Asymptotic Analysis for Target Asset Portfolio Allocation with Small Transaction Costs

Cong Liu and Harry Zheng*

Abstract. In this paper we discuss the asset allocation in the presence of small proportional transaction costs. The objective is to keep the asset portfolio close to a target portfolio and at the same time to reduce the trading cost in doing so. We derive the variational inequality and prove a verification theorem. Furthermore, we apply the second order asymptotic expansion method to characterize explicitly the optimal no transaction region when the transaction cost is small and show that the boundary points are asymmetric in relation to the target portfolio position, in contrast to the symmetric relation when only the first order asymptotic expansion method is used, and the leading order is a constant proportion of the cubic root of the small transaction cost. In addition, we use the asymptotic results for the boundary points and obtain an expansion for the value function. The results are illustrated in the numerical example.

Keywords. Target asset portfolio allocation, tracking error, proportional transaction costs, Magnus expansion, asymptotic analysis.

JEL Classification. G11.

1 Introduction

In pension and insurance fund management it is often necessary to allocate the fund in different asset classes with fixed proportions of wealth invested in each one of them. This may be due to the regulatory requirement, asset diversification, liability structure, etc., see Meyer and Meyer (2004) and Dionne (2013). Such a strategy is called constant mix (or rebalancing of investments) trading strategy, see Ang (2014). If the market is complete, it is easy to achieve the fixed proportion of wealth in a specific asset by continuously trading the underlying asset, but it is impractical in the presence of transaction costs (brokerage fees, taxes, etc.). The fund manager then faces two conflicting objectives: reducing the total transaction cost and reducing the tracking error (dispersion from the target), see Grinold and Kahn (2000). This paper discusses optimal trading strategies in the presence of small transaction costs. The problem is related to utility maximization with transaction costs. We next give a literature review on the subject.

The work of Merton (1969) is the starting point of continuous-time utility based portfolio theory. With the help of the stochastic control theory, the portfolio problem can be formulated as a Hamilton-Jacobi-Bellman (HJB) equation, which can be solved explicitly for a hyperbolic absolute risk aversion investor. The corresponding optimal investment strategy involves continuously rebalancing the portfolio to maintain a constant fraction of

*Department of Mathematics, Imperial College, London SW7 2BZ, UK. cong.liu11@imperial.ac.uk and h.zheng@imperial.ac.uk.

total wealth in each asset during the whole investment period. However, this optimal policy is unrealistic in the presence of transaction costs. Magill and Constantinides (1976) are the first to incorporate proportional transaction costs into Merton's model. Their heuristic analysis for the infinite horizon investment and consumption problem gives a fundamental insight into the optimal strategy and the existence of the no transaction region. Davis and Norman (1990) provide a rigorous mathematical analysis for the same problem by applying the stochastic control theory. Using "continuous control" (consumption) and "singular control" (transaction), they show that the investor's optimal trading strategy is to maintain the portfolio position inside the no transaction region. If the initial portfolio position is outside the no transaction region, the investor should immediately sell or buy stock in order to move to its boundary. The investor then trades only when the portfolio position is at the boundary of the no transaction region, and only as much as necessary to keep it from exiting the no transaction region, while no trading occurs in the interior of the region. The optimal policies are determined by the solution of a free boundary problem, where the free boundaries correspond to the optimal buying and selling policies. Shreve and Soner (1994) generalize the results of Davis and Norman (1990) with the theory of viscosity solutions.

In practice transaction costs are small relative to values of transactions. In the limit of small transaction costs, Atkinson and Wilmott (1995) apply the perturbation method to derive an approximate solution to a model with transaction cost of a fixed fraction of portfolio value in Morton and Pliska (1995). Janeček and Shreve (2004) provide a rigorous derivation of the asymptotic expansions of the value function and boundaries of the no transaction region for an investor with the power utility. Bichuch (2011) presents a rigorous proof for a finite horizon case. All aforementioned papers use the stochastic control methods. Some recent papers obtain the power series expansions of arbitrary order for the optimal value function and the boundaries of the no transaction region with the duality theory and the shadow price method, see Gerhold et al. (2012, 2014) for details and references therein.

Rogers (2004) observes that the impact of small transaction costs consists of two parts: the direct cost incurred by actual trading and the displacement cost due to deviating from the frictionless target position. Leland (2000) postulates a "cost function" as the discounted sum of the trading cost and the tracking error cost. Inspired by these works we formulate the target asset allocation problem with transaction costs as a cost minimization problem made of two parts, similar to those of Leland (2000). We prove a verification theorem for optimality of the local-time trading strategy. We use the Magnus expansion to characterise the solution of non-autonomous and non-homogeneous systems of ordinary differential equations (ODEs). We apply the first and second order asymptotic expansion method to describe explicitly the optimal no transaction region when the transaction cost is small, which is not discussed in Leland (2000). We show that the boundary points are asymmetric in relation to the target portfolio position, in contrast to the symmetric relations when only the first order asymptotic expansion method is used, and the leading order is a constant proportion (depending on the target asset portfolio) of the cubic root of the small transaction cost. The results and methods discussed in this paper can provide useful insights for insurance and pension fund managers in making asset allocation decisions in the presence of proportional transaction costs.

The paper is organized as follows. Section 2 describes the model and the cost minimization problem. Section 3 discusses the HJB variational inequality and the verification theorem and applies the Magnus expansion method to characterize the optimal solution in the no transaction region. Section 4 performs asymptotic analysis of the no transaction region and the value function with respect to the transaction cost parameter and shows that the boundary points are symmetric to the given target portfolio level with the first order asymptotic expansion method but are asymmetric with the second order asymptotic expansion method. Section 5 gives numerical examples of the main results. Section 6 concludes. Appendix contains the proofs of the theorems.

2 Problem Formulation

Assume $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ is a filtered probability space and the market consists of two securities: one riskless asset S^0 paying a fixed interest rate r, i.e., $S^0_t = e^{rt}$, and one risky asset S following a geometric Brownian motion process

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion with $\{\mathscr{F}_t\}_{t\geq 0}$ being the natural filtration of W and satisfying the usual conditions, μ ($\mu > r$) and σ^2 are positive constants representing the instantaneous rate of return and variance of the stock, respectively. Assume the initial position of an agent is x dollars in the money market and y dollars in stock. Assume there is a proportional transaction cost in the sense that the investor pays a fixed fraction ϵ of the amount transacted on buying or selling the stock.

A *trading strategy* is any pair $(L_t, M_t)_{t\geq 0}$ of non-decreasing and right continuous adapted processes with $L_{0^-} = M_{0^-} = 0$. L_t and M_t represent the cumulative dollar values of buying and selling the stock respectively up to time t.

Denote by X_t and Y_t the monetary values of the riskless and risky positions, respectively. The self-financing condition and the dynamics of S^0_t and S_t imply that

$$\begin{aligned} dX_t &= rX_t dt - (1+\epsilon)dL_t + (1-\epsilon)dM_t, \\ dY_t &= \mu Y_t dt + \sigma Y_t dW_t + dL_t - dM_t. \end{aligned}$$

with $X_{0^-} = x$, $Y_{0^-} = y$, where $\epsilon \in (0,1)$ is a constant proportional transaction cost per dollar trading. The dollar transaction cost at time t is given by $\epsilon dL_t + \epsilon dM_t$. Note that

$$\begin{aligned} X_0 &= x - (1+\epsilon)L_0 + (1-\epsilon)M_0, \\ Y_0 &= y + L_0 - M_0 \end{aligned}$$

may differ from X_{0^-}, Y_{0^-} because of the initial transaction at time 0.

Define the *solvency region*

$$\mathscr{S} := \{(x,y) \in \mathbb{R}^2 : x + (1+\epsilon)y \geq 0, x + (1-\epsilon)y \geq 0\}.$$

We can re-parameterize the problem by introducing new variables $w_t = X_t + Y_t$ (the total wealth at time t) and $\pi_t = Y_t/w_t$ (the fraction of total wealth held in stock at time t). The return of wealth, using the dynamics for X_t and Y_t, can be calculated to follow

$$dw_t = [r + (\mu - r)\pi_t]w_t dt + \sigma\pi_t w_t dW_t - \epsilon dL_t - \epsilon dM_t.$$

with $w_{0^-} = x+y$, denoted by w. When there is an initial transaction $w_0 = w - \epsilon(L_0 + M_0)$. The proportion of wealth in stock, π_t, satisfies the following stochastic differential equation

$$d\pi_t = (\mu - r - \sigma^2\pi_t)\pi_t(1-\pi_t)dt + \sigma\pi_t(1-\pi_t)dW_t + d\Pi^+_t - d\Pi^-_t,$$

where $d\Pi_t^+ = (1+\epsilon\pi_t)dL_t/w_{t^-}$, $d\Pi_t^- = (1-\epsilon\pi_t)dM_t/w_{t^-}$, and $\pi_{0^-} = y/w$, denoted by π. It can be easily checked that

$$\begin{aligned}(w_{t^-} - \epsilon dL_t)(\pi_{t^-} + d\Pi_t^+) - w_{t^-}\pi_{t^-} &= dL_t \\ (w_{t^-} - \epsilon dM_t)(\pi_{t^-} - d\Pi_t^-) - w_{t^-}\pi_{t^-} &= -dM_t,\end{aligned}$$

which imply that $d\Pi_t^+$ and $d\Pi_t^-$ are the instantaneous absolute changes of π_t at time t as a result of buying and selling. Note that Π_0^+ and Π_0^- may not be zero due to possible transactions at time 0.

The solvency region $\mathscr{S}$ can be transformed as an interval with respect to π:

$$\mathscr{S} = \{\pi \in \mathbb{R} : -1/\epsilon \leq \pi \leq 1/\epsilon\}.$$

A trading strategy (Π^+, Π^-) is *admissible* if (Π^+, Π^-) ensures $\pi_t \in \mathscr{S}$ for all $t \geq 0$ and satisfies

$$\mathbb{E}_\pi\left[\int_0^\infty e^{-\rho t} d\Pi_t^+\right] < \infty \text{ and } \mathbb{E}_\pi\left[\int_0^\infty e^{-\rho t} d\Pi_t^-\right] < \infty, \tag{1}$$

where ρ is a discount factor and $\mathbb{E}_\pi$ the conditional expectation operator with $\pi_{0^-} = \pi$. Condition (1) guarantees finite expected value of discounted total changes of risky proportions due to transactions. It rules out strategies with infinite discounted amount of transactions. The set of all admissible strategies given initial position π is denoted by $\mathcal{A}(\pi)$. Note that ρ is not necessarily equal to r, the riskfree interest rate, as ρ is a subjective discount factor used by a portfolio manager for future transactions or opportunity costs, whereas r is an objective one used in the market as a whole.

The trading strategy (Π^+, Π^-) is a control on the state process π. It requires a portfolio manager to monitor the risky proportion process and make the trading decision based on the shift of the portfolio's risky position to the target position. In this paper we focus on the asset allocation problem in which the target asset ratio π^* (for the risky asset) is given exogenously.

Assumption 2.1. *Assume $0 < \pi^* < 1$, i.e., hedging, rather than leveraging or short-selling, is the main objective when there is no transaction cost.*

We do not assume any specific utility function over wealth or intermediate consumption since it can rarely be specified by portfolio managers. Instead, we postulate a "cost function" which may be more financially sensible to investment managers (see, for example, Grinold and Kahn (2000)). The total cost due to the existence of proportional transaction costs consists of two parts: the trading cost and the cost associated with the *tracking error* – the divergence from the desired target proportions, π^*, to the actual ratios π_t. The tracking error cost can be reduced by trading more frequently, which leads to greater transactions costs.

The expected value of the cost at time 0^- depends on the trading strategy (Π^+, Π^-) and on the initial asset proportion π. Following Leland (2000) and Pliska and Suzuki (2004), we define the cost function as

$$J(\pi; \Pi^+, \Pi^-) = \mathbb{E}_\pi\left[\int_0^\infty e^{-\rho t}\lambda\sigma^2(\pi_t - \pi^*)^2 dt + \epsilon\int_0^\infty e^{-\rho t} d\Pi_t^+ + \epsilon\int_0^\infty e^{-\rho t} d\Pi_t^-\right],$$

where $\lambda\sigma^2(\pi_t - \pi^*)^2 dt$ is the incremental tracking error cost at time t, which is assumed to be proportional to the variance of the tracking error and λ is the "price of tracking error", and $\epsilon d\Pi_t^+ + \epsilon d\Pi_t^-$ is the incremental cost due to transactions at time t. Similar to Pliska and Suzuki (2004), we model the transaction cost to be proportional to the movement in the risky position, not proportional to the dollar amount change as commonly used in the literature. (Note that $\epsilon d\Pi_t^+$ is approximately equal to $\epsilon dL_t/w_{t-}$, the incremental transaction cost per unit wealth at time t, when the higher order term $\mathcal{O}(\epsilon^2)$ is ignored.)

The portfolio manager seeks an admissible trading strategy (Π^{+*}, Π^{-*}) that minimizes $J(\pi; \Pi^+, \Pi^-)$ over the set $\mathcal{A}(\pi)$ and the value function is defined by

$$f(\pi) = \inf_{(\Pi^+,\Pi^-)\in\mathcal{A}(\pi)} J(\pi; \Pi^+, \Pi^-).$$

3 Verification Theorem and Magnus Expansion

To get some idea as to the nature of optimal policies, we consider a restricted class of policies in which Π^+ and Π^- are constrained to be absolutely continuous with respect to the Lebesgue measure with bounded derivatives, i.e.

$$\Pi_t^+ = \int_0^t \eta_s^+ ds, \quad \Pi_t^- = \int_0^t \eta_s^- ds, \quad 0 \le \eta_s^+, \eta_s^- \le \kappa,$$

where κ is a positive constant and η^+ and η^- are predictable processes. The optimality equation for the function f is

$$\inf_{0\le\eta^+,\eta^-\le\kappa} \left\{ \mathcal{L}f(\pi) - \rho f(\pi) + \lambda\sigma^2(\pi - \pi^*)^2 + \eta^+[\epsilon + f'(\pi)] + \eta^-[\epsilon - f'(\pi)] \right\} = 0,$$

where $\mathcal{L}$ is the infinitesimal generator of π, defined by

$$\mathcal{L}f(\pi) = \frac{1}{2}\sigma^2\pi^2(1-\pi)^2 f''(\pi) + (\mu - r - \sigma^2\pi)\pi(1-\pi)f'(\pi).$$

The infimum is achieved as following points:

$$\eta^+ = \begin{cases} \kappa & \text{if} \quad \epsilon + f'(\pi) \le 0 \\ 0 & \text{if} \quad \epsilon + f'(\pi) > 0 \end{cases}, \quad \eta^- = \begin{cases} \kappa & \text{if} \quad \epsilon - f'(\pi) \le 0 \\ 0 & \text{if} \quad \epsilon - f'(\pi) > 0 \end{cases}.$$

This indicates that buying and selling either take place at maximum rate or not at all. The solvency region $\mathscr{S}$ splits into three regions, "buy", $(\mathcal{B})$, "sell"$(\mathcal{S})$ and "no transact"$(\mathcal{NT})$. At the boundary between $\mathcal{B}$ and $\mathcal{NT}$ regions,

$$f'(\pi) = -\epsilon,$$

whereas at the boundary between $\mathcal{NT}$ and $\mathcal{S}$ regions,

$$f'(\pi) = \epsilon.$$

The $\mathcal{NT}$ region is captured by inequalities

$$-\epsilon < f'(\pi) < \epsilon. \tag{2}$$

Assuming the no transaction region coincides with some interval $\pi_- < \pi < \pi_+$ to be determined, and noting that at π_- the left inequality in (2) holds as equality, while at π_+ the right inequality holds as equality, and accordingly, $\mathcal{B} = [-1/\epsilon, \pi_-)$ and $\mathcal{S} = (\pi_+, 1/\epsilon]$.

Using the same approach as in Harrison and Taksar (1983), we have that the value function f satisfies the following HJB variational inequality

$$\min\{\mathcal{L}f(\pi) - \rho f(\pi) + \lambda(\pi - \pi^*)^2, \epsilon + f'(\pi), \epsilon - f'(\pi)\} = 0.$$

We next state a verification theorem that characterizes the optimal trading strategies and the optimal portfolio proportion process. Specifically, the optimal Π^{+*} and Π^{-*} are the local times at the boundary points of $\mathcal{NT}$ and the optimal π is a reflecting diffusion process in the interval $\mathcal{NT}$.

Theorem 3.1. *Suppose there are constants π_- and π_+ ($\pi_- < \pi_+$) and a function $f \in \mathscr{C}^2(\mathscr{S})$ such that*

$$f'(\pi) \geq -\epsilon \quad \text{on } \mathscr{S} \text{ with equality on } [-1/\epsilon, \pi_-], \tag{3}$$

$$f'(\pi) \leq \ \epsilon \quad \text{on } \mathscr{S} \text{ with equality on } [\ \pi_+, 1/\epsilon], \tag{4}$$

$$\mathcal{L}f(\pi) - \rho f(\pi) + \lambda^2(\pi - \pi^*)^2 \geq 0 \text{ on } \mathscr{S} \text{ with equality on } [\pi_-, \pi_+]. \tag{5}$$

Let $\mathcal{NT}$ denote the closed interval $[\pi_-, \pi_+]$. Then for any initial endowment $\pi \in \mathcal{NT}$, the trading strategy $(\Pi^{+(*)}, \Pi^{-(*)})$, defined by

$$\Pi_t^{+(*)} = \int_0^t \mathbb{1}_{\{\pi_s = \pi_-\}} d\Pi_s^{+(*)} \text{ and } \Pi_t^{-(*)} = \int_0^t \mathbb{1}_{\{\pi_s = \pi_+\}} d\Pi_s^{-(*)},$$

is optimal. If $\pi \in \mathscr{S} \backslash \mathcal{NT}$, then an immediate transaction to the closest endpoints of $\mathcal{NT}$, i.e. $\{\pi_-, \pi_+\}$, followed by application of this policy is optimal. The value function is $f(\pi)$.

Proof. See Appendix A. $\square$

The value function f in the no transaction region satisfies the following equation

$$\frac{1}{2}\sigma^2\pi^2(1-\pi)^2 f''(\pi) + (\mu - r - \sigma^2\pi)\pi(1-\pi)f'(\pi) - \rho f(\pi) = -\lambda\sigma^2(\pi - \pi^*)^2 \tag{6}$$

with two free boundary conditions

$$f'(\pi_-) = -\epsilon, \qquad f'(\pi_+) = \epsilon. \tag{7}$$

Condition (7) is not enough to uniquely determine a solution as π_- and π_+ are free boundary points. The optimal boundaries are the ones that also satisfy the following smooth-pasting condition, see Dumas(1991),

$$f''(\pi_-) = 0, \qquad f''(\pi_+) = 0. \tag{8}$$

We may compute $f(\pi_-)$ and $f(\pi_+)$ by substituting (7) and (8) into (6) with $\pi = \pi_-$ and π_+, which gives

$$f(\pi_-) = \zeta(-\epsilon, \pi_-), \qquad f(\pi_+) = \zeta(\epsilon, \pi_+),$$

where

$$\zeta(x, y) = \frac{x[y(1-y)(\mu - r - \sigma^2 y)] + \lambda\sigma^2(y - \pi^*)^2}{\rho}. \tag{9}$$

We can write (6) equivalently as

$$\phi'(\pi) = A(\pi)\phi(\pi) + \psi(\pi), \tag{10}$$

where

$$\phi(\pi) = \begin{bmatrix} f(\pi) \\ f'(\pi) \end{bmatrix}, \quad A(\pi) = \begin{bmatrix} 0 & 1 \\ \frac{2\rho}{\sigma^2\pi^2(1-\pi)^2} & -\frac{2(\mu-r-\sigma^2\pi)}{\sigma^2\pi(1-\pi)} \end{bmatrix}, \quad \psi(\pi) = \begin{bmatrix} 0 \\ -\frac{2\lambda(\pi-\pi^*)^2}{\pi^2(1-\pi)^2} \end{bmatrix}. \tag{11}$$

The free boundary conditions become

$$\phi_0 := \phi(\pi_-) = \begin{bmatrix} \zeta(-\epsilon, \pi_-) \\ -\epsilon \end{bmatrix}, \quad \phi_1 := \phi(\pi_+) = \begin{bmatrix} \zeta(\epsilon, \pi_+) \\ \epsilon \end{bmatrix}. \tag{12}$$

(10) is a 2-dimensional first order non-autonomous and non-homogeneous free boundary ODE. The matrix function A is continuous on $[\pi_-, \pi_+]$, if Assumption 2.1 is satisfied and ϵ is small, see Remark 3.2.

Remark 3.2. *Note that, by writing the original free boundary problem (6) into the linear system (10), we introduce two singular points $\pi = 0$ and $\pi = 1$. However, with Assumption 2.1 and a sufficiently small ϵ, the no transaction region $[\pi_-, \pi_+]$ is a closed subinterval of $(0, 1)$ and there is no singularity in (6). This can be checked by asymptotic results in Theorem 4.1 and Theorem 4.3, and by numerical results in Section 5.*

We can use the Magnus expansion method to solve equation (10). The fundamental solution to $\phi'(\pi) = A(\pi)\phi(\pi)$ is given by a matrix exponential (see Blanes et al. (2009))

$$\Phi(\pi) = \exp(\Omega(\pi)) = \sum_{i=0}^{\infty} \frac{1}{i!}\Omega^i(\pi) \tag{13}$$

with $\Phi(\pi_-) = \mathbb{I}_2$, where $\mathbb{I}_2$ is an identity matrix and $\Omega(\pi)$ is given by the matrix series expansion

$$\Omega(\pi) = \sum_{k=1}^{\infty} \Omega_k(\pi) \tag{14}$$

with

$$\begin{aligned}
\Omega_1(\pi) &= \int_{\pi_-}^{\pi} A(\pi_1)d\pi_1, \\
\Omega_2(\pi) &= \frac{1}{2}\int_{\pi_-}^{\pi} d\pi_1 \int_{\pi_-}^{\pi_1} d\pi_2 \, [A(\pi_1), A(\pi_2)], \\
\Omega_3(\pi) &= \frac{1}{6}\int_{\pi_-}^{\pi} d\pi_1 \int_{\pi_-}^{\pi_1} d\pi_2 \int_{\pi_-}^{\pi_2} d\pi_3 \, ([A(\pi_1), [A(\pi_2), A(\pi_3)]] + [A(\pi_3), [A(\pi_2), A(\pi_1)]]), \\
&\vdots
\end{aligned}$$

where $[M_1, M_2] \equiv M_1M_2 - M_2M_1$ is a commutator operator of two matrices M_1 and M_2. The series (14) is called the *Magnus expansion*. Note that $\Omega_k(\pi) = 0$ for $k \geq 2$ if A is a constant matrix or a matrix of dimension 1. In that case, $\Phi(\pi)$ is a standard matrix exponential. However, A defined in (11) is a 2×2 non-constant matrix, we have to use the Magnus expansion to solve equation (10). It is known (see Blanes et al. (2009)) that the Magnus expansion (14) converges and the equality (13) holds if $A(\pi)$ is invertible and $\int_{\pi_-}^{\pi} \|A(\nu)\|_2 d\nu < \boldsymbol{\pi}$, where $\boldsymbol{\pi}$ is the mathematical constant $3.1415926\ldots$. Under the assumption of small transaction cost ϵ, the above sufficient condition is satisfied and thus the convergence of the Magnus expansion is guaranteed. The inverse of $\Phi(\pi)$ is equal to $\Phi^{-1}(\pi) = \exp(-\Omega(\pi))$.

The general solution to (10) is given by

$$\phi(\pi) = \Phi(\pi)\phi_0 + \Phi(\pi) \int_{\pi_-}^{\pi} \Phi^{-1}(\nu)\psi(\nu)d\nu \tag{15}$$

with boundary conditions (12), which leads to

$$\int_{\pi_-}^{\pi_+} \Phi^{-1}(\pi)\psi(\pi)d\pi = \Phi^{-1}(\pi_+)\phi_1 - \phi_0. \tag{16}$$

4 Main Results on Asymptotic Analysis

In this section we discuss the asymptotic expansion of π_+ and π_- in terms of ϵ. Define

$$\delta_+ = \pi_+ - \pi^*, \quad \delta_- = \pi^* - \pi_-, \quad \delta = \delta_+ + \delta_-, \tag{17}$$

the width of the right and left part of the no transaction region around π^* and the total width of the no transaction region, respectively,

We now state the main result on the first order expansion of the no transaction region.

Theorem 4.1. *Let $\Phi^{-1}(\pi)$ be expanded to its first order such that $\Phi^{-1}(\pi) = \mathbb{I}_2 - \Omega_1(\pi)$ and δ_+ and δ_- be defined in (17). Then δ_+ and δ_- are given by*

$$\delta_+ = \alpha\epsilon^{1/3} + o\left(\epsilon^{1/3}\right), \quad \delta_- = \alpha\epsilon^{1/3} + o\left(\epsilon^{1/3}\right),$$

where

$$\alpha = \left(\frac{3(\pi^*)^2(1-\pi^*)^2}{4\lambda}\right)^{1/3}. \tag{18}$$

Proof. See Appendix B. □

Theorem 4.1 says that the effect of proportional transaction costs to the width of the no transaction region is of order $\mathcal{O}(\epsilon^{1/3})$. The no transaction region is symmetric, up to leading order $\mathcal{O}(\epsilon^{1/3})$, with respect to the target strategy π^*.

Remark 4.2. *Note that, strictly speaking, the first order expansion of $\Phi^{-1}(\pi)$ should be $\Phi^{-1}(\pi) = \mathbb{I}_2 - \Omega(\pi) = \mathbb{I}_2 - \sum_{k=1}^{\infty} \Omega_k(\pi)$. However, components of $\Omega_k(\pi), k \geq 2$, are with orders of at least $\mathcal{O}(\epsilon^{5/3})$ (see the proof of Theorem 4.3 in Appendix C) and therefore do not affect the asymptotic result in Theorem 4.1. Note also that, in the presence of proportional transaction costs, the width of the no transaction region has leading order $\mathcal{O}(\epsilon^{1/3})$. This result is obtained by directly applying asymptotic analysis to the Magnus representation of the solution to (10), not by first assuming that the width of the no trade region can be expanded in powers of $\epsilon^{1/3}$ and then calculating the leading order coefficient.*

We next state the result on the second order expansion of the no transaction region.

Theorem 4.3. *Let $\Phi^{-1}(\pi)$ be expanded to its second order such that $\Phi^{-1}(\pi) = \mathbb{I}_2 - \Omega_1(\pi) + \frac{1}{2}\Omega_1^2(\pi)$ and assume δ_+ and δ_-, defined in (17), can be represented in terms of powers of $\epsilon^{1/3}$. Then δ_+ and δ_- are given by*

$$\delta_+ = \alpha\epsilon^{1/3} + \beta\epsilon^{2/3} + \gamma\epsilon + \mathcal{O}(\epsilon), \quad \delta_- = \alpha\epsilon^{1/3} - \beta\epsilon^{2/3} + \gamma\epsilon + \mathcal{O}(\epsilon), \tag{19}$$

where α is defined in (18) and

$$\beta = -\frac{\pi^*(1-\pi^*)(\mu - r - \sigma^2\pi^*)}{2\alpha\lambda\sigma^2} \tag{20}$$

and γ is some constant.

Proof. See Appendix C. □

Theorems 4.1 and 4.3 tell us that the transaction cost has the same 1/3-order effect on the width of both intervals $[\pi_-, \pi^*]$ and $[\pi^*, \pi_+]$, but asymmetric 2/3-order effects. In case of $0 < \pi^* < 1$, this asymmetry is related to the relationship of π^*, the target strategy, and $(\mu - r)/\sigma^2$, the relative local risk premium. If $\pi^* > (\mu - r)/\sigma^2$, we have $\beta > 0$ and $\delta_+ > \delta_-$, up to order ϵ, which implies that the portfolio manager gives relatively more tolerance to the trading strategy being greater than the target proportion. If $\pi^* = (\mu - r)/\sigma^2$, then $\delta_+ = \delta_-$, which means that the manager gives equal tolerance to the divergence of trading strategy away from the target.

To the best of our knowledge, there is no such research that expands the boundaries of the no transaction region to order $\mathcal{O}(\epsilon^{2/3})$ and derives explicitly coefficient of $\epsilon^{2/3}$- order term for both the left and right boundaries.

Using the results of Theorems 4.1 and 4.3, we perform a similar analysis as above to the expansion of the value function in the no transaction region. The main result is the following theorem.

Theorem 4.4. *Let the expansion of δ_+ and δ_- be given as in (19). Then the value function f in the no transaction region can be written as,*

$$\begin{aligned} f(\pi) \;=\; & -\frac{\lambda}{6(\pi^*)^2(1-\pi^*)^2}(\pi - \pi^*)^4 + \left[\frac{\lambda\sigma^2\alpha^2}{\rho} + \frac{\lambda\alpha^2}{(\pi^*)^2(1-\pi^*)^2}(\pi - \pi^*)^2\right]\epsilon^{2/3} \\ & + \left[\frac{\lambda\sigma^2}{\rho}(\beta^2 + 2\alpha\gamma) + \frac{(\mu - r - \sigma^2\pi^*)(1 - 2\pi^*) - \sigma^2\pi^*(1-\pi^*)}{\rho}\alpha - \frac{5\alpha}{8}\right]\epsilon^{4/3} \\ & + \mathcal{O}(\epsilon^{5/3}). \end{aligned} \tag{21}$$

Proof. See Appendix D. □

Theorem 4.4 indicates that the presence of proportional transaction costs has a uniform impact to the value function $f(\pi)$ in the order $\mathcal{O}(\epsilon^{2/3})$ and has no impact of order $\mathcal{O}(\epsilon)$.

Remark 4.5. *In special cases when π equals π_+, π_- and π^*, we have, ignoring the higher order term $\mathcal{O}(\epsilon^{5/3})$,*

$$f(\pi_+) = f(\pi_-) = \frac{\lambda\sigma^2\alpha^2}{\rho}\epsilon^{2/3} + \xi\epsilon^{4/3} \text{ and } f(\pi^*) \;= f(\pi_+) - \frac{5\alpha}{8}\epsilon^{4/3},$$

where

$$\xi = \frac{\lambda\sigma^2(\beta^2 + 2\alpha\gamma)}{\rho} + \frac{(\mu - r - \sigma^2\pi^*)(1-2\pi^*) - \sigma^2\pi^*(1-\pi^*)}{\rho}\alpha.$$

Since $\alpha > 0$ we have $f(\pi^*) < f(\pi_-)$ and $f(\pi^*) < f(\pi_+)$.

5 Numerical Results

The effect of proportional transaction costs on the width of the no transaction region is of order $\epsilon^{1/3}$, see (19). The true buying and selling boundaries (π_-^* and π_+^*) are solutions to equality (16), as functions of ϵ. However, we only have a series rather than an explicit expression for the fundamental solution $\Phi(\pi)$. A direct comparison between $\{\pi_-^*, \pi_+^*\}$ and $\{\pi_-, \pi_+\}$, obtained using results in (38), is impossible. Instead, we turn to an alternative $\{\pi'_-, \pi'_+\}$ to $\{\pi_-^*, \pi_+^*\}$, which is obtained by solving (16) with approximation $\Phi^{-1}(\pi) = \exp(-\Omega_1(\pi))$. It is clear that $\{\pi'_-, \pi'_+\}$ is, though very close, not equal to $\{\pi_-^*, \pi_+^*\}$ and thus can only be used as a reference.

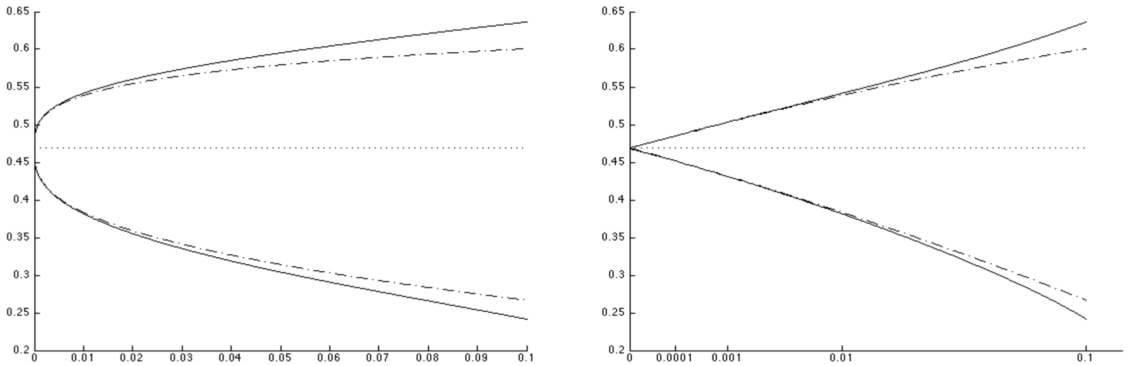

Figure 1: Buy (lower) and sell (upper) boundaries (vertical axis, as risky weights) as functions of the transaction costs ϵ, in linear scale (left panel) and cubic scale (right panel). Together with target strategy π^* (dotted line). The plot compares $\{\pi_-, \pi_+\}$ (dashed) and the $\{\pi'_-, \pi'_+\}$ (solid). Parameters are $\mu = 0.2$, $\sigma = 0.4$, $\lambda = 1$, $\rho = 0.1$ and $r = 0.05$. Target proportion $\pi^* = 0.5 * (\mu - r)/\sigma^2 = 0.469$.

In Figure 1, we compare the asymptotic (to the order $\epsilon^{2/3}$) boundaries $\{\pi_-, \pi_+\}$ and reference boundaries $\{\pi'_-, \pi'_+\}$ both in linear scale and in cubic scale. The asymptotic no transaction region is narrower than the reference no transaction region, due to insufficient expansion. We include also in Figure 1 the target strategy π^* to show the asymmetry of the upper and lower "half" of the no transaction region. In this example, we have chosen π^* such that $\pi^* < (\mu - r)/\sigma^2$. Thereby β defined in (20) is negative and $\delta_+ < \delta_-$ (see (19)). This coincides with the plots in Figure 1. In addition, for a given $\pi^* \in (0, 1)$ and s sufficiently small ϵ, the no transaction region $[\pi_-, \pi_+]$ is indeed a closed sub-interval of $(0, 1)$. Thus we have checked numerically the correctness of Remark 3.2.

It can be observed that, when transaction cost is sufficiently small, especially when $\epsilon \leq 0.01$, π'_+ and π_+ (same for π'_- and π_-) almost coincide. Their closeness can also be observed in Figure 2, where we plot the absolute and relative error between approximated boundaries and the reference boundaries.

Recall that the asymptotic expansion for the value function f in the no transaction region is given in (21), where coefficient of term involving $\epsilon^{4/3}$ includes an unknown constant γ. However, this term is a constant once transaction cost parameter ϵ is given and thus is only a shift to f and does not change the shape of the latter. Therefore, we can plot the value function f in the no transaction region numerically up to order $\epsilon^{4/3}$ by using (21) and ignoring the terms of order $\epsilon^{4/3}$ and higher.

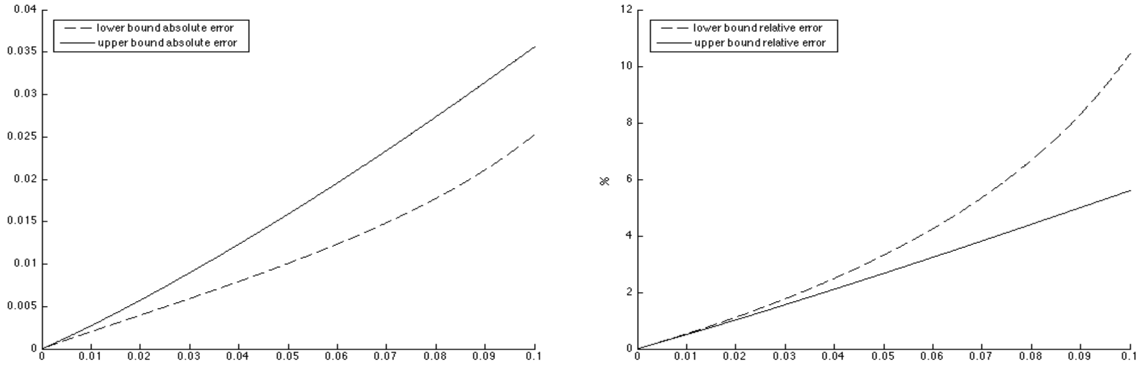

Figure 2: Absolute (left panel) and relative (right panel) error between the approximated weights and the reference weights. Parameters are $\mu = 0.2$, $\sigma = 0.4$, $\lambda = 1$, $\rho = 0.1$ and $r = 0.05$.

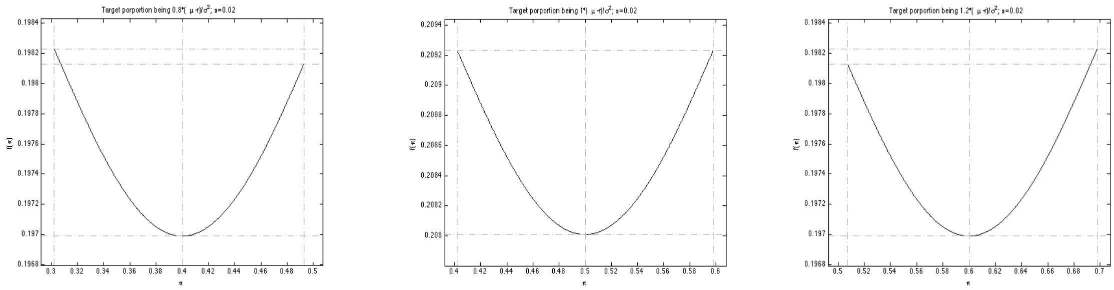

Figure 3: Plots of f with $\pi^* = 0.8 * (\mu - r)/\sigma^2$ (left panel), $\pi^* = (\mu - r)/\sigma^2$ (middle panel) and $\pi^* = 1.2 * (\mu - r)/\sigma^2$ (right panel). Parameters are $\mu = 0.q$, $\sigma = 0.4$, $\lambda = 1$, $\rho = 0.1$ and $r = 0.02$ and $\epsilon = 0.02$. Horizontally dashed lines highlight levels of $f(\pi_-)$, $f(\pi_+)$ and $f(\pi^*)$, and vertically dashed lines highlight π_-, π_+ and π^*.

In Figure 3, we choose the market parameters to cover the cases of $\pi^* > (\mu - r)/\sigma^2$, $\pi^* = (\mu - r)/\sigma^2$ and $\pi^* < (\mu - r)/\sigma^2$, which corresponds to the asymmetric no transaction region with bigger right part, the symmetric no transaction region with respect to π^* and the asymmetric no transaction region with bigger left part, respectively. π_+ and π_- are approximated by $\tilde{\pi}_+ = \alpha\epsilon^{1/3} + \beta\epsilon^{2/3}, \tilde{\pi}_- = \alpha\epsilon^{1/3} - \beta\epsilon^{2/3}$, and f is plotted on $[\tilde{\pi}_-, \tilde{\pi}_+]$, using (21) and ignoring the terms of order $\epsilon^{4/3}$ and higher.

6 Conclusions

In this paper we discuss the asset allocation in the presence of transaction costs. The objective is to keep the asset portfolio close to the target portfolio π^* and at the same time to reduce the trading cost. We prove a verification theorem for optimality of a local time trading strategy. Furthermore, we apply the asymptotic expansion method to characterize explicitly the optimal no transaction region $[\pi^* - \delta_-, \pi^* + \delta_+]$ when the transaction cost ϵ is small and show that δ_- and δ_+ are approximately equal to $\alpha\epsilon^{1/3}$ when the first order expansion method is used while $\delta_+ - \delta_-$ is approximately equal to $2\beta\epsilon^{2/3}$ when the second order expansion method is used. The boundary points of the no transaction region are asymmetric in relation to the target portfolio π^* to the order $\epsilon^{2/3}$. We also provide the asymptotic result for the value function f up to the order $\mathcal{O}(\epsilon^{4/3})$ and perform some numerical tests and plot corresponding graphs.

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A Proof of Theorem 3.1

Let (Π^+, Π^-) be an admissible policy and $(\pi_t)_{t\geq 0}$ the corresponding controlled state process with the initial state $\pi_{0-} = \pi$. Denote by $\Delta\Pi_t^+ = \Pi_t^+ - \Pi_{t-}^+$ the jump of Π^+ at time t, $\Pi_t^+ = 0$ for $t < 0$, which implies $\Delta\Pi_0^+ = \Pi_0^+$, and $\Pi^{+(c)}$ the continuous part of Π^+, that is, $\Pi_t^{+(c)} \equiv \Pi_t^+ - \sum_{0\leq s\leq t} \Delta\Pi_s^+$ for $t \geq 0$. Similar notations are defined for Π^-. We have $\Pi^{+(c)}$ and $\Pi^{-(c)}$ are continuous and nondecreasing with $\Pi_0^{+(c)} = \Pi_0^{-(c)} = 0$. Define a stochastic process for $T \geq 0$ by

$$\mathcal{M}_T(\pi, \Pi^+, \Pi^-) := \int_0^T e^{-\rho t}\lambda\sigma^2(\pi_t - \pi^*)^2 dt + \int_0^T e^{-\rho t}\epsilon d\Pi_t^+ + \int_0^T e^{-\rho t}\epsilon d\Pi_t^- + e^{-\rho T} f(\pi_T),$$

where f satisfies (5), (3) and (4). Note that

$$\mathcal{M}_0(\pi, \Pi^+, \Pi^-) = f(\pi_0) + \epsilon\Delta\Pi_0^+ + \epsilon\Delta\Pi_0^-.$$

An application of the Itô formula gives

$$\begin{aligned}
\mathbb{E}_\pi[\mathcal{M}_T(\pi, \Pi^+, \Pi^-)] =& f(\pi) + \mathbb{E}_\pi\left[\int_0^T e^{-\rho t}(\mathcal{L}f(\pi_t) - \rho f(\pi_t) + \lambda\sigma^2(\pi_t - \pi^*)^2)dt\right] \\
&+ \mathbb{E}_\pi\left[\int_0^T e^{-\rho t}[\epsilon + f'(\pi_t)]d\Pi_t^{+(c)}\right] \\
&+ \mathbb{E}_\pi\left[\int_0^T e^{-\rho t}[\epsilon - f'(\pi_t)]d\Pi_t^{-(c)}\right] \\
&+ \mathbb{E}_\pi\left[\sum_{0\leq t\leq T} e^{-\rho t}[f(\pi_t) - f(\pi_{t-}) + \epsilon\Delta\Pi_t^+ + \epsilon\Delta\Pi_t^-]\right] \\
&=: f(\pi) + I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{22}$$

Here we have used the fact that $\int_0^T e^{-\rho t} f'(\pi_t)\sigma\pi_t(1 - \pi_t)dW_t$ is a martingale since f' is bounded on $\mathscr{S}$ and π_t stays in $\mathscr{S}$ by definition of admissible policy (Π^+, Π^-) .

Suppose that $\pi \in \mathcal{NT}$ and $(\Pi^+, \Pi^-) = (\Pi^{+*}, \Pi^{-*})$ as defined in the theorem statement. Since f satisfies (5), (3) and (4), it follows immediately that $I_1 = I_2 = I_3 = 0$ and I_4 also vanishes since $(\pi_t)_{t\geq 0}$ is continuous. Hence, letting $T \to \infty$ and using the fact that f is bounded on $\mathscr{S}$, we have

$$f(\pi) = \lim_{T\to\infty} \mathbb{E}_\pi\left[\mathcal{M}_T(\pi, \Pi^{+*}, \Pi^{-*})\right] = J(\pi; \Pi^{+*}, \Pi^{-*}).$$

For $\pi \in \mathscr{S}\backslash\mathcal{NT}$, take $\pi \in \mathcal{B}$ for example. An initial transaction of $\Delta\Pi_0^{+*} = \pi_- - \pi$ should be made. By definition of cost function,

$$J(\pi; \Pi^{+*}, \Pi^{-*}) = J(\pi_-; \Pi^{+*}, \Pi^{-*}) + \epsilon\Delta\Pi_0^{+*}.$$

Since f' satisfies the equality in (3), integrating from π to π_- yields

$$f(\pi) = f(\pi_-) + \epsilon \Delta \Pi_0^{+*}.$$

Since we have already shown $f(\pi_-) = J(\pi_-; \Pi^{+*}, \Pi^{-*})$, then the above two equalities imply $f(\pi) = J(\pi; \Pi^{+*}, \Pi^{-*})$. Similar argument applies for $\pi \in \mathcal{S}$. It follows that $f(\pi) = J(\pi; \Pi^{+*}, \Pi^{-*})$ holds throughout $\mathscr{S}$.

We next show that $f(\pi) \le J(\pi; \Pi^+, \Pi^-)$ for any admissible (Π^+, Π^-). For an arbitrary $(\Pi^+, \Pi^-) \in \mathcal{A}(\pi)$, it is clear that (5), (3) and (4) imply $I_i \ge 0$ for $i = 1, 2, 3$. Furthermore, suppose $\Delta \Pi_t^+ \ge 0$ and $\Delta \Pi_t^- = 0$. Then we have

$$f(\pi_t) - f(\pi_{t-}) + \epsilon \Delta \Pi_t^+ + \epsilon \Delta \Pi_t^- = f(\pi_t) - f(\pi_t - \Delta \Pi_t^+) + \epsilon \Delta \Pi_t^+ = \int_{\pi_t - \Delta \Pi_t^+}^{\pi_t} \left[f'(\pi) + \epsilon \right] d\pi \ge 0$$

by (3). From (4) we get a similar inequality at time t where $\Delta \Pi_t^+ = 0$ and $\Delta \Pi_t^- \ge 0$. Therefore, $I_4 \ge 0$. Letting $T \to \infty$ and using the fact that f is bounded on $\mathscr{S}$, we have, from (22), that

$$f(\pi) \le \lim_{T \to \infty} \mathbb{E}_\pi [\mathcal{M}_T(\pi, \Pi^+, \Pi^-)] = J(\pi; \Pi^+, \Pi^-).$$

This confirms the optimality of (Π^{+*}, Π^{-*}).

B Proof of Theorem 4.1

We first give preliminary estimates for some expressions that will be used for later calculation. For small transaction costs we would expect the trading strategy π to be close to the target level π^*, which implies that

$$\begin{aligned}
\frac{1}{\pi(1-\pi)} &= \frac{1}{(\pi^* + (\pi - \pi^*))(1 - \pi^* - (\pi - \pi^*))} \\
&= \frac{1}{\pi^*(1-\pi^*) + (1 - 2\pi^*)(\pi - \pi^*) - (\pi - \pi^*)^2} \\
&= \frac{1}{\pi^*(1-\pi^*)} - \frac{1 - 2\pi^*}{(\pi^*)^2 (1-\pi^*)^2} (\pi - \pi^*) + \mathcal{O}((\pi - \pi^*)^2).
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
\frac{1}{\pi^2(1-\pi)^2} &= \frac{1}{(\pi^*)^2(1-\pi^*)^2} - \frac{2(1-2\pi^*)}{(\pi^*)^3(1-\pi^*)^3} (\pi - \pi^*) + \mathcal{O}((\pi - \pi^*)^2) \qquad (23) \\
\frac{\mu - r - \sigma^2 \pi}{\pi(1-\pi)} &= \frac{\mu - r - \sigma^2 \pi^*}{\pi^*(1-\pi^*)} - \frac{(\mu - r)(1 - 2\pi^*) + \sigma^2 (\pi^*)^2}{(\pi^*)^2 (1-\pi^*)^2} (\pi - \pi^*) + \mathcal{O}((\pi - \pi^*)^2).
\end{aligned}$$

We can give the following estimates for $\zeta(\epsilon, \pi_+)$ and $\zeta(-\epsilon, \pi_-)$, see (9) for definition,

$$\begin{aligned}
\zeta(\epsilon, \pi_+) = {} & \frac{\lambda \sigma^2}{\rho} \delta_+^2 + \frac{\pi^*(1-\pi^*)(\mu - r - \sigma^2 \pi^*)}{\rho} \epsilon \\
& + \frac{(\mu - r - \sigma^2 \pi^*)(1 - 2\pi^*) - \sigma^2 \pi^* (1-\pi^*)}{\rho} \epsilon \delta_+ + \mathcal{O}(\epsilon \delta_+^2),
\end{aligned} \qquad (24)$$

and

$$
\begin{aligned}
\zeta(-\epsilon, \pi_-) = & \frac{\lambda\sigma^2}{\rho}\delta_-^2 - \frac{\pi^*(1-\pi^*)(\mu - r - \sigma^2\pi^*)}{\rho}\epsilon \\
& + \frac{(\mu - r - \sigma^2\pi^*)(1-2\pi^*) - \sigma^2\pi^*(1-\pi^*)}{\rho}\epsilon\delta_- - \mathcal{O}(\epsilon\delta_-^2).
\end{aligned} \tag{25}
$$

We now look at the last term in the expansion of $\Phi^{-1}(\pi)$, i.e., $\Omega_1(\pi)$, which is the integral with respect to coefficient matrix $A(\nu)$,

$$
\Omega_1(\pi) = \int_{\pi_-}^{\pi} A(\nu)d\nu = \begin{bmatrix} \omega_{11}^1(\pi) & \omega_{12}^1(\pi) \\ \omega_{21}^1(\pi) & \omega_{22}^1(\pi) \end{bmatrix}, \tag{26}
$$

where $\omega_{11}^1(\pi) = 0$, $\omega_{12}^1(\pi) = \pi - \pi_-$, and

$$
\omega_{21}^1(\pi) = \int_{\pi_-}^{\pi} \frac{2\rho}{\sigma^2\nu^2(1-\nu)^2}d\nu, \qquad \omega_{22}^1(\pi) = \int_{\pi_-}^{\pi} -\frac{2(\mu - r - \sigma^2\nu)}{\sigma^2\nu(1-\nu)}d\nu.
$$

Using (23), we have

$$
\omega_{21}^1(\pi) = \frac{2\rho(\pi-\pi_-)}{\sigma^2(\pi^*)^2(1-\pi^*)^2} - \frac{2\rho(1-2\pi^*)}{\sigma^2(\pi^*)^3(1-\pi^*)^3}[(\pi-\pi^*)^2 - \delta_-^2] + \mathcal{O}((\pi-\pi^*)^3) + \mathcal{O}(\delta_-^3) \tag{27}
$$

and

$$
\begin{aligned}
\omega_{22}^1(\pi) \quad = \quad & -\frac{2(\mu - r - \sigma^2\pi^*)}{\sigma^2\pi^*(1-\pi^*)}(\pi - \pi_-) + \frac{(\mu - r)(1-2\pi^*) + \sigma^2(\pi^*)^2}{\sigma^2(\pi^*)^2(1-\pi^*)^2}[(\pi-\pi^*)^2 - \delta_-^2] \\
& + \mathcal{O}((\pi-\pi^*)^3) + \mathcal{O}(\delta_-^3).
\end{aligned} \tag{28}
$$

Consequently, with the expansion of $\Phi^{-1}(\pi) = \mathbb{I}_2 - \Omega_1(\pi)$, (16) becomes

$$
\int_{\pi_-}^{\pi_+} (\mathbb{I}_2 - \Omega_1(\pi))\psi(\pi)d\pi = (\mathbb{I}_2 - \Omega_1(\pi_+))\phi_1 - \phi_0. \tag{29}
$$

The LHS of (29) is, see (11) and (26),

$$
\text{LHS} = \begin{bmatrix} 0 \\ \int_{\pi_-}^{\pi_+} -\frac{2\lambda(\pi-\pi^*)^2}{\pi^2(1-\pi)^2}d\pi \end{bmatrix} - \begin{bmatrix} \int_{\pi_-}^{\pi_+} -\frac{2\lambda(\pi-\pi^*)^2}{\pi^2(1-\pi)^2}w_{12}^1(\pi)d\pi \\ \int_{\pi_-}^{\pi_+} -\frac{2\lambda(\pi-\pi^*)^2}{\pi^2(1-\pi)^2}w_{22}^1(\pi)d\pi \end{bmatrix} =: \begin{bmatrix} 0 \\ \mathcal{I}_1 \end{bmatrix} - \begin{bmatrix} \mathcal{I}_2 \\ \mathcal{I}_3 \end{bmatrix}.
$$

Using (23) and (28), after some lengthy but straightforward calculation, we get

$$
\begin{aligned}
\mathcal{I}_1 &= -\frac{2\lambda(\delta_+^3 + \delta_-^3)}{3(\pi^*)^2(1-\pi^*)^2} + \frac{\lambda(1-2\pi^*)}{(\pi^*)^3(1-\pi^*)^3}(\delta_+^4 - \delta_-^4) + \mathcal{O}(\delta^5), \\
\mathcal{I}_2 &= -\frac{\lambda}{6(\pi^*)^2(1-\pi^*)^2}(3\delta_+^4 + 4\delta_+^3\delta_- + \delta_-^4) + \mathcal{O}(\delta^5), \\
\mathcal{I}_3 &= \frac{\lambda(\mu - r - \sigma^2\pi^*)}{3\sigma^2(\pi^*)^3(1-\pi^*)^3}(3\delta_+^4 + 4\delta_+^3\delta_- + \delta_-^4) + \mathcal{O}(\delta^5).
\end{aligned}
$$

On the other hand, the right hand side of (29) equals, see (12) and (26),

$$
\text{RHS} = \begin{bmatrix} \zeta(\epsilon, \pi_+) - \zeta(-\epsilon, \pi_-) - \omega_{12}^1(\pi_+)\epsilon \\ 2\epsilon - \omega_{21}^1(\pi_+)\zeta(\epsilon, \pi_+) - \omega_{22}^1(\pi_+)\epsilon \end{bmatrix} =: \begin{bmatrix} \mathcal{I}_4 \\ \mathcal{I}_5 \end{bmatrix}.
$$

Using (24), (25) and (27), (28), we can derive

$$\begin{aligned}
\mathcal{I}_4 &= \frac{\lambda\sigma^2}{\rho}(\delta_+^2 - \delta_-^2) + \frac{2\pi^*(1-\pi^*)(\mu - r - \sigma^2\pi^*)}{\rho}\epsilon - \epsilon(\delta_+ + \delta_-) \\
&\quad + \frac{(\mu - r - \sigma^2\pi^*)(1-2\pi^*) - \sigma^2\pi^*(1-\pi^*)}{\rho}\epsilon(\delta_+ - \delta_-) + \mathcal{O}(\epsilon\delta^2), \\
\mathcal{I}_5 &= 2\epsilon - \frac{2\lambda(\delta_+^3 + \delta_+^2\delta_-)}{(\pi^*)^2(1-\pi^*)^2} + \frac{2\lambda(1-2\pi^*)(\delta_+^4 - \delta_+^2\delta_-^2)}{(\pi^*)^3(1-\pi^*)^3} + \mathcal{O}(\epsilon\delta^2) + \mathcal{O}(\delta^5).
\end{aligned}$$

Comparing the components of LHS and RHS of (29) we have

$$\begin{aligned}
&\frac{\lambda\sigma^2}{\rho}(\delta_+^2 - \delta_-^2) + \frac{2\pi^*(1-\pi^*)(\mu - r - \sigma^2\pi^*)}{\rho}\epsilon - \frac{\lambda}{6(\pi^*)^2(1-\pi^*)^2}(3\delta_+^4 + 4\delta_+^3\delta_- + \delta_-^4) \\
&- \epsilon(\delta_+ + \delta_-) + \frac{(\mu - r - \sigma^2\pi^*)(1-2\pi^*) - \sigma^2\pi^*(1-\pi^*)}{\rho}\epsilon(\delta_+ - \delta_-) + \mathcal{O}(\epsilon\delta^2) + \mathcal{O}(\delta^5) = 0
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
&2\epsilon - \frac{2\lambda(\delta_+^3 + \delta_+^2\delta_-)}{(\pi^*)^2(1-\pi^*)^2} + \frac{2\lambda(\delta_+^3 + \delta_-^3)}{3(\pi^*)^2(1-\pi^*)^2} + \frac{\lambda(\mu - r - \sigma^2\pi^*)}{3\sigma^2(\pi^*)^3(1-\pi^*)^3}(3\delta_+^4 + 4\delta_+^3\delta_- + \delta_-^4) \\
&+ \frac{2\lambda(1-2\pi^*)(\delta_+^4 - \delta_+^2\delta_-^2)}{(\pi^*)^3(1-\pi^*)^3} - \frac{\lambda(1-2\pi^*)(\delta_+^4 - \delta_-^4)}{(\pi^*)^3(1-\pi^*)^3} + \mathcal{O}(\epsilon\delta^2) + \mathcal{O}(\delta^5) = 0.
\end{aligned} \tag{31}$$

Equality (31) suggests that ϵ should have an order as high as that of δ^3. Applying this observation to (30), we obtain

$$\lim_{\epsilon\to 0}\frac{\delta_+}{\delta} = \lim_{\epsilon\to 0}\frac{\delta_-}{\delta} = \frac{1}{2}.$$

Therefore,

$$\delta_+ = \frac{1}{2}\delta + o(\delta) \quad \text{and} \quad \delta_- = \frac{1}{2}\delta + o(\delta). \tag{32}$$

Recall that ϵ is of order higher than or equal to δ^3. If we assume $\epsilon = o\big(\delta^3\big)$ and substitute (32) back into (31), the two δ^3 order terms in (31), i.e.,

$$-\frac{2\lambda(\delta_+^3 + \delta_+^2\delta_-)}{(\pi^*)^2(1-\pi^*)^2} \quad \text{and} \quad \frac{2\lambda(\delta_+^3 + \delta_-^3)}{3(\pi^*)^2(1-\pi^*)^2},$$

now become

$$-\frac{\lambda\delta^3}{2(\pi^*)^2(1-\pi^*)^2} \quad \text{and} \quad \frac{\lambda\delta^3}{6(\pi^*)^2(1-\pi^*)^2},$$

which cannot cancel each other. Therefore, ϵ can only be of the same order with δ^3, or equivalently,

$$\delta_- = \alpha\epsilon^{1/3} + o\Big(\epsilon^{1/3}\Big), \quad \delta_+ = \alpha\epsilon^{1/3} + o\Big(\epsilon^{1/3}\Big).$$

Substituting the above two expressions for δ_+ and δ_- back to (31) and compare $\mathcal{O}(\epsilon)$ order terms, we end up with the following equation with α being unknown,

$$2\epsilon - \frac{4\lambda\alpha^3\epsilon}{(\pi^*)^2(1-\pi^*)^2} + \frac{4\lambda\alpha^3\epsilon}{3(\pi^*)^2(1-\pi^*)^2} + o(\epsilon) = 0.$$

Solving for α, we obtain (18).

C Proof of Theorem 4.3

The second order inner layer expansion of the Magnus expansion requires the calculation of $\Omega_2(\pi)$. Note that

$$
\begin{aligned}
[A(\pi_1), A(\pi_2)] &= A(\pi_1)A(\pi_2) - A(\pi_2)A(\pi_1) \\
&= \begin{bmatrix} \frac{2\rho}{\sigma^2}\left(\frac{1}{\pi_2^2(1-\pi_2)^2} - \frac{1}{\pi_1^2(1-\pi_1)^2}\right) & -\frac{2}{\sigma^2}\left(\frac{\mu-r-\sigma^2\pi_2}{\pi_2(1-\pi_2)} - \frac{\mu-r-\sigma^2\pi_1}{\pi_1(1-\pi_1)}\right) \\ -\frac{4\rho}{\sigma^4}\left(\frac{\mu-r-\sigma^2\pi_1}{\pi_1(1-\pi_1)\pi_2^2(1-\pi_2)^2} - \frac{\mu-r-\sigma^2\pi_2}{\pi_2(1-\pi_2)\pi_1^2(1-\pi_1)^2}\right) & -\frac{2\rho}{\sigma^2}\left(\frac{1}{\pi_2^2(1-\pi_2)^2} - \frac{1}{\pi_1^2(1-\pi_1)^2}\right) \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{A}_{11}(\pi_1,\pi_2) & \mathbb{A}_{12}(\pi_1,\pi_2) \\ \mathbb{A}_{21}(\pi_1,\pi_2) & \mathbb{A}_{22}(\pi_1,\pi_2). \end{bmatrix} =: \mathbb{A}.
\end{aligned}
$$

Applying the results in (23), we have $\mathbb{A}_{11} = -\mathbb{A}_{22}$ and

$$
\begin{aligned}
\mathbb{A}_{11} &= \frac{2\rho}{\sigma^2}\left[-\frac{2(1-2\pi^*)}{(\pi^*)^3(1-\pi^*)^3}(\pi_2-\pi_1) + \mathcal{O}((\pi_2-\pi^*)^2) - \mathcal{O}((\pi_1-\pi^*)^2)\right] \\
&= a_1(\pi_2-\pi_1) + \mathcal{O}((\pi_2-\pi^*)^2) - \mathcal{O}((\pi_1-\pi^*)^2), \\
\mathbb{A}_{12} &= -\frac{2\rho}{\sigma^2}\left[-\frac{(\mu-r)(1-2\pi^*)+\sigma^2(\pi^*)^2}{(\pi^*)^2(1-\pi^*)^2}(\pi_2-\pi_1) + \mathcal{O}((\pi_2-\pi^*)^2) - \mathcal{O}((\pi_1-\pi^*)^2)\right] \\
&= a_2(\pi_2-\pi_1) + \mathcal{O}((\pi_2-\pi^*)^2) - \mathcal{O}((\pi_1-\pi^*)^2), \\
\mathbb{A}_{21} &= -\frac{4\rho}{\sigma^4}\left[-\frac{(\mu-r)(1-2\pi^*)-2\sigma^2\pi^*+3\sigma^2(\pi^*)^2}{(\pi^*)^4(1-\pi^*)^4}(\pi_2-\pi_1) + \mathcal{O}((\pi_2-\pi^*)^2) - \mathcal{O}((\pi_1-\pi^*)^2)\right] \\
&= a_3(\pi_2-\pi_1) + +\mathcal{O}((\pi_2-\pi^*)^2) - \mathcal{O}((\pi_1-\pi^*)^2).
\end{aligned}
$$

Each element of $\mathbb{A}$ has its leading term being $\pi_2 - \pi_1$ multiplied by some constants. From

$$\int_{\pi_-}^{\pi}\int_{\pi_-}^{\pi_1}(\pi_2-\pi_1)d\pi_2 d\pi_1 = -\frac{(\pi-\pi_-)^3}{6},$$

we see easily that

$$\Omega_2(\pi) = \begin{bmatrix} \mathcal{O}((\pi-\pi_-)^3) & \mathcal{O}((\pi-\pi_-)^3) \\ \mathcal{O}((\pi-\pi_-)^3) & \mathcal{O}((\pi-\pi_-)^3) \end{bmatrix}.$$

Substituting $\Phi^{-1} = \mathbb{I}_2 - \Omega_1(\pi) - \Omega_2(\pi)$ into (16) we get

$$\int_{\pi_-}^{\pi_+}(\mathbb{I}_2 - \Omega_1(\pi))\psi(\pi)d\pi - \int_{\pi_-}^{\pi_+}\Omega_2(\pi)\psi(\pi)d\pi = (\mathbb{I}_2 - \Omega_1(\pi_+))\phi_1 - \phi_0 - \Omega_2(\pi_+)\phi_1.$$

Compared to the first order inner layer expansion (29), we get two extra terms from second order inner layer expansion, the last term on each side. We are only interested in the order of these two terms.

$$\int_{\pi_-}^{\pi_+}\Omega_2(\pi)\psi(\pi)d\pi = \int_{\pi_-}^{\pi_+}\begin{bmatrix} \mathcal{O}((\pi-\pi_-)^3)\mathcal{O}((\pi-\pi^*)^2) \\ \mathcal{O}((\pi-\pi_-)^3)\mathcal{O}((\pi-\pi^*)^2) \end{bmatrix} d\pi = \begin{bmatrix} \mathcal{O}(\delta^6) \\ \mathcal{O}(\delta^6) \end{bmatrix} = \begin{bmatrix} \mathcal{O}(\epsilon^2) \\ \mathcal{O}(\epsilon^2) \end{bmatrix}$$

and

$$\Omega_2(\pi)\phi_1 = \begin{bmatrix} \mathcal{O}((\pi-\pi_-)^3) & \mathcal{O}((\pi-\pi_-)^3) \\ \mathcal{O}((\pi-\pi_-)^3) & \mathcal{O}((\pi-\pi_-)^3) \end{bmatrix}\begin{bmatrix} \mathcal{O}(\epsilon^{2/3}) \\ \mathcal{O}(\epsilon) \end{bmatrix} = \begin{bmatrix} \mathcal{O}(\epsilon^{5/3}) \\ \mathcal{O}(\epsilon^{5/3}) \end{bmatrix}.$$

Similar calculations show that Ω_k, $k \geq 3$, have higher orders than $\mathcal{O}(\epsilon^{5/3})$. Combining the results above, we see that $\Omega_k, k = 2, 3, \ldots$ do not have $\mathcal{O}(\epsilon^{4/3})$ effect to equality (16).

With the second order expansion $\Phi^{-1}(\pi) = \mathbb{I}_2 - \Omega_1(\pi) + \frac{1}{2}\Omega_1^2(\pi)$, (16) becomes

$$\int_{\pi_-}^{\pi_+} (\mathbb{I}_2 - \Omega_1(\pi) + \frac{1}{2}\Omega_1^2(\pi))\psi(\pi)d\pi = (\mathbb{I}_2 - \Omega_1(\pi_+) + \frac{1}{2}\Omega_1^2(\pi_+))\phi_1 - \phi_0. \tag{33}$$

Compared to the first order expansion (29), there are two extra terms, one on each side. For the last term on the LHS of (33),

$$\int_{\pi_-}^{\pi_+} \frac{1}{2}\Omega_1^2(\pi)\psi(\pi)d\pi = \int_{\pi_-}^{\pi_+} \begin{bmatrix} \mathcal{O}((\pi - \pi_-)^4) \\ \mathcal{O}((\pi - \pi_-)^4) \end{bmatrix} d\pi = \begin{bmatrix} \mathcal{O}(\delta^5) \\ \mathcal{O}(\delta^5) \end{bmatrix}. \tag{34}$$

For the last term on the RHS of (33),

$$\frac{1}{2}\Omega_1^2(\pi_+)\phi_1 = \begin{bmatrix} \frac{\lambda}{(\pi^*)^2(1-\pi^*)^2}(\delta_+^4 + 2\delta_+^3\delta_- + \delta_+^2\delta_-^2) + \mathcal{O}(\delta^5) \\ -\frac{2\lambda(\mu - r - \sigma^2\pi^*)}{\sigma^2(\pi^*)^3(1-\pi^*)^3}(\delta_+^4 + 2\delta_+^3\delta_- + \delta_+^2\delta_-^2) + \mathcal{O}(\delta^5) \end{bmatrix}. \tag{35}$$

Including (34) and (35) in (30) and (31), the latter two respectively become

$$\begin{aligned} &\frac{\lambda\sigma^2}{\rho}(\delta_+^2 - \delta_-^2) + \frac{2\pi^*(1-\pi^*)(\mu - r - \sigma^2\pi^*)}{\rho}\epsilon - \frac{\lambda}{6(\pi^*)^2(1-\pi^*)^2}(3\delta_+^4 + 4\delta_+^3\delta_- + \delta_-^4) \\ &+ \frac{\lambda(\delta_+^4 + 2\delta_+^3\delta_- + \delta_+^2\delta_-^2)}{(\pi^*)^2(1-\pi^*)^2} - \epsilon(\delta_+ + \delta_-) + \mathcal{O}(\delta^5) = 0 \end{aligned} \tag{36}$$

and

$$\begin{aligned} &2\epsilon - \frac{2\lambda(\delta_+^3 + \delta_+^2\delta_-)}{(\pi^*)^2(1-\pi^*)^2} + \frac{2\lambda(\delta_+^3 + \delta_-^3)}{3(\pi^*)^2(1-\pi^*)^2} + \frac{\lambda(\mu - r - \sigma^2\pi^*)}{3\sigma^2(\pi^*)^3(1-\pi^*)^3}(3\delta_+^4 + 4\delta_+^3\delta_- + \delta_-^4) \\ &- \frac{2\lambda(\mu - r - \sigma^2\pi^*)}{\sigma^2(\pi^*)^3(1-\pi^*)^3}(\delta_+^4 + 2\delta_+^3\delta_- + \delta_+^2\delta_-^2) + \mathcal{O}(\delta^5) = 0. \end{aligned} \tag{37}$$

Since δ_+ and δ_- are assumed to be powers of $\epsilon^{1/3}$, we can write

$$\begin{cases} \delta_+ = \alpha\epsilon^{1/3} + \beta_+\epsilon^{2/3} + \gamma_+\epsilon + o(\epsilon), \\ \delta_- = \alpha\epsilon^{1/3} + \beta_-\epsilon^{2/3} + \gamma_-\epsilon + o(\epsilon). \end{cases} \tag{38}$$

Substituting (38) into (36), also noting α in (18), we get the equality

$$\frac{2\alpha\lambda\sigma^2}{\rho}(\beta_+ - \beta_-)\epsilon + \frac{2\pi^*(1-\pi^*)(\mu - r - \sigma^2\pi^*)}{\rho}\epsilon + \frac{\lambda\sigma^2}{\rho}[(\beta_+^2 - \beta_-^2) + 2\alpha(\gamma_+ - \gamma_-)]\epsilon^{4/3} + o\Big(\epsilon^{4/3}\Big) = 0,$$

which implies

$$\begin{cases} \beta_+ - \beta_- = -\dfrac{\pi^*(1-\pi^*)(\mu - r - \sigma^2\pi^*)}{\alpha\lambda\sigma^2}, \\ (\beta_+^2 - \beta_-^2) + 2\alpha(\gamma_+ - \gamma_-) = 0. \end{cases} \tag{39}$$

Substituting (38) into (37), we get

$$-\frac{2\lambda(5\alpha^2\beta_+ + \alpha^2\beta_-)}{(\pi^*)^2(1-\pi^*)^2}\epsilon^{4/3} + \frac{2\lambda(3\alpha^2\beta_+ + 3\alpha^2\beta_-)}{3(\pi^*)^2(1-\pi^*)^2}\epsilon^{4/3} - \frac{16\lambda(\mu - r - \sigma^2\pi^*)}{3\sigma^2(\pi^*)^3(1-\pi^*)^3}\alpha^4\epsilon^{4/3} + o\Big(\epsilon^{4/3}\Big) = 0.$$

To have the coefficient of order $\epsilon^{4/3}$ zero, we must have

$$\beta_+ = \beta = -\frac{\pi^*(1-\pi^*)(\mu - r - \sigma^2\pi^*)}{2\alpha\lambda\sigma^2}.$$

Combining (39) and (20), we get $\beta_- = -\beta$ and $\gamma_+ = \gamma_- = \gamma$. Finding the exact value of γ requires further expansion of $\Phi^{-1}(\pi)$.

D Proof of Theorem 4.4

The general solution to linear system (10) is given by (15), which can be written explicitly in matrix form as

$$\begin{bmatrix} f(\pi) \\ f'(\pi) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(\pi) & \Phi_{12}(\pi) \\ \Phi_{12}(\pi) & \Phi_{22}(\pi) \end{bmatrix} \left(\begin{bmatrix} \phi_0^{(1)} \\ \phi_0^{(2)} \end{bmatrix} + \int_{\pi_-}^{\pi} \begin{bmatrix} \Phi_{11}^{-1}(\nu) & \Phi_{12}^{-1}(\nu) \\ \Phi_{21}^{-1}(\nu) & \Phi_{22}^{-1}(\nu) \end{bmatrix} \begin{bmatrix} \psi_1(\nu) \\ \psi_2(\nu) \end{bmatrix} d\nu \right).$$

The cost function $f(\pi)$ can be calculated by

$$f(\pi) \quad = \quad \Phi_{11}(\pi)(\phi_0^{(1)} + \mathcal{I}_7) + \Phi_{12}(\pi)(\phi_0^{(2)} + \mathcal{I}_8), \tag{40}$$

where $\mathcal{I}_7 := \int_{\pi_-}^{\pi}(\Phi_{11}^{-1}(\nu)\psi_1(\nu)+\Phi_{12}^{-1}(\nu)\psi_2(\nu))d\nu$ and $\mathcal{I}_8 := \int_{\pi_-}^{\pi}(\Phi_{21}^{-1}(\nu)\psi_1(\nu)+\Phi_{22}^{-1}(\nu)\psi_2(\nu))d\nu$. Similar to the case of the asymptotic analysis for no trade region boundaries, we expand here $\Phi(\pi)$ to second order outer layer and first order inner layer,

$$\Phi(\pi) = \mathbb{I}_2 + \Omega_1(\pi) + \frac{1}{2}\Omega_1^2(\pi),$$

and accordingly

$$\Phi^{-1}(\pi) = \mathbb{I}_2 - \Omega_1(\pi) + \frac{1}{2}\Omega_1^2(\pi).$$

Following a similar calculation in Section 4, we can write the relevant components of $\Phi(\pi)$ and $\Phi^{-1}(\pi)$ shown in (40) as

$$\begin{aligned} \Phi_{11}(\pi) &= 1 + \frac{\rho}{\sigma^2(\pi^*)^2(1-\pi^*)^2}(\pi-\pi_-)^2 + \mathcal{O}(\delta^3), \\ \Phi_{12}(\pi) &= (\pi-\pi_-) - \frac{(\mu - r - \sigma^2\pi^*)}{\sigma^2\pi^*(1-\pi^*)}(\pi-\pi_-)^2 + \mathcal{O}(\delta^3), \end{aligned}$$

and

$$\begin{aligned} \Phi_{11}^{-1}(\nu) &= 1 + \frac{\rho}{\sigma^2(\pi^*)^2(1-\pi^*)^2}(\nu-\pi_-)^2 + \mathcal{O}(\delta^3), \\ \Phi_{12}^{-1}(\nu) &= -(\nu-\pi_-) - \frac{(\mu - r - \sigma^2\pi^*)}{\sigma^2\pi^*(1-\pi^*)}(\nu-\pi_-)^2 + \mathcal{O}(\delta^3), \\ \Phi_{21}^{-1}(\nu) &= -\frac{2\rho}{\sigma^2(\pi^*)^2(1-\pi^*)^2}(\nu-\pi_-) + \frac{2\rho(1-2\pi^*)}{\sigma^2(\pi^*)^3(1-\pi^*)^3}[(\nu-\pi^*)^2 - \delta_-^2] \\ &\quad -\frac{2\rho(\mu - r - \sigma^2\pi^*)}{\sigma^4(\pi^*)^3(1-\pi^*)^3}(\nu-\pi_-)^2 + \mathcal{O}(\delta^3), \\ \Phi_{22}^{-1}(\nu) &= 1 + \frac{2(\mu - r - \sigma^2\pi^*)}{\sigma^2\pi^*(1-\pi^*)}(\nu-\pi_-) - \frac{(\mu-r)(1-2\pi^*) + \sigma^2(\pi^*)^2}{\sigma^2(\pi^*)^2(1-\pi^*)^2}[(\nu-\pi^*)^2 - \delta_-^2] \\ &\quad +\frac{2(\mu - r - \sigma^2\pi^*)^2 + \rho\sigma^2}{\sigma^4(\pi^*)^2(1-\pi^*)^2}(\nu-\pi_-)^2 + \mathcal{O}(\delta^3). \end{aligned}$$

With the components of $\Phi^{-1}(\nu)$ and the expression for $\psi(\nu)$, we can then substitute them into the expressions for $\mathcal{I}_7$ and $\mathcal{I}_8$,

$$\begin{aligned} \mathcal{I}_7 &= \frac{\lambda}{2(\pi^*)^2(1-\pi^*)^2}[(\pi-\pi^*)^4 - \delta_-^4] + \frac{2\lambda}{3(\pi^*)^2(1-\pi^*)^2}[(\pi-\pi^*)^3\delta_- + \delta_-^4] + \mathcal{O}(\delta^5), \\ \mathcal{I}_8 &= -\frac{2\lambda}{3(\pi^*)^2(1-\pi^*)^2}[(\pi-\pi^*)^3 + \delta_-^3] - \frac{\lambda[(\mu - r - \sigma^2\pi^*) - \sigma^2(1-2\pi^*)]}{\sigma^2(\pi^*)^3(1-\pi^*)^3}[(\pi-\pi^*)^4 - \delta_-^4] \\ &\quad -\frac{4\lambda(\mu - r - \sigma^2\pi^*)}{3\sigma^2(\pi^*)^3(1-\pi^*)^3}[(\pi-\pi^*)^3\delta_- + \delta_-^4] + \mathcal{O}(\delta^5). \end{aligned}$$

We also know that $\phi(\pi_-) = \phi_0$ and the components of ϕ_0 are $\phi_0^{(1)} = \zeta(-\epsilon, \pi_-)$ which has an asymptotic expansion (25) and $\phi_0^{(2)} = -\epsilon$. Substituting all expressions above into (40), after some lengthy but straightforward calculation, we can derive (21), the asymptotic expansion of f.

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