribution from t to T, and let

$$\tilde{X}^\pi(t) = X^\pi(t) + D(t). \tag{3.14}$$

Using (2.1), we have

$$d\tilde{X}^\pi(t) = (r\tilde{X}^\pi(t) + \pi^\top(t)\sigma\xi)dt + \pi^\top(t)\sigma dW(t), \tilde{X}^\pi(0) = \tilde{x}_0 \geq 0, \tag{3.15}$$

with $\tilde{x}_0 = x_0 + D(0) = x_0 + \int_0^T c(s)e^{-rs}ds$. Note that $\tilde{X}^\pi(T) = X^\pi(T)$. Therefore, the optimization problem (2.2) is equivalent to the following problem:

$$\begin{cases} \max\limits_{\pi\in\mathcal{A}} E[U(\tilde{X}^\pi(T) - L)], \\ s.t.\ \tilde{X}^\pi(t) \text{ satisfies (3.15)},\ \ \tilde{X}^\pi(T) \geq L. \end{cases} \tag{3.16}$$

Following a similar argument in Bian et al. (2011), we take a nonnegative supermartingale process Y^ν such that

$$Z^\nu(t) := \tilde{X}^\pi(t)Y^\nu(t), 0 \leq t \leq T,$$

is a supermartingale. Then the dual process is given by

$$dY^\nu(t) = Y^\nu(t)(-rdt - (\sigma^{-1}\nu(t) + \xi)^\top dW(t)), Y^\nu(0) = y_0,$$

where $\nu \in \mathcal{A}_0$. Since Z^ν is a supermartingale, also noting V^c is the dual function of U^c, we have

$$\begin{aligned} E[U^c(\tilde{X}^\pi(T) - L)] &\leq E[V^c(Y^\nu(T)) + (\tilde{X}^\pi(T) - L)Y^\nu(T)], \\ &\leq E[V^c(Y^\nu(T)) + \tilde{x}_0 y_0 - LY^\nu(T)]. \end{aligned}$$

Consider the dual minimization problem

$$\inf_{\nu\in\mathcal{A}_0} E[V^c(Y^\nu(T)) - LY^\nu(T)].$$

For $0 \leq t \leq T$ and $y > 0$, the dual value function is defined by

$$v(t,y) = \inf_{\nu\in\mathcal{A}_0} E[V^c(Y^\nu(T)) - LY^\nu(T)|Y^\nu(t) = y].$$

The dual HJB equation is given by

$$\begin{cases} \frac{\partial v}{\partial t}(t,y) - ry\frac{\partial v}{\partial y}(t,y) + \frac{1}{2}y^2 \min\limits_{\nu\in[0,\infty)^2} \|\xi + \sigma^{-1}\nu\|^2 \frac{\partial^2 v}{\partial y^2}(t,y) = 0, y > 0, t < T, \\ v(T,y) = V(y) - Ly. \end{cases} \tag{3.17}$$

Here we have used $V(y) = V^c(y)$ for $y > 0$. Let

$$f(\nu) := \|\xi + \sigma^{-1}\nu\|^2 = (\frac{\nu_1}{\sigma_I} + \vartheta_I)^2 + \frac{1}{\bar{\rho}^2}(\frac{\nu_2}{\sigma_S} + \vartheta_S - \rho(\frac{\nu_1}{\sigma_I} + \vartheta_I))^2. \tag{3.18}$$

Lemma 3.3. *Under the assumption that $r_I + i > r, \mu_S > r$, we have that f has a unique minimizer $\hat{\nu} \in [0,\infty)^2$ satisfying $\|\xi + \sigma^{-1}\hat{\nu}\|^2 > 0$ and $\min\{\hat{\nu}_1, \hat{\nu}_2\} = 0$.*

Proof. Since f is a continuous, strictly convex and coercive function, f has a unique minimizer $\hat{\nu} \in [0,\infty)^2$. To derive the unique minimizer of (3.18), we define a Lagrange function as

$$L(\nu_1, \nu_2, u_1, u_2) = (\frac{\nu_1}{\sigma_I} + \vartheta_I)^2 + \frac{1}{\bar{\rho}^2}(\frac{\nu_2}{\sigma_S} + \vartheta_S - \rho(\frac{\nu_1}{\sigma_I} + \vartheta_I))^2 - u_1\nu_1 - u_2\nu_2.$$

The Kuhn-Tucker condition implies that

$$\begin{cases} L_{\nu_1} = \frac{2}{\sigma_I}(\frac{\nu_1}{\sigma_I} + \vartheta_I) - \frac{2\rho}{\bar{\rho}^2\sigma_I}(\frac{\nu_2}{\sigma_S} + \vartheta_S - \rho(\frac{\nu_1}{\sigma_I} + \vartheta_I)) - u_1 = 0, \\ L_{\nu_2} = \frac{2}{\bar{\rho}^2\sigma_S}(\frac{\nu_2}{\sigma_S} + \vartheta_S - \rho(\frac{\nu_1}{\sigma_I} + \vartheta_I)) - u_2 = 0, \\ u_i\nu_i = 0, u_i \geq 0, \nu_i \geq 0, i = 1, 2. \end{cases} \tag{3.19}$$

If $u_1 = 0, u_2 = 0$, solving (3.19) yields

$$\begin{cases} \hat{\nu}_1 = -\vartheta_I\sigma_I < 0, \\ \hat{\nu}_2 = -\vartheta_S\sigma_S < 0, \end{cases} \tag{3.20}$$

which contradicts the condition that $\nu_1 \geq 0, \nu_2 \geq 0$. Therefore, u_1, u_2 cannot be both zero and the solution (3.20) is impossible, which implies $\|\xi + \sigma^{-1}\hat{\nu}\|^2 > 0$. From the complementary slackness condition $u_iv_i = 0, i = 1, 2$, we conclude that $\hat{\nu}_1 = 0$ or $\hat{\nu}_2 = 0$. □

Denote by $\hat{\xi} = \xi + \sigma^{-1}\hat{\nu}$. Then $\hat{\xi} \neq 0$ and the solution to (3.17) is given by

$$\begin{cases} v(t,y) = E[V(Y^{\hat{\nu}}(T)) - LY^{\hat{\nu}}(T)|Y^{\hat{\nu}}(t) = y], \\ Y^{\hat{\nu}}(s) = ye^{-(r+\frac{\|\hat{\xi}\|^2}{2})(s-t) - \hat{\xi}^\top(W(s)-W(t))}, t \leq s \leq T. \end{cases} \tag{3.21}$$

Remark 3.4. *If the guaranteed threshold value $L = L(T)$ is stochastic, then we can also follow the same idea as above to define the dual value function as follows:*

$$v(t,y,L) = \inf_{\nu \in \mathcal{A}_0} E[V^c(Y^\nu(T)) - L(T)Y^\nu(T)|Y^\nu(t) = y, L(t) = L].$$

However, solving such an optimization problem explicitly is difficult because the dual HJB equation is a nonlinear PDE and we may have to rely on numerical methods. Although L in our paper is deterministic, we can choose it by considering the average rate of inflation and the level of economic growth to reduce the inflation risk.

Define the value functions of the primal problem and the concavified version to be

$$u(t,\tilde{x}) = \max_{\pi \in \mathcal{A}} E[U(\tilde{X}^\pi(T) - L)|\tilde{X}^\pi(t) = \tilde{x}], \tag{3.22}$$

and

$$u^c(t,\tilde{x}) = \max_{\pi \in \mathcal{A}} E[U^c(\tilde{X}^\pi(T) - L)|\tilde{X}^\pi(t) = \tilde{x}]. \tag{3.23}$$

We first calculate $u^c(t,\tilde{x})$. Bian et al. (2011) and Xu and Shreve (1992) give the relationship between the primal value function $u^c(t,\tilde{x})$ and the dual value function $v(t,y)$.

Theorem 3.5. *(Bian et al. (2011)) Assume $v(t,y)$ is given by (3.21) and the conditions (3.3) and (3.4) hold. Then we have*

$$u^c(t,\tilde{x}) = v(t, y(t,\tilde{x})) + \tilde{x}y(t,\tilde{x}), \tilde{x} \geq 0,$$

where $y = y(t,\tilde{x})$ satisfies

$$v_y(t,y) + \tilde{x} = 0. \tag{3.24}$$

Moreover, the optimal feedback control is given by

$$\pi^*(t) = (\sigma^\top)^{-1}\hat{\xi}y(t,\tilde{x})v_{yy}(t, y(t,\tilde{x})) \in [0,\infty)^2. \tag{3.25}$$

To see how the short-selling constraints impact the optimal wealth process, we express $Y^{\hat{\nu}}(T)$ as

$$Y^{\hat{\nu}}(T) = Y^{\hat{\nu}}(t)\frac{H^{\hat{\nu}}(T)}{H^{\hat{\nu}}(t)},$$

where

$$H^{\hat{\nu}}(t) = e^{-(r+\frac{\|\hat{\xi}\|^2}{2})t - \hat{\xi}^\top W(t)} \tag{3.26}$$

is a state-price density process in a fictitious market (see Cox and Huang (1989)). If there is no limitation on the trading strategy, then $\hat{\nu} = (0,0)^\top$ and $H^{\hat{\nu}}(t) = e^{-(r+\frac{\|\xi\|^2}{2})t-\xi^\top W(t)}$, which is exactly the pricing kernel in a complete market. The following result is useful in deriving analytic expressions for the optimal wealth process and the optimal investment strategy.

Lemma 3.6. *Let $H^{\hat{\nu}}(t)$ be defined by (3.26). Then for any $\lambda \in \mathbb{R}, h > 0, y > 0$, we have*

$$E[(H^{\hat{\nu}}(T))^\lambda 1_{\{H^{\hat{\nu}}(T)<h\}}|H^{\hat{\nu}}(t) = y] = y^\lambda e^{\frac{\lambda^2\alpha^2(t)}{2} - \lambda\beta(t)}\Phi(d_1(h,y,t) - \lambda\alpha(t)), \tag{3.27}$$

where

$$\alpha(t) = \|\hat{\xi}\|\sqrt{T-t}, \beta(t) = (r + \frac{\|\hat{\xi}\|^2}{2})(T-t), d_1(h,y,t) = \frac{\ln(h/y) + \beta(t)}{\alpha(t)}, \tag{3.28}$$

and Φ is the cumulative distribution function of a standard normal variable.

Proof. We rewrite $H^{\hat{\nu}}(T)$ as

$$H^{\hat{\nu}}(T) = H^{\hat{\nu}}(t)e^Z, Z = -\beta(t) - \hat{\xi}^\top(W(T) - W(t)) \sim N(-\beta(t), \alpha^2(t)). \tag{3.29}$$

Then

$$E[(H^{\hat{\nu}}(T))^\lambda 1_{\{H^{\hat{\nu}}(T)<h\}}|H^{\hat{\nu}}(t) = y] = y^\lambda \int_{-\infty}^{\ln(h/y)} e^{\lambda z}\frac{1}{\sqrt{2\pi}\alpha(t)}e^{-\frac{(z+\beta(t))^2}{2\alpha^2(t)}}dz.$$

Simple calculus leads to (3.27). □

Proposition 3.7. *Under the utility function (3.5) and the condition $x_0 + \int_0^T c(s)e^{-rs}ds \geq Le^{-rT}$, the optimal terminal wealth and the optimal wealth process at time $0 \leq t < T$ are given by*

$$X^{\pi^*}(T) = \begin{cases} I_1(y_0H^{\hat{\nu}}(T)) + \theta + L, & H^{\hat{\nu}}(T) < \frac{k}{y_0}, \\ L, & H^{\hat{\nu}}(T) \geq \frac{k}{y_0}, \end{cases} \tag{3.30}$$

and

$$X^{\pi^*}(t) = \frac{1}{H^{\hat{\nu}}(t)} E[H^{\hat{\nu}}(T) X^{\pi^*}(T)|\mathcal{F}_t] - \int_t^T c(s) e^{-r(s-t)} ds, \tag{3.31}$$

where k is given by (3.6), $y_0 = Y^{\hat{\nu}}(0)$ is the solution to the budget constraint

$$E[H^{\hat{\nu}}(T) X^{\pi^*}(T)] = x_0 + \int_0^T c(s) e^{-rs} ds, \tag{3.32}$$

and

$$\frac{1}{H^{\hat{\nu}}(t)} E[H^{\hat{\nu}}(T) X^{\pi^*}(T)|\mathcal{F}_t] = G(k, y_0 H^{\hat{\nu}}(t), t) + \theta e^{-r(T-t)} \Phi(d_2(k, y_0 H^{\hat{\nu}}(t), t)) + L e^{-r(T-t)}, \tag{3.33}$$

with

$$G(k, y, t) = \int_{-\infty}^{d_1(k,y,t)} I_1(y e^{\alpha(t) z - \beta(t)}) e^{\alpha(t) z - \beta(t)} \varphi(z) dz, \tag{3.34}$$

φ the standard normal density function, $d_2(k, y, t) = d_1(k, y, t) - \alpha(t)$, $\alpha(t), \beta(t)$ and $d_1(k, y, t)$ defined by (3.28).

The optimal investment strategy is given by

$$\pi^*(t) = (\sigma^\top)^{-1} \hat{\xi} \Lambda(t, Y^{\hat{\nu}}(t)) \in [0, \infty)^2, \tag{3.35}$$

where

$$\Lambda(t, y) = \theta e^{-r(T-t)} \frac{\varphi(d_2(k, y, t))}{\alpha(t)} - y G_y(k, y, t) > 0. \tag{3.36}$$

Proof. Assume $\hat{\nu}^\top = (\hat{\nu}_1, \hat{\nu}_2)$ with $\hat{\nu}_1 \geq 0, \hat{\nu}_2 \geq 0$ is a solution to (3.19). Note that $V(y)$ is continuous for $y > 0$ and continuously differentiable for $y > 0$ except at finitely many points. Using (3.12) and pathwise differentiation, we have

$$v_y(t, y) = -E[x^*(Y^{\hat{\nu}}(T)) \frac{Y^{\hat{\nu}}(T)}{y} | Y^{\hat{\nu}}(t) = y] - L e^{-r(T-t)}, \tag{3.37}$$

where $x^*(y)$ is defined in (3.11). From (3.24), we obtain that under the utility function (3.5),

$$\begin{aligned} \tilde{X}^{\pi^*}(t) &= -v_y(t, y_0 H^{\hat{\nu}}(t)) \\ &= E[x^*(y_0 H^{\hat{\nu}}(T)) \frac{H^{\hat{\nu}}(T)}{H^{\hat{\nu}}(t)} | \mathcal{F}_t] + L e^{-r(T-t)}. \end{aligned} \tag{3.38}$$

In particular,

$$X^{\pi^*}(T) = \tilde{X}^{\pi^*}(T) = x^*(y_0 H^{\hat{\nu}}(T)) + L, \tag{3.39}$$

which yields (3.30) and y_0 satisfies the budget constraint (3.32). Substituting (3.39) into (3.38) yields

$$\tilde{X}^{\pi^*}(t) = E[\tilde{X}^{\pi^*}(T) \frac{H^{\hat{\nu}}(T)}{H^{\hat{\nu}}(t)} | \mathcal{F}_t]. \tag{3.40}$$

Equation (3.31) is a direct consequence of (3.40) and (3.14). Equation (3.33) follows from Lemma 3.6. It remains to show that there is a unique root y_0 to (3.32). Note that for any $\omega \in \Omega$, $y_0 \to$

$X^{\pi^*}(T)$ is a decreasing function of y_0 since I_1 is strictly decreasing. Then $V(y_0) = E[H^{\hat{\nu}}(T)X^{\pi^*}(T)]$ is continuous and strictly decreasing in y_0. Furthermore, for any $\omega \in \Omega$, we have $\lim_{y_0 \to 0^+} X^{\pi^*}(T) = \infty$ and $\lim_{y_0 \to \infty} X^{\pi^*}(T) = L$, which yields

$$\lim_{y_0 \to 0^+} V(y_0) = \infty, \lim_{y_0 \to \infty} V(y_0) = Le^{-rT}.$$

Thus, for $Le^{-rT} < x_0 + \int_0^T e^{-rs}ds$, there exists a unique solution $y_0 \in (0, \infty)$ to equation (3.32); for $Le^{-rT} = x_0 + \int_0^T e^{-rs}ds$, there is only one admissible solution $X^{\pi^*}(T) = L$, which implies that one should only invest in the risk-free cash bond to attain the minimum guarantee at retirement.

The optimal control $\pi^*(t)$ in (3.35) can be easily derived from (3.25), (3.37) and (3.33). In particular, (3.25) guarantees that $\pi^*(t) \in [0, \infty)^2$, which implies Λ given by (3.36) is nonnegative. $\square$

Remark 3.8. *Similar to Basak and Shapiro (2001), it is easy to see that if $x_0 + \int_0^T c(s)e^{-rs}ds < Le^{-rT}$, then the optimization problem (2.2) is infeasible.*

Proposition 3.9. *Under the utility function (3.1) and the condition $x_0 + \int_0^T c(s)e^{-rs}ds \geq Le^{-rT}$, the value function of the primal problem (3.22) is*

$$u(t, \tilde{x}) = u^c(t, \tilde{x}),$$

where $u^c(t, \tilde{x})$ is defined by (3.23). Furthermore, the optimal wealth process and the optimal investment strategy are given by (3.31) and (3.35), respectively. The optimal terminal wealth is given by (3.30) and

$$P(X^{\pi^*}(T) \in (L, L + z)) = 0, \tag{3.41}$$

where z is define by (3.7).

Proof. Since $H^{\hat{\nu}}(T)$ has no atom, Theorem 5.1 of Reichlin (2013) gives that $u(t, \tilde{x}) = u^c(t, \tilde{x})$. Therefore, the optimal wealth process and the optimal investment strategy under the utility (3.1) are the same as those under the utility (3.5). Note that $\{x | U(x) < U^c(x)\} = (0, z)$ and $\{x^*(y) \in \{U(x) < U^c(x)\}\} = \emptyset$. Then from Proposition 5.3 of Reichlin (2013), we have that the optimal terminal wealth under the utility (3.1) is the same as that under the utility (3.5) given by (3.30). Since $I_1(y)$ is a strictly decreasing function, we conclude that for $H^{\hat{\nu}}(T) \leq \frac{k}{y_0}$,

$$I_1(y_0 H^{\hat{\nu}}(T)) + \theta + L \geq I_1(U_1'(z - \theta)) + \theta + L = z + L.$$

Therefore, $P(X^{\pi^*}(T) \in (L, z + L)) = 0$. $\square$

Equation (3.30) implies that the PI constraint ensures that the optimal terminal wealth is always above L. If the economy is good, then the members also gain an additional amount over L by $\theta + I_1(y_0 H^{\hat{\nu}}(T))$ from participating in the financial market.

Similar to the formulas for the optimal wealth process derived in Chen et al. (2017) in a complete market, the optimal wealth process $X^{\pi^*}(t)$ given by (3.31) is obtained by the pricing theory and consists of two components. The first part $\frac{1}{H^{\hat{\nu}}(t)} E[H^{\hat{\nu}}(T)X^{\pi^*}(T)|\mathcal{F}_t]$ is the t-price of the random variable $X^{\pi^*}(T)$ in an incomplete market. The second part $\int_t^T c(s)e^{-r(s-t)}ds$, which can also be expresses as $\frac{1}{H^{\hat{\nu}}(t)} E[\int_t^T H^{\hat{\nu}}(s)c(s)ds|\mathcal{F}_t]$, represents the price of the aggregated contribution from t

to T. As mentioned in Zhang and Ewald (2010), in a complete market, the non-negative terminal wealth constraint does not guarantee non-negative wealth at all times due to the presence of a positive income stream, which is the problem of liquidity constraint. In our considered market, the wealth may also become negative. However, the wealth $X^{\pi^*}(t)$ plus the discounted value of future contributions $D(t)$ is no less than the discounted value of minimum performance at each time.

Equation (3.35) is similar to the result in the portfolio selection literature, which considers a general investment problem. Comparing (3.35) with (29) in Chen et al. (2017), the contribution part and the minimum guarantee are not included in the investment strategy under our model. The reason is that the guarantee L and the contribution rate $c(t)$ are both deterministic and these two parts do not bring hedge demands in the investment strategy. When the contribution rate $c(t)$ and L both depend on the stock and indexed-bond market, we can still formulate the dual optimization problem. However, this generalization leads to a nonlinear dual HJB equation which is equally difficult to solve as the primal HJB equation.

Remark 3.10. *For a utility defined in Example 3.1, when there are no short-selling and PI constraints, the optimal terminal wealth (3.30) has the same form as (3.5) in Guan and Liang (2016); when there is only PI constraint, the optimal terminal wealth has the same form as (27) in Chen et al. (2017). Therefore, extending Guan and Liang (2016) and Chen et al. (2017), we include both PI and short-selling constraints in our framework. Due to incompleteness introduced through short-selling constraints, the martingale method used in Guan and Liang (2016) and Chen et al. (2017) does not work in our model.*

We next analyze how the reference point θ and the protection level L impact the optimal terminal wealth. If θ is 0, then the utility function (3.1) degenerates to $U(x) = U_1(x)$ for $x \geq 0$. We have $z = 0$, $k = \infty$ and $d_1(k, y, t) = \infty$, which results in simplified expressions for the optimal wealth and control processes in Proposition 3.7 and the optimal terminal wealth is given by

$$X^{\pi^*}(T) = I_1(y_0 H^{\hat{\nu}}(T)) + L. \tag{3.42}$$

Comparing (3.30) with (3.42), we can see that the optimal terminal wealth $X^{\pi^*}(T)$ under a concave utility is continuous, while $X^{\pi^*}(T)$ under an S-shaped utility takes a two-region form with a point mass at L. When $H^{\hat{\nu}}(T)$ is low, the optimal terminal wealth under loss aversion is similar to the smooth concave utility and is above $\theta + L$; when $H^{\hat{\nu}}(T)$ is high, $X^{\pi^*}(T)$ equals to L since the loss aversion states a risk-seeking preference under $\theta + L$.

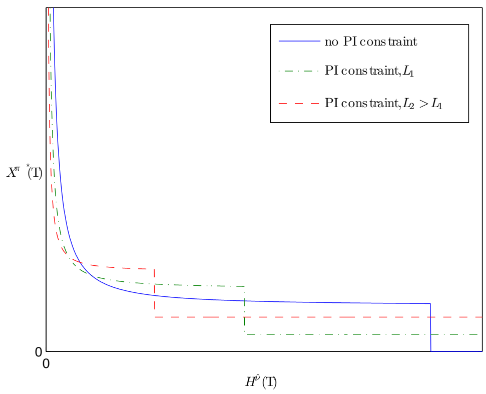

Figure 2: $X^{\pi^*}(T)$ versus $H^{\hat{\nu}}(T)), \theta > 0$

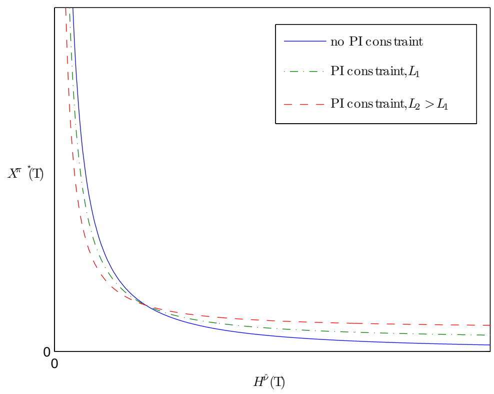

Figure 3: $X^{\pi^*}(T)$ versus $H^{\hat{\nu}}(T)), \theta = 0$

For a given $\theta \geq 0$, it is easy to conclude that y_0 increases with L from the budget constraint (3.32). which implies that for a quite low $H^{\hat{\nu}}(T)$, the optimal terminal wealth $X^{\pi^*}(T)$ decreases with L. Figures 2 and 3 display the relationship between the optimal terminal wealth and $H^{\hat{\nu}}(T)$ for an S-shaped utility and a concave utility, respectively. We can observe that in good economic states the PI constraint decreases the optimal terminal wealth and in bad economic states the PI constraint keeps the terminal wealth above L, which implies the protection for the members in bad economic states is at the expense of the optimal terminal wealth in good economic states. We can also note that a higher protection level L leads to a lower value of $X^{\pi^*}(T)$ in good economic states. Therefore, putting the PI constraint improves the risk management since it keeps the optimal terminal above L and makes the optimal terminal wealth less volatile.

The PI constraint provides a minimum guarantee for bad economic states. The next result measures the region of the protected states.

Proposition 3.11. *For a given $\theta > 0$, we have that*

$$P(X^{\pi^*}(T) = L) = 1 - \Phi(d_1(k, y_0, 0)),$$

where k is given by (3.6) and y_0 is determined by (3.32). Furthermore, $P(X^{\pi^*}(T) = L)$ is an increasing function of L.

Proof. From (3.30), we have

$$P(X^{\pi^*}(T) = L) = P(H^{\hat{\nu}}(T) \geq \frac{k}{y_0}).$$

The fact that y_0 increases in L concludes the result. □

Remark 3.12. *The design of the minimum guarantee is important for the DC pension plan. If L is too high, then more states need to be insured against, which brings a prudent investment process and a small expected optimal terminal wealth. If L is too low, then the effect of the PI constraint is not significant. Similar to Boulier et al. (2001), Liang and Guan (2014) and Chen et al. (2017), we can set L to be the annual guarantee, that is, $L = \int_T^{\varpi} e^{-r(s-T)} a(s)_{s-T}p_T ds$, where ϖ is the largest survival age, $a(s) = a(T)e^{g(s-T)}, s \in [T, \tau]$ is the annuity at time s, g is a constant that reflects the average inflation rate and the increasing standard of living, τ is the date of death and random, ${}_{s-T}p_T = e^{-\int_T^s \lambda(v)dv}$ is the probability that the contributor will survive to s given that she is still alive at T and $\lambda(v)$ is the deterministic force of mortality.*

From Propositions 3.7, 3.9, we can easily obtain the optimal wealth process for a special class of utility function, including power and non-HARA utilities.

Example 3.13. *Let the dual function of U_1 be given by*

$$V_1(y) = \sum_{i=1}^{m} c_i y^{q_i}, \tag{3.43}$$

where $q_i < 0, c_i > 0, i = 1, 2, \cdots, m$, and $m \geq 1$ is an integer. (3.43) covers many utility functions, including power utility

$$U_1(x) = Bx^{\gamma}, 0 < \gamma < 1, B > 0,$$

with dual function given by

$$V_1(y) = B(1-\gamma)(\frac{y}{B\gamma})^{\frac{\gamma}{\gamma-1}},$$

and non-HARR utility (see Bian and Zheng (2015))

$$U_1(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x),$$

where $H(x) = (\frac{2}{-1+\sqrt{1+4x}})^{\frac{1}{2}}$, *with dual function given by*

$$V_1(y) = \frac{1}{3}y^{-3} + y^{-1}.$$

Propositions 3.7, 3.9 say that the optimal wealth process is given by (3.31) with

$$G(k, y, t) = -\sum_{i=1}^{m} c_i q_i y^{q_i-1} e^{\Gamma_i(t,T)} \Phi(d^i(k, y, t))),$$

where $\Gamma_i(t, T) = -q_i\beta(t) + \frac{q_i^2}{2}\alpha^2(t)$, *and* $d^i(k, y, t) = d_1(k, y, t) - q_i\alpha(t)$.

Next we derive the explicit formulas for the dual control $\hat{\nu}$, the optimal wealth process and the optimal investment strategy under the utility function (3.1).

Proposition 3.14. *Consider the utility function (3.1).*

Case A: If $\vartheta_I > \rho\vartheta_S$ *and* $\vartheta_S > \rho\vartheta_I$, *then the optimal wealth process is given by (3.31) with* $\hat{\nu}$ *and* $\hat{\xi}$ *replaced by* $(0, 0)^\top$ *and* $(\vartheta_I, (\vartheta_S - \rho\vartheta_I)/\overline{\rho})^\top$, *respectively. The optimal investment strategy is*

$$\pi^*(t) = \Lambda(t, Y^{\hat{\nu}}(t)) \begin{pmatrix} \frac{\vartheta_I-\rho\vartheta_S}{\sigma_I\overline{\rho}^2} \\ \frac{\vartheta_S-\rho\vartheta_I}{\sigma_S\overline{\rho}^2} \end{pmatrix},$$

where $\Lambda(t, y)$ *is defined in (3.36).*

Case B: If $\rho\vartheta_I \geq \vartheta_S$, *then the optimal wealth process is given by (3.31) with* $\hat{\nu}$ *and* $\hat{\xi}$ *replaced by* $(0, \sigma_S(\rho\vartheta_I - \vartheta_S))^\top$ *and* $(\vartheta_I, 0)^\top$, *respectively. The optimal investment strategy is*

$$\pi^*(t) = \begin{pmatrix} \frac{\vartheta_I\Lambda(t,Y^{\hat{\nu}}(t))}{\sigma_I} \\ 0 \end{pmatrix}.$$

Case C: If $\rho\vartheta_S \geq \vartheta_I$, *then the optimal wealth process is given by (3.31) with* $\hat{\nu}$ *and* $\hat{\xi}$ *replaced by* $(\sigma_I(\rho\vartheta_S - \vartheta_I), 0)^\top$ *and* $(\rho\vartheta_S, \overline{\rho}\vartheta_S)^\top$, *respectively. The optimal investment strategy is*

$$\pi^*(t) = \begin{pmatrix} 0 \\ \frac{\vartheta_S\Lambda(t,Y^{\hat{\nu}}(t))}{\sigma_S} \end{pmatrix}.$$

Proof. To derive the explicit formulas for the optimal wealth, we need to find the unique minimizer of (3.18), $\hat{\nu}$, with $\min\{\hat{\nu}_1, \hat{\nu}_2\} = 0$. We solve the equation (3.19) according to the following three cases.

Case A: If $u_1 > 0, u_2 > 0$, then $\hat{\nu}_1 = 0, \hat{\nu}_2 = 0$, and

$$\begin{cases} u_1 = \frac{2(\vartheta_I-\rho\vartheta_S)}{\sigma_I\overline{\rho}^2}, \\ u_2 = \frac{2(\vartheta_S-\rho\vartheta_I)}{\sigma_S\overline{\rho}^2}. \end{cases}$$

Since $u_1 > 0, u_2 > 0$, we must have $\vartheta_I > \rho\vartheta_S, \vartheta_S > \rho\vartheta_I$, and $\hat{\xi} = \sigma^{-1}\hat{\nu} + \xi = \xi = (\vartheta_I, \frac{\vartheta_S-\rho\vartheta_I}{\overline{\rho}})^\top$. Substituting the expressions for $\hat{\nu}$ and $\hat{\xi}$ into (3.31) and (3.35), we obtain the result.

Case B: If $u_1 > 0, u_2 = 0$, then $\hat{\nu}_1 = 0$, and

$$\begin{cases} \frac{2\vartheta_I}{\sigma_I} - \frac{2\rho}{\sigma_I \overline{\rho}^2}(\frac{\nu_2}{\sigma_S} + \vartheta_S - \rho\vartheta_I) - u_1 = 0, \\ \frac{2}{\sigma_S \overline{\rho}^2}(\frac{\nu_2}{\sigma_S} + \vartheta_S - \rho\vartheta_I) = 0. \end{cases}$$

Solving the above equation, we have

$$\begin{cases} u_1 = \frac{2\vartheta_I}{\sigma_I} \\ \hat{\nu}_2 = (\rho\vartheta_I - \vartheta_S)\sigma_S. \end{cases}$$

Since $u_1 > 0, \hat{\nu}_2 \geq 0$, we have $\rho\vartheta_I \geq \vartheta_S$, and $\hat{\xi} = \sigma^{-1}\hat{\nu} + \xi = (\vartheta_I, 0)^\top$.

Case C: If $u_1 = 0, u_2 > 0$, then $\hat{\nu}_2 = 0$, and

$$\begin{cases} \frac{2}{\sigma_I}(\frac{\nu_1}{\sigma_I} + \vartheta_I) - \frac{2\rho}{\sigma_I \overline{\rho}^2}(\vartheta_S - \rho(\frac{\nu_1}{\sigma_I} + \vartheta_I)) = 0, \\ \frac{2\rho}{\sigma_S \overline{\rho}^2}(\vartheta_S - \rho(\frac{\nu_1}{\sigma_I} + \vartheta_I)) - u_2 = 0. \end{cases}$$

Solving the above equation, we obtain

$$\begin{cases} \hat{\nu}_1 = \sigma_I(\rho\vartheta_S - \vartheta_I), \\ u_2 = \frac{2\vartheta_S}{\sigma_S}. \end{cases}$$

From $\hat{\nu}_1 \geq 0, u_2 > 0$, we have $\rho\vartheta_S \geq \vartheta_I$ and $\hat{\xi} = \sigma^{-1}\hat{\nu} + \xi = (\rho\vartheta_S, \overline{\rho}\vartheta_S)^\top$. □

Remark 3.15. *To gain some economic intuition of the optimal strategies, we may express the dynamics of $I(t)$ and $S(t)$ as*

$$\frac{dI(t)}{I(t)} = rdt + \sigma_I(\vartheta_I dt + dW_1(t)),$$

$$\frac{dS(t)}{S(t)} = rdt + \sigma_S\rho(\vartheta_I dt + dW_1(t)) + \sigma_S\overline{\rho}(\frac{\vartheta_S - \rho\vartheta_I}{\overline{\rho}}dt + dW_2(t)),$$

where ϑ_I is the market price of $W_1(t)$ and $(\vartheta_S - \rho\vartheta_I)/\overline{\rho}$ is the market price of $W_2(t)$, or similarly,

$$\frac{dS(t)}{S(t)} = rdt + \sigma_S(\vartheta_S dt + dW_S(t)),$$

$$\frac{dI(t)}{I(t)} = rdt + \sigma_I\rho(\vartheta_S dt + dW_S(t)) + \sigma_I\overline{\rho}(\frac{\vartheta_I - \rho\vartheta_S}{\overline{\rho}}dt + dW_3(t)),$$

where $W_S(t)$ and $W_3(t)$ are two independent Brownian motions, ϑ_S is the market price of $W_S(t)$ and $(\vartheta_I - \rho\vartheta_S)/\overline{\rho}$ is the market price of $W_3(t)$.

We may explain the optimal strategies as follows: if the market price of the risk $W_2(t)$ is non-positive, then we should not invest the money in the stock; if the market price of the risk $W_3(t)$ is non-positive, then we should not invest the money in the indexed bond; if the market prices of the risk of all Brownian motions are positive, then we should invest in both the indexed bond and the stock.

Remark 3.16. *For a utility defined in Example 3.1, if we set the contribution rate $c(t) = 0$ and $L = 0$, our model degenerates to the no pension case studied in Berkelaar et al. (2004). When the parameters satisfy the condition $\vartheta_I > \rho\vartheta_S$ and $\vartheta_S > \rho\vartheta_I$, Case A of Proposition 3.14 coincides with theirs by considering an indexed bond and a stock as two risky assets. Note that, if there are* *no short-selling constraints, then $\hat{\xi} = \xi$. Therefore, when $\vartheta_I > \rho\vartheta_S$ and $\vartheta_S > \rho\vartheta_I$, the optimal control naturally meets the short-selling constraints, which implies that the short-selling constraints are not binding. If the reference point θ is also set to be 0, then the percentage of wealth invested in the risky assets becomes $\frac{\pi^*(t)}{X^{\pi^*}(t)} = \frac{(\sigma^\top)^{-1}\hat{\xi}}{1-\gamma_2}$. If we set $\rho = 0$, then we have $\frac{\pi_1^*(t)}{X^{\pi^*}(t)} = \frac{\vartheta_I}{(1-\gamma_2)\sigma_I}$ and $\frac{\pi_2^*(t)}{X^{\pi^*}(t)} = \frac{\vartheta_S}{(1-\gamma_2)\sigma_S}$, which are Merton's portfolio, see Merton (1969, 1971).*

4 Numerical analysis

In this section, we carry out some numerical analysis for the optimal investment problem under the short-selling and the PI constraints. Since the impacts of the parameters on the optimal strategies have been investigated in a lot of literature, we mainly study the influence of the reference point, the short-selling and PI constraints on the optimal terminal wealth.

Consider the utility function defined in Example 3.1. For all the computations, the values of certain parameters are held fixed except otherwise indicated: $T = 40, r_I = 0.02, r = 0.05, i = 0.04, \mu_S = 0.08, \sigma_I = 0.2, \sigma_S = 0.25, \rho = 0.8$. The pension account's initial wealth is $x_0 = 1$. The amount of the money contributed to the pension is set to be $c(t) = 0.1$. Assume $A = 2.25$ and $B = 1$, as estimated in Tversky and Kahneman(1992), and choose $\gamma_1 = 0.15$ and $\gamma_2 = 0.2$.

4.1 Impact of reference point and short-selling constraints on terminal wealth

In this section, we perform some numerical calculations to investigate the effect of the reference point θ and the short-selling constraints on the optimal terminal wealth. To better illustrate how θ impacts on the optimal terminal wealth, we let $L = 0$. Figure 4 displays the relationship between the optimal terminal wealth $P(X^{\pi^*}(T) = 0)$ and θ. From it we can observe that as the reference point θ increases, the probability that $X^{\pi^*}(T)$ achieves 0 increases. The reason is that $P(X^{\pi^*}(T) = 0)$ measures the region of bad economic states $\{H^{\hat{\nu}}(T) \geq \frac{k}{y_0}\}$ and a larger value of θ is sure to enlarge the region of the domain of the losses. Note that the manager is endowed with $x_0 = 1$ and receives a continuous contribution payment $c = 0.1$. We note that the probability $P(X^{\pi^*}(T) = 0)$ is very low for a small θ, since it is easy to achieve. We can also see that the probability $P(X^{\pi^*}(T) = 0)$ under short-selling constraints is larger than that without short-selling constraints. The intuitive reason is that the manager has less choices of the investment strategies under short-selling constraints.

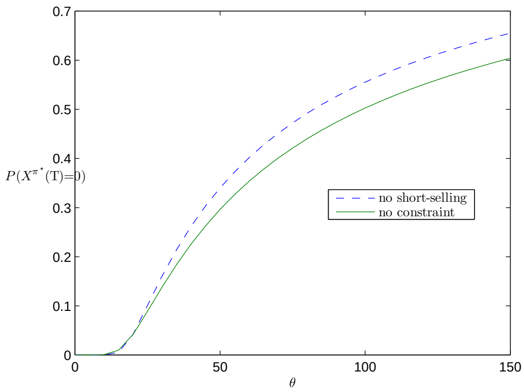

Figure 4: $P(X^{\pi^*}(T) = 0)$ versus $\theta, L = 0$

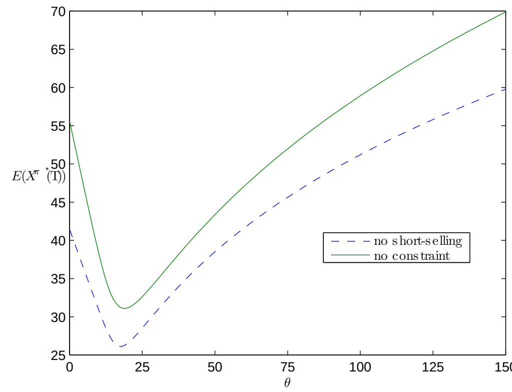

Figure 5: $E(X^{\pi^*}(T))$ versus $\theta, L = 0$

Figure 5 represents the relationship between $E(X^{\pi^*}(T))$ and θ. We can see that the unconstrained expectation is larger than the constrained one. We can also observe that the expectation has a V-shape pattern in the reference point. The threshold value is about $x_0e^{rT}+\int_0^T ce^{rs}ds = 20.1$. This is because if the manager puts all of his initial surplus and the contribution into the risk-free bond, then he will reach a level about 20.1. As explained in Chen et al. (2017), when the reference point is quite low (less than the threshold), a decrease in the reference point leads to an increase in the proportion of wealth invested in each risky asset, since the low reference point is very easy to attain by investing in the risk-free asset. When the reference point is larger than this threshold, the loss-averse manager in the domain of losses seeks more risk and therefore put much more money into risky assets to achieve his goal.

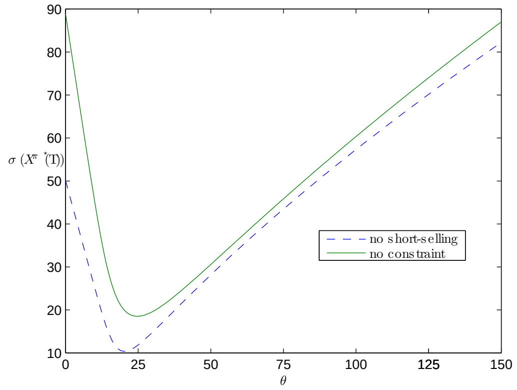

Figure 6: $\sigma(X^{\pi^*}(T))$ versus $\theta, L = 0$

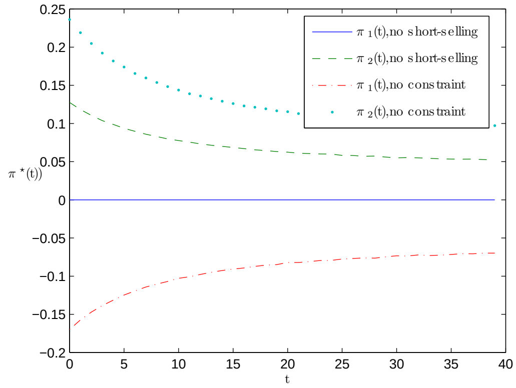

Figure 7: optimal portfolio weight, $L = 0, \theta = 20$

Figure 6 represents the relationship between the standard deviation of $X^{\pi^*}(T)$ and θ. Note that the curve of Figure 6 is similar to that of Figure 5. The variance also has a V-shaped pattern in the reference point. From it we also see that the short-selling constraints induce a preference for less volatility.

Figure 7 presents the optimal investment strategies with $\theta = 20$ (for notational convenience, we still use $\pi^*(t)$ to denote the optimal proportion of wealth invested in the risky assets in this section). We can observe that the proportions invested in the risky assets under short-selling or no short-selling constraints are both very low, which complies with the observations from Figures 5-6: when θ is set to be about $x_0e^{rT} + \int_0^T ce^{rs}ds$, the reference point can be easily obtain by investing a large proportion of wealth in the cash bond. We can also see that the manager can only invest in the stock under the short-selling constraints, while he can short sell the indexed bond under no short-selling constraints. Therefore, the manager may earn more gains when there are no restrictions on short-selling, which explains the observation from Figure 5.

Table 1 presents the probability $P(X^{\pi^*}(T) = 0)$, mean, standard deviation, and quantile values at low end and high ends of $X^{\pi^*}(T)$ for different θ with and without short-selling constraints to further illustrate the impact of θ on the optimal terminal wealth. We note that numerical results presented in Table 1 are consist with the observations from Figures 4-6. The probability $P(X^{\pi^*}(T) > 0)$, mean, standard deviation, and quantile values at high end of $X^{\pi^*}(T)$ under short-selling constraints are all less than those without short-selling constraints, since the manager has less choices of investment strategies under short-selling constraints. Quantile values at high end illustrate that the optimal terminal wealth under loss aversion in good economic states increases

Table 1: Means, standard deviations, quantile values and probabilities

	θ(short-selling constraints)				θ (no constraint)			
	0	20	40	100	0	20	40	100
mean	41.4	26.5	34.9	51.2	55.6	31.2	39.3	58.9
std dev	50.1	10.4	21.6	57.3	88.8	20.1	24.6	60.3
0.025 quantile	4.1	0	0	0	3.2	0	0	0
0.975 quantile	169.6	50.2	62.1	127.7	267.9	77.9	79.9	144.4
$P(X^{\pi^*}(T)=0)$	0	0.042	0.262	0.555	0	0.042	0.226	0.503

with θ, which is at the expense of enlarging the region of domain of losses, since $P(X^{\pi^*}(T)=0)$ increases with θ. We also see that there exists a threshold such that $E(X^{\pi^*}(T))$ and $\sigma(X^{\pi^*}(T))$ decrease with θ when it is less than this threshold and then increase with θ when it is larger than the threshold. Therefore, the choice of reference point θ is very important for the pension manager. A relatively low value of θ may bring an unsatisfactory utility of terminal wealth, while a too high value of θ will lead the pension manager to take more risk to achieve a terminal wealth higher than θ, which makes the members be more likely to get nothing.

4.2 Impact of PI constraint on terminal wealth

In this section, we analyze the effect of the PI constraint on the terminal wealth. Let $L=10, \theta=100$ and the other parameters are the same as those in the previous section.

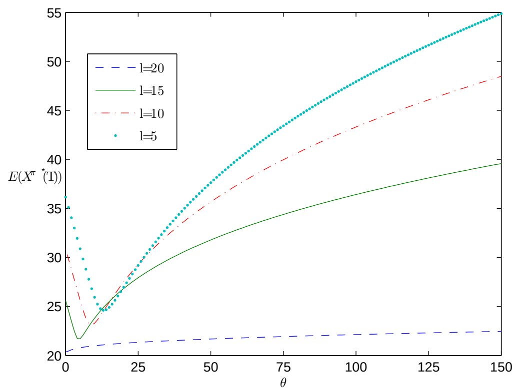

Figure 8: $E(X^{\pi^*}(T))$ versus θ for different L, short-selling constraints

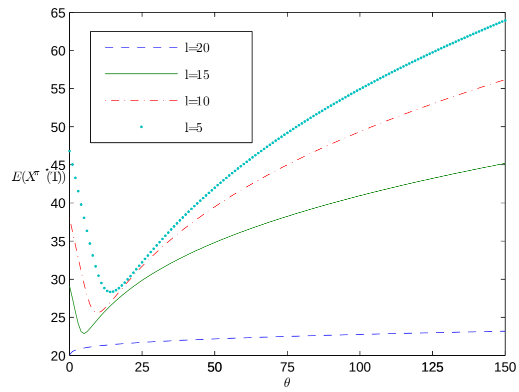

Figure 9: $E(X^{\pi^*}(T))$ versus θ for different L, no short-selling constraints

Figures 8-9 show the impacts of the PI and short-selling constraints on $E(X^{\pi^*}(T))$. From them we can see that similar to Figure 5, Figures 8-9 both have V-shaped curves and $E(X^{\pi^*}(T))$ without short-selling constraints is larger than that with short-selling constraints. Furthermore, an increase of L leads to a left shift of the threshold value of the V-shaped curve. The intuitive reason is that in order to reach the minimum guarantee, the pension manager puts aside Le^{-rT} at the initial time and if all of the net surplus is invested in the cash, then the net account value is $x_0e^{rT}+\int_0^T ce^{rs}ds-L$, which is about the threshold value. We can also note that on average an increase in the minimum guarantee L will lead to a decrease of $E(X^{\pi^*}(T))$, since the manager will hold a portfolio with less risk to achieve a higher L. In particular, for $L=20$, $E(X^{\pi^*}(T))$ is not sensitive to θ and is always a bit larger than 20. This is due to the fact that L is very close to the initial surplus $x_0 + \int_0^T ce^{-rs}ds$ and then only little money can be invested in the risky assets.

Figure 10 presents the optimal investment strategies for $L = 20$. From it we can see that whether or not there are short-selling constraints, the pension manager invests a very low proportion of money in the risk assets at initial time. Then with time passing, she/he invests almost all of the wealth in the cash bond so that she/he can reach the guarantee 20, which explains the curve of $E(X^{\pi^*}(T))$ for $L = 20$ in Figures 8-9.

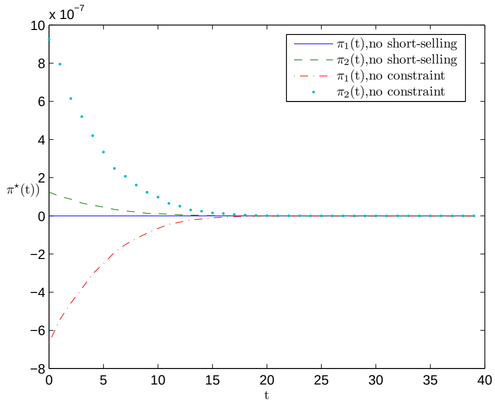

Figure 10: optimal portfolio weight for $\theta = 100, L = 20$

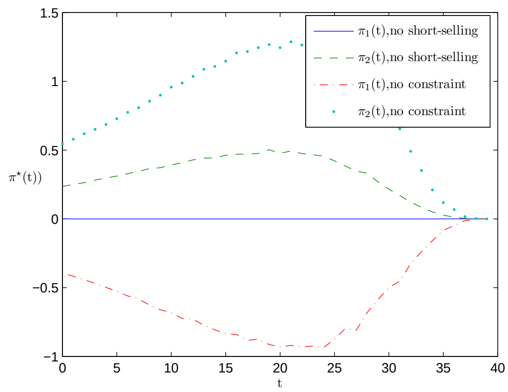

Figure 11: optimal portfolio weight for $\theta = 100, L = 10$

Figure 11 plots the optimal investment strategies for $L = 10$. Since the guarantee is easy to attain, the pension manager takes more risk to achieve the reference point $\theta = 100$. We note that when there are no short-selling constraints, the manager shorts about 40% of the indexed bond and invests about 50% of the stock at the beginning of the pension fund. Gradually the proportion of money in the stock increases about to 130% and the proportion of money in the indexed bond decreases to about -100%. When the pension manager achieves a wealth higher than θ in some states at time about 23, the proportions of money in the risky assets become very low and the proportion of money in the cash bond is very high so that she/he can obtain the guarantee L at retirement time. when short-selling is not allowed, the manager can only invest in the stock. Similarly, at the beginning of the pension plan, she/he increases the the proportion of money in the stock slowly and the proportion is much lower than that without short-selling constraints. Then from about time 23, the proportion of money in the stock decreases quickly and instead the proportion of money in the cash bond increases quickly to attain the guarantee.

Figure 12 displays the impacts of the initial surplus x_0 and the contribution rate c on $E(X^{\pi^*}(T))$ for a given $L = 10$. From them we can see that the impact of x_0 is similar to that of c. Increasing x_0 or c leads to an increase in the threshold of the V-shaped curves of $E(X^{\pi^*}(T))$ and $\sigma(X^{\pi^*}(T))$, since the threshold is about $x_0e^{rT} + \int_0^T ce^{rs}ds - L$. We can also observe that $E(X^{\pi^*}(T))$ increases with the initial surplus x_0 and the contribution rate c.

Figure 13 represents the effect of the correlation between the indexed bond and the stock on the expectation of $X^{\pi^*}(T)$. We can see that for a low ρ, the optimal investment strategy naturally satisfies the short-selling constraints and therefore the constrained expectation of $X^{\pi^*}(T)$ is the same as the unconstrained expectation. When ρ grows to $\frac{\vartheta_I}{\vartheta_S}$, the short-selling constraints lead the manager to only invest in the stock. Therefore, the constrained and unconstrained expectations of $X^{\pi^*}(T)$ are different, and $E(X^{\pi^*}(T))$ under short-selling constraints is less than that under no

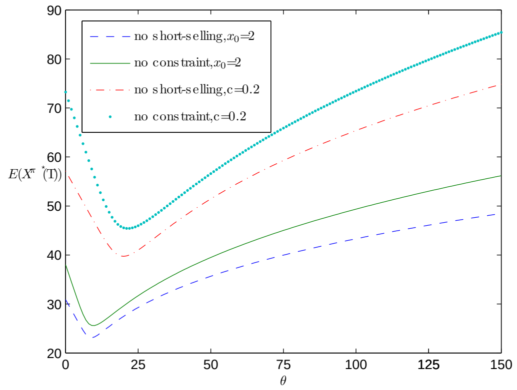

Figure 12: $E(X^{\pi^*}(T))$ versus θ for different $x_0, c, L = 10$

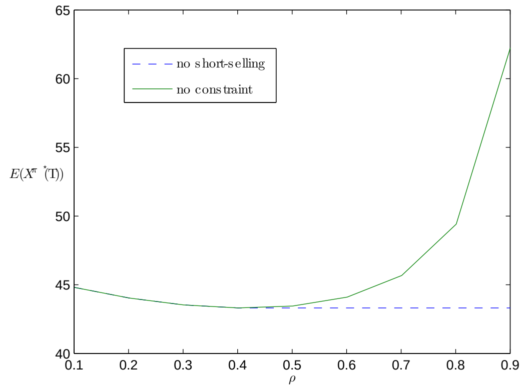

Figure 13: Effect of ρ on $E(X^{\pi^*}(T))$, $\theta = 100, L = 10$

Table 2: Means, standard deviations, quantile values and probabilities

	L(short-selling constraints)				L (no constraint)			
	5	10	15	20	5	10	15	20
mean	47.9	43.3	36.4	22.1	54.9	49.4	41.0	22.8
std dev	55.6	52.2	45.0	16.8	58.7	55.5	48.6	18.6
0.025 quantile	5	10	15	20	5	10	15	20
0.975 quantile	128.8	128.6	128.4	123.9	141.1	137.8	133.8	124.4
$P(X^{\pi^*}(T) = L)$	0.627	0.712	0.818	0.986	0.575	0.664	0.779	0.980

short-selling constraints, since the manager under no short-selling constraints can make much more gains by short selling risky assets.

Table 2 presents the probability $P(X^{\pi^*}(T) = L)$, mean, standard deviation, and quantile values at low end and high ends of $X^{\pi^*}(T)$ for a given $\theta = 100$ and different L with and without short-selling constraints to deeply analyze the impact of the guarantee on the optimal terminal wealth. From it we see that the probability $P(X^{\pi^*}(T) > L)$, mean, standard deviation, and quantile values at high end of $X^{\pi^*}(T)$ under short-selling constraints are all less than those without short-selling constraints, which implies short-selling constraints can reduce the investment risk. We can also observe that as L increases, the probability $P(X^{\pi^*}(T) > L)$, mean, standard deviation, and quantile values at high end all decrease, since the pension manager will become more prudent and make the optimal terminal wealth less volatile in order to achieve a higher guarantee.

Therefore, numerical results illustrate that by putting the short-selling and PI constraints can improve the risk management for the investors and the regulators.

5 Conclusions

The S-shaped utility can better reflect the pension plan manager's attitude toward risk since it includes loss aversion and risk seeking for losses. We investigate an investment problem for a DC plan manager under an S-shaped utility function. To better protect the member from the manager's gambling investment strategies, we incorporate the short-selling and PI constraints into the modelling. By using a concavification technique and a dual control method, we derive the explicit expressions for the optimal wealth process and the optimal investment strategy. Theoretical and numerical results show that the short-selling and PI constraints strictly improve the risk management. In particular, the PI constraint can well protect the members' benefits for the loss states by keeping the terminal wealth no less than the minimum guarantee.

The present work might be extended. One possible extension is that the continuous contribution payment and the minimum guarantee follow some stochastic processes. Another possible extension is to consider the optimal investment for a DC pension plan under both the short-selling and the VaR constraints. We leave these to future research.

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