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1      **WEAK CONVERGENCE OF UTILITY-RISK PORTFOLIOS**

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*In Celebration of Alain Bensoussan on the Occasion of his 85<sup>th</sup> Birthday*

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**ABSTRACT.** In this article, we establish the validity of the Donsker invariance principle for weak convergence of the optimal utility-downside-risk portfolio payoff. Thus, the optimal solution is robust to changes in the market model via the pricing kernel. The convergence result is valid for all feasible combinations of utility functions satisfying the Inada conditions, convex downside deviation risk measures, and pricing kernels satisfying integrability conditions. Thus, our results provide a unified framework to understand whether utility-downside-risk solutions are stable under mild misspecifications of asset price processes. As another application, we further obtain a result that the optimal utility-risk values and policies driven by discrete-time binomial models converge weakly to those of the limiting continuous-time Black-Scholes model. We demonstrate the convergence result using numerical examples and find that the optimal utility-risk value converges as the time interval of the binomial model gets smaller.

2    1. **Introduction.** Since the celebrated work of Markowitz [30], the portfolio se-  
 3    lection problem has become one of the most important research topics in finance  
 4    and economics. Expected utility and mean-variance are two common criteria for  
 5    assessing portfolio performance. Merton [32] and Samuelson [36] investigate utility  
 6    maximisation problems in continuous-time and multi-period settings, respectively,  
 7    by considering them as stochastic control problems. The advantage of formulating  
 8    the stochastic control problems allows a direct application of the dynamic program-  
 9    ming principle (DPP) to get the Hamilton-Jacobi-Bellman (HJB) equation, leading  
 10   to a non-linear partial differential equation (PDE) by invoking the inherent tower  
 11   property; see, for examples, [5, 8, 32, 37] for a detailed introduction. Without as-  
 12   suming a specific form of utility functions, such as hyperbolic or constant absolute  
 13   risk aversion, the solvability of non-linear PDEs is barely discussed in general. Ben-  
 14   soussan et al. [5] established a special transformation to linearise the primal HJB  
 15   equation into a semi-linear PDE, which can be solved via the weak solution ap-  
 16   proach in a weighted Sobolev space. This novel method is further applied to resolve

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1 more advanced optimisation problems with general utility functions and model set-  
 2 tings, like [29]. Alternatively, the martingale method can be applied to solve utility  
 3 maximisation problems in a complete market, where the existence of the optimal  
 4 terminal payoff can be shown by convex analysis and that of the optimal trading  
 5 strategy by the Martingale Representation Theorem; see [12, 23, 27, 34] for details.

6 As an aftermath of several major financial crises in the last two decades, tighter  
 7 government regulations, including adequate liquidity requirements, have been en-  
 8 forced to cope with the possibility of extreme losses in the market. As a conse-  
 9 quence, there is a trend towards incorporating enterprise risk management practice  
 10 into portfolio selection problems. Value-at-Risk (V@R) constraints have been im-  
 11 plemented into portfolio optimisation in light of the prevailing regulation using  
 12 V@R as the method of risk assessment; see [3, 13, 40, 41]. Furthermore, more  
 13 general goal-reaching criteria, such as concerning whether or not the terminal cash  
 14 flow exceeds a certain benchmark, have also been applied to portfolio selection;  
 15 see [14, 16, 23, 28]. Apart from the quantile-based risk measures, another more  
 16 commonly used risk measure is the deviation risk measure, which measures the de-  
 17 viation of the random return from its own mean. Variance is a special example  
 18 of the deviation risk measure. Markowitz [30] adopted the mean-variance criteria  
 19 for evaluating portfolio performance. Li and Ng [26] and Richardson [35] extended  
 20 the mean-variance problem into dynamic Black-Scholes models. From that point  
 21 on, more sophisticated mean-variance frameworks have also been investigated, such  
 22 as [1, 9, 10, 11]. In particular, Bensoussan et al. [10] established a time-consistent  
 23 mean-variance equilibrium portfolio under short-selling prohibition in both discrete  
 24 and continuous models and showed that discrete-time equilibrium portfolios con-  
 25 verge to the continuous-time counterpart in a linear order. Apart from the usual  
 26 dynamic programming approach or martingale approach, Bensoussan and his collab-  
 27 orators innovated a novel mean-field technique to solve the classical continuous-time  
 28 mean-variance problem [9] and a mean-variance capital investment optimisation [7],  
 29 which cannot be solved by the classical stochastic control methods.

30 A main criticism against variance as a risk measure is that the variance incurs  
 31 penalty on the upside profitable return. In contrast, the downside-risk measures,  
 32 which penalise only when the return is below its expectation, can remedy the com-  
 33 mon criticism against variance, thus Markowitz [31] also claimed that semivariance,  
 34 an example of downside risk measures, could be a more sensible risk measure than  
 35 variance. However, the continuous-time mean-downside-risk problem under the  
 36 Black-Scholes model was shown to have no optimal solution in Jin et al. [20], as the  
 37 optimal value function is unattainable by any of the admissible portfolios.

38 Wong et al. [39] first proposed applying the deviation-risk measure to the utility-  
 39 risk<sup>1</sup> portfolio selection framework, in which the objective function is the difference  
 40 of the deviation risk from the utility function, under the continuous-time Black-  
 41 Scholes market. Wong et al. [39] showed the existence of the optimal solutions  
 42 for various utility-risk problems, including the utility-semivariance problem, which  
 43 is in sharp contrast to the non-existence for continuous-time mean-downside-risk  
 44 problem in [20]. The obtained existence result in [39] suggests the revival of using  
 45 semivariance as a proper risk measure in the portfolio management paradigm. The  
 46 general utility-deviation-risk framework in Wong et al. [39] is developed into more

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<sup>1</sup>Jin et al. [20] called their problem formulation mean-risk problem, though they only considered deviation risk measure in their work, so to avoid ambiguity, we use a single word “risk” to stand for the deviation risk measure in the rest of this article.

1 interesting studies, like time-consistent strategies in Gu et al. [18] and consump-  
 2 tion and investment settings in Bensoussan et al. [6]. In particular, Bensoussan  
 3 et al. [6] converted the utility-variance problem with consideration of intermediate  
 4 consumption into a mean field-type control problem, which can be tackled using  
 5 the Hamilton-Jacobi-Bellman and Fokker-Planck equations framework.

6 To bring the dynamic utility-semivariance framework to use in practice, we could  
 7 calibrate the model in [39] using past market data, yet the estimated parameters  
 8 may be different from their actual values during the investment horizon due to  
 9 the limitation of estimation from past data on future predictions. This deviation  
 10 could be a subtle issue because most theoretical claims in portfolio selection are  
 11 mostly model-dependent, and so any tiny deviations in the parameters could lead to  
 12 substantial inaccuracy in the optimal solution if it is sensitive. However, the lack of  
 13 explicit forms of the optimal solutions in [39] to study their functional dependence  
 14 on parameters leads to an extra hurdle to the investigation of the robustness of  
 15 optimal utility-risk solutions.

16 To address the foregoing issue, we shall establish the weak convergence of optimal  
 17 utility-risk payoffs, each corresponding to an approximating market such that the  
 18 sequence of the corresponding pricing kernels converges to a limiting one. This  
 19 convergence result implies that the optimal solution is insensitive to changes in  
 20 the underlying market parameter values. In other words, if we slightly perturb the  
 21 market model, the projections on the average generated by the model do not change  
 22 drastically.

23 In comparison with some existing results of weak convergence in the literature,  
 24 the convergence of optimal payoff for utility maximisation (without other risk mea-  
 25 sures) was studied by He [19] for complete market setups and then by Larsen and  
 26 Zitkovic [25] for incomplete market setups. In particular, Larsen and Zitkovic [25]  
 27 investigated the stability of the utility maximisation, which is how small perturba-  
 28 tions of the market coefficient processes can cause a change in the optimal solution.  
 29 More research studies regarding the convergence and stability of utility maximisa-  
 30 tion in incomplete markets have been done; to name a few, see [4, 22, 24, 33]. As  
 31 the utility-deviation-risk setting is recently established in [39], its convergence and  
 32 stability have not been studied.

33 As an application of the weak convergence theorem established in this article, we  
 34 shall verify that the conditions for the theorem are satisfied by the pricing kernels  
 35 in connection with the discrete binomial tree model. Hence, the corresponding  
 36 optimal terminal payoffs converge in distribution to that under the continuous-  
 37 time Black-Scholes setting. This convergence result will be further illustrated by  
 38 numerical examples, in which the optimal values converge in a linear order as the  
 39 time interval of the binomial model gets smaller – the same convergence order found  
 40 in [10]. Furthermore, under some regularity assumptions, we can further establish  
 41 the weak convergence of the optimal portfolio and the wealth process by using a  
 42 similar argument as in [19].

43 The rest of the paper is organised as follows. In Section 2, we first introduce  
 44 the problem formulation and some assumptions on the market model. Then, we  
 45 establish the unique existence of the optimal solution for the utility-downside-risk  
 46 framework. Section 3 lists certain conditions on a sequence of pricing kernels under  
 47 which the optimal terminal payoffs converge in distribution. In Section 4, we shall  
 48 demonstrate the verification of the aforementioned conditions by the binomial tree  
 49 models. Hence, with our main claim, we can conclude that the optimal terminal

1 wealth for utility-risk problems under the Black-Scholes model can be approximated  
 2 by that under binomial tree models. Furthermore, under some mild regularity  
 3 assumptions, we further establish weak convergence of the optimal portfolio weight  
 4 and wealth processes. Section 5 demonstrates our established convergence results.  
 5 The technical proofs will be provided in the Appendix.

## 6 2. Problem setting and preliminaries.

7 **2.1. Utility-risk portfolio selection.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a fixed complete proba-  
 8 bility space. Suppose that the market has  $m + 1$  assets with the joint price process  
 9  $(S(t))_{t \geq 0}$  with  $S(t) := (S_0(t), \dots, S_m(t))^t$ ; here  $M^t$  denotes the transpose of a ma-  
 10 trix  $M$ . Define the information filtration up to time  $t$  as  $\mathcal{F}_t := \sigma(S(s) : s \leq t)$ ,  
 11 satisfying the usual conditions.

12 Let  $\pi(t) := (\pi_1(t), \dots, \pi_m(t))^t$ , where  $\pi_k(t)$  is the money amount invested in the  
 13  $k$ -th risky asset of the portfolio at time  $t$ . The dynamics of controlled wealth process  
 14  $(X^\pi(t))_{t \geq 0}$  is given by

$$dX^\pi(t) = (X^\pi(t) - \pi(t)^t \mathbf{1}_m) \frac{dS_0(t)}{S_0(t)} + \sum_{k=1}^m \pi_k(t) \frac{dS_k(t)}{S(t)}, \quad (1)$$

15 and  $X^\pi(0) = x_0 > 0$ , where  $\mathbf{1}_m$  is an  $m$ -dimensional vector with all components 1.

16 Let  $U$  be a utility function satisfying the following property:

17 **Property 2.1.** The utility function  $U : (0, \infty) \rightarrow \mathbb{R}$  is strictly increasing, strictly  
 18 concave, continuously differentiable and  $U'(0) := \lim_{x \downarrow 0} U'(x) = \infty$  and  $U'(\infty) :=$   
 19  $\lim_{x \uparrow \infty} U'(x) = 0$ , i.e. the Inada condition holds.

20 For the ease of discussion, we extend the domain of  $U$  from  $\mathbb{R}_+$  to  $\mathbb{R}$  so that  
 21  $U(0) := \lim_{x \downarrow 0} U(x)$  and  $U(x) := -\infty$  for  $x < 0$ .

22 The deviation risk function  $D$  is used to measure the difference of the random  
 23 portfolio return and its own expected value. In the present article, we focus on  
 24 downside risk functions defined as follows:

25 **Definition 2.2.**  $D : \mathbb{R} \rightarrow \mathbb{R}_+$  is said to be a downside risk function if it is positive,  
 26 strictly increasing, strictly convex, and continuously differentiable on  $(0, \infty)$  and  
 27  $D(x) = 0$  for  $x \leq 0$  such that  $D'(\infty) = \infty$  and  $D'(0) = 0$ .

28 The objective functional is given by

$$J(\pi(\cdot)) := \mathbb{E}[U(X^\pi(T))] - \gamma \mathbb{E}[D(\mathbb{E}[X^\pi(T)] - X^\pi(T))], \quad (2)$$

29 where the terminal time  $T < \infty$  and  $\gamma > 0$  denotes the risk aversion coefficient.  
 30 From that point on, we denote the process  $(X(t))_{t \in [0, T]}$  in short as  $X(\cdot)$  unless  
 31 otherwise specified.

32 The investor aims to simultaneously maximize the expected value of the utility  
 33 and minimize the deviation risk. For example, if the utility function is  $U(x) = \frac{1}{p} x^p$   
 34 while the downside risk function is  $D(x) = x_+^2$ , then our problem becomes the power-  
 35 utility-semivariance problem. Given  $p \geq 1$ , let  $\mathcal{L}^p := \{Z \mid \|Z\|_p := (\mathbb{E}[|Z|^p])^{\frac{1}{p}} < \infty\}$ .  
 36 Define  $\mathcal{H}^2$  to be the class of all  $\mathcal{F}_t$ -adapted processes equipped with the norm  
 37  $\|\pi(\cdot)\|_{\mathcal{H}^2} := (\mathbb{E}[\int_0^T \pi(t)^t \pi(t) dt])^{\frac{1}{2}} < \infty$ . As a mild technical requirement, we require  
 38 the terminal wealth of every admissible control to satisfy the following conditions:

$$X \geq 0 \text{ a.s.}; \quad U(X) \in \mathcal{L}^1; \quad D(\mathbb{E}[X] - X) \in \mathcal{L}^1. \quad (3)$$

1 The non-negativity of terminal wealth  $X$  ensures that  $U(X)$  can be properly eval-  
 2 uated. The integrability of both  $U(X)$  and  $D(\mathbb{E}[X] - X)$  guarantees both terms in  
 3 the objective functional (2) take finite values. With these conditions on the terminal  
 4 wealth, we consider the class of all admissible controls

$$\mathcal{A} := \{\pi(\cdot) \in \mathcal{H}^2 \mid X^\pi(T) \in \mathcal{L}^2 \cap \mathcal{F}_T \text{ and } X^\pi(T) \text{ satisfies (3)}\}.$$

5 Our utility risk problem can then be formulated as follows:

6 **Problem 1.** Maximize  $J(\pi(\cdot))$ , subject to  $\pi(\cdot) \in \mathcal{A}$  and  $(X^\pi(\cdot), \pi(\cdot))$  follows (1)  
 7 with an initial wealth  $x_0$ .

8 We impose further assumptions in the market:

9 **Assumption 2.3** (Pricing kernel). There exists a pricing kernel  $\xi(\cdot) \in \mathcal{H}^2$ ,  $\xi(0) = 1$   
 10 and  $\xi(t) \in \mathbb{R}_+$  a.s. for all  $t \in [0, T]$ , such that  $\xi(T) \in \mathcal{L}^2$  and  $(\xi(t)X^\pi(t))_{t \in [0, T]}$  is a  
 11 martingale for all  $\pi(\cdot) \in \mathcal{A}$ .

12 **Assumption 2.4** (Market completeness). For every  $X \in \mathcal{L}^2 \cap \mathcal{F}_T$ , there exists a  
 13 self-financing strategy  $\pi(\cdot) \in \mathcal{H}^2$  such that  $X^\pi(T) = X$ .

14 Moreover, we assume the following on the utility function which has also been  
 15 adopted in [39].

16 **Assumption 2.5.** There exists  $k_0 > 0$  so that the inverse of the first-order deriv-  
 17 ative of  $U$ ,  $(U')^{-1}$ , satisfies  $(U')^{-1}(k_0\xi(T)) \in \mathcal{L}^2$ .

18 2.1.1. *Examples of market models.*

19 **Example 2.6** (Continuous-time Black-Scholes model). Suppose that the market  
 20 has one bond and one stock, we consider the bond price and stock price satisfying  
 21 the following:

$$\begin{cases} dB(t) = rB(t)dt, & B(0) = b_0 > 0, \\ dS(t) = \mu S(t)dt + \sigma S(t)dW(t), & S(0) = s_0 > 0, \end{cases}$$

22 where  $W(\cdot)$  is the standard Brownian motion. Let  $\pi(t)$  be the money amount of  
 23 risky investment at time  $t$ , so that the dynamics of the wealth process become:

$$dX^\pi(t) = (rX^\pi(t) + \alpha\pi(t))dt + \sigma\pi(t)dW(t), \quad X^\pi(0) = x_0 > 0,$$

24 where  $\alpha := \mu - r$ . The continuous-time pricing kernel is a process  $\xi^*(\cdot)$  such that:

$$\xi^*(t) := \exp \left[ -rt - \frac{\alpha^2}{2\sigma^2}t - \frac{\alpha}{\sigma}W_t \right]. \quad (4)$$

25 Clearly,  $\xi^*(\cdot)$  satisfies Assumption 2.3. Assumption 2.4 can be validated by an  
 26 application of the Martingale Representation Theorem. Note that all assumptions  
 27 are satisfied even when the market parameters are  $\mathcal{F}_t$ -adapted processes in general.

28 **Example 2.7** (Binomial tree model). We next consider a  $N$ -period discrete time  
 29 framework with time interval  $\Delta t := \frac{T}{N}$ . The price processes of bond  $B^{(N)}(\cdot)$  and  
 30 stock  $S^{(N)}(\cdot)$  are given by  $B^{(N)}(t) := B_{\lfloor t/\Delta t \rfloor}^{(N)}$  and  $S^{(N)}(t) := S_{\lfloor t/\Delta t \rfloor}^{(N)}$  respectively,  
 31 where  $B_n^{(N)}$  and  $S_n^{(N)}$  have the following dynamics:

$$\begin{aligned} B_{n+1}^{(N)} &= (1 + r\Delta t) B_n^{(N)}, & B_0^{(N)} &= b_0; \\ S_{n+1}^{(N)} &= \left( 1 + \mu\Delta t + \sigma\sqrt{\Delta t}Z_{n+1}^{(N)} \right) S_n^{(N)}, & S_0^{(N)} &= s_0; \end{aligned}$$

1 where  $Z_n^{(N)}$ 's are iid Bernoulli random variables for  $n = 1, \dots, N$ , with probability:  
 2  $\mathbb{P}[Z_n^{(N)} = 1] = \mathbb{P}[Z_n^{(N)} = -1] = \frac{1}{2}$ . Let  $\pi_n^{(N)}$  be the money amount invested in the  
 3 risky asset over the  $n^{\text{th}}$  period, the corresponding wealth process  $X^{(N),\pi}(\cdot)$  with  
 4  $X^{(N),\pi}(t) := X_{\lfloor t/\Delta t \rfloor}^{(N),\pi}$  has the following dynamics:

$$X_{n+1}^{(N),\pi} = (1 + r\Delta t) X_n^{(N),\pi} + \left( \alpha\Delta t + \sigma\sqrt{\Delta t} Z_{n+1}^{(N)} \right) \pi_n^{(N)}, \quad (5)$$

5  $X_0^{(N),\pi} = x_0$ , where  $\alpha = \mu - r$ .

6 In this binomial tree model, the pricing kernel process  $\xi^{(N)}(\cdot)$  is given by  $\xi^{(N)}(t) :=$   
 7  $\xi_{\lfloor t/\Delta t \rfloor}^{(N)}$ , such that:

$$\xi_n^{(N)} := \prod_{k=1}^n \left( \frac{1 - \frac{\alpha}{\sigma}\sqrt{\Delta t} Z_k^{(N)}}{1 + r\Delta t} \right) \text{ for } n=1, \dots, N. \quad (6)$$

8 Hence, Assumption 2.3 is obviously satisfied.

9 For Assumption 2.4, given a  $\mathcal{F}_T$ -measurable  $X \in \mathcal{L}^2$ , the corresponding attain-  
 10 able wealth process can be obtained by

$$X_n^{(N),\pi} = \mathbb{E} \left[ \frac{\xi_n^{(N)}}{\xi_n^{(N)}} X \middle| \xi_n^{(N)} \right] = \mathbb{E} \left[ \prod_{k=n+1}^N \left( \frac{1 - \frac{\alpha}{\sigma}\sqrt{\Delta t} Z_k^{(N)}}{1 + r\Delta t} \right) X \right],$$

11 and the underlying trading strategy can be obtained in accordance with (5).

12 **Example 2.8** (Stochastic interest rate model). Adopting the market setting as  
 13 specified in [2], suppose that the market has one riskless money account, one stock,  
 14 and one zero coupon bond fund; their prices at  $t$  are denoted by  $S_0(t)$ ,  $S_1(t)$ , and  
 15  $S_2(t)$ , respectively. The asset prices satisfy the following:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, & S_0(t) = s_0 > 0, \\ dS_1(t) = (r(t) + \alpha_1)S_1(t)dt + \sigma_{11}S_1(t)dW_1(t) + \sigma_{12}S_1(t)dW_2(t), & S_1(0) = s_1 > 0, \\ dS_2(t) = (r(t) + \alpha_2)S_2(t)dt + \sigma_{21}S_2(t)dW_1(t), & S_2(0) = s_2 > 0, \end{cases}$$

16 where  $W_1(\cdot)$  and  $W_2(\cdot)$  are two independent standard Brownian motions, and the  
 17 interest rate  $r(\cdot)$  follows an Ornstein-Uhlenbeck process with constant parameters  
 18 given by

$$dr(t) = a_r(b_r - r(t))dt - \sigma_r dW_1(t), \quad r(0) = r_0.$$

19 Let  $\pi(\cdot) := (\pi_1(t), \pi_2(t))_{t \in [0, T]}$  and  $\pi_1(t)$  and  $\pi_2(t)$  be the respective money  
 20 amount invested in the stock and the bond at time  $t$ , so that the dynamics of  
 21 wealth process becomes:

$$\begin{cases} dX^\pi(t) = (X^\pi(t) - \pi_1(t) - \pi_2(t)) \frac{dS_0(t)}{S_0(t)} + \pi_1(t) \frac{dS_1(t)}{S_1(t)} + \pi_2(t) \frac{dS_2(t)}{S_2(t)} \\ \quad = (r(t)X^\pi(t) + \alpha_1\pi_1(t) + \alpha_2\pi_2(t))dt + (\sigma_{11}\pi_1(t) + \sigma_{21}\pi_2(t))dW_1(t) + \sigma_{12}\pi_1(t)dW_2(t), \\ X^\pi(0) = x_0 > 0 \end{cases}$$

22 The pricing kernel for the market setting becomes the following:

$$\xi(t) := \exp \left[ - \int_0^t r(s)ds - \frac{1}{2} \left( \frac{(\alpha_1 - \alpha_2 \frac{\sigma_{12}}{\sigma_{21}})^2}{\sigma_{11}^2} + \frac{\alpha_2^2}{\sigma_{21}^2} \right) t - \frac{\alpha_2}{\sigma_{21}} W_1(t) - \frac{\alpha_1 - \alpha_2 \frac{\sigma_{11}}{\sigma_{21}}}{\sigma_{12}} W_2(t) \right],$$

23  $(\xi(t)X^\pi(t))_{t \in [0, T]}$  is a martingale in light of an application of Itô's lemma, and so

24 Assumption 2.3 follows.

1 Both Brownian noises  $W_1(\cdot)$  and  $W_2(\cdot)$  can be hedged out perfectly through trad-  
 2 ing the stock and the zero-coupon bond simultaneously, so the market is complete  
 3 and Assumption 2.4 is satisfied.

4 **2.2. Unique existence of optimal solution.** In this subsection, we state the  
 5 existence results of present utility-risk problems. Though our present setting is  
 6 more general than the one in [39], all results in this subsection can be established  
 7 using the same arguments. Our Problem 1 can be converted into the following static  
 8 one:

**Problem 2.**

$$\text{Maximize } \Psi(X), \quad \text{subject to } X \in \mathcal{X} \text{ and } \mathbb{E}[\xi(T)X] = x_0,$$

9 where  $\Psi : \mathcal{X} \rightarrow \mathbb{R}$  is defined by  $\Psi(X) := \mathbb{E}[U(X)] - \gamma\mathbb{E}[D(\mathbb{E}[X] - X)]$  and  $\mathcal{X} :=$   
 10  $\{X \in \mathcal{L}^2 | X \in \mathcal{F}_T \text{ and } X \text{ satisfies (3)}\}$ .

11 Under Assumptions 2.3 and 2.4, each feasible solution to Problem 2 is attainable  
 12 by an admissible control with the initial wealth  $x_0$ , so it can be shown that the  
 13 optimal solution of Problem 2 is the optimal terminal wealth of Problem 1:

14 **Theorem 2.9.** *If  $\pi(\cdot) \in \mathcal{A}$  is optimal for Problem 1, then  $X^\pi(T)$  is optimal for*  
 15 *Problem 2. Conversely, if  $X \in \mathcal{X}$  is optimal for Problem 2, there exists  $\pi(\cdot) \in \mathcal{A}$*   
 16 *such that  $X^\pi(T) = X$  and  $\pi(\cdot)$  is optimal for Problem 1.*

17 Note that the maximisation in Problem 2 is confined to the set  $\mathcal{X}$ , so that the  
 18 solution obtained in Problem 2 is an admissible terminal wealth in Problem 1. Since  
 19 Theorem 2.9 provides the equivalence between Problems 1 and 2, our present paper  
 20 now aims to establish an admissible terminal wealth  $X \in \mathcal{X}$  that maximizes  $\Psi(X)$ .

21 **Theorem 2.10.** *Under Assumption 2.5 so that  $U$  satisfies Property 2.1 and  $D$  is*  
 22 *given by Definition 2.2,  $\hat{X} = I(M, \gamma R + Y\xi(T))$  is the unique optimal terminal*  
 23 *wealth of the utility-downside-risk problem where  $I$  is an implicit function  $I : \mathbb{R} \times$   
 24  $(0, \infty) \rightarrow (0, \infty)$  satisfying:*

$$U'(I(m, y)) + \gamma D'(m - I(m, y)) - y = 0, \quad \text{for any } (m, y) \in \mathbb{R} \times (0, \infty), \quad (7)$$

25 and  $(Y, M, R) \in \mathbb{R}^3$  is the unique solution to the system of nonlinear equations

$$\mathbb{E}[\xi(T)I(M, \gamma R + Y\xi(T))] = x_0, \quad (8)$$

$$\mathbb{E}[I(M, \gamma R + Y\xi(T))] = M, \quad (9)$$

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi(T)))] = R. \quad (10)$$

26 *Proof.* The outline of the proof is given in Appendix A. □

27 **3. Main result: weak convergence of optimal payoffs.** In this section, we  
 28 first establish the convergence of optimal terminal payoffs for a sequence of utility-  
 29 risk problems, namely, if a sequence of pricing kernels converges weakly to a limit,  
 30 then the respective optimal terminal payoffs will converge in distribution to that  
 31 under the limiting pricing kernel.

32 We consider a sequence of markets, each of which satisfies Assumptions 2.3 and  
 33 2.4; denote them by  $\{\Pi^{(N)}\}_{N \in \mathbb{N}}$ , and each market  $\Pi_N$  has its own unique pricing  
 34 kernel  $\xi^{(N)}(\cdot)$ . Meanwhile, we have a sequence of the random terminal values of pric-  
 35 ing kernels (or simply call it the terminal pricing kernel for convenience), namely,  
 36  $\{\xi^{(N)}\}_{N \in \mathbb{N}}$  with  $\xi^{(N)} := \xi^{(N)}(T)$ . By Theorem 2.10, we know that each market  
 37 possesses a unique optimal terminal wealth for the utility risk problem, denoted by

1  $\hat{X}^{(N)} \in \mathcal{L}^2$ . We aim to show that if this sequence of terminal pricing kernels con-  
 2 verges in distribution to a terminal pricing kernel  $\xi^* \in \mathcal{L}^2$  corresponding to a market  
 3  $\Pi^*$ , then this limiting optimal terminal wealth solves for the utility-downside-risk  
 4 problem under the market  $\Pi^*$ .

5 **Theorem 3.1.** *Given that  $U$  and  $D$  respectively satisfy Property 2.1 and Definition*  
 6 *2.2. Under all markets  $\Pi^{(N)}$  and  $\Pi^*$  satisfying Assumptions 2.3, 2.4, and 2.5*  
 7 *with their corresponding unique terminal pricing kernels  $\xi^{(N)}$  and  $\xi^* \in \mathcal{L}^2$ , and*  
 8 *optimal terminal wealths are given by  $\hat{X}^{(N)}$  and  $\hat{X}^*$  in Theorem 2.10. Suppose*  
 9 *that a sequence of their terminal pricing kernels  $\{\xi^{(N)}\}_{N \in \mathbb{N}}$  satisfy the following*  
 10 *conditions:*

- 11 (i)  $\{\xi^{(N)}\}_{N \in \mathbb{N}}$  converges in distribution to  $\xi^*$  as  $N \rightarrow \infty$ .  
 12 (ii) For any  $K > 0$ ,  $\{\xi^{(N)}((U')^{-1}(\xi^{(N)}) + K)\}_{N \in \mathbb{N}}$  is uniformly integrable.

13 Then the sequence  $\{\hat{X}^{(N)}\}_{N \in \mathbb{N}}$  converges in distribution to  $\hat{X}^*$ . Hence,  $\Psi(\hat{X}^{(N)}) \rightarrow$   
 14  $\Psi(\hat{X}^*)$ , i.e., the sequence of optimal value functions of the utility-risk problem under  
 15  $\Pi^{(N)}$  converges to that under  $\Pi^*$ .

16 *Proof.* By Theorem 2.10, the unique optimal terminal wealth for the utility-downside-  
 17 risk problem under  $\Pi^{(N)}$  is given by  $\hat{X}^{(N)} = I(M^{(N)}, \gamma R^{(N)} + Y^{(N)}\xi^{(N)})$ , where  
 18 the function  $I$  is given in (7) and the triple  $(Y^{(N)}, M^{(N)}, R^{(N)})$  satisfies the system  
 19 of equations in (8)-(10) under  $\xi(T) = \xi^{(N)}$ . Similarly, the unique optimal terminal  
 20 wealth under  $\Pi^*$  is given by  $\hat{X}^* = I(M^*, \gamma R^* + Y^*\xi^*)$ . Note that the choice of  $I$   
 21 in (7) is independent of the choice of pricing kernel  $\xi$ .

22 By Skorokhod's representation theorem and condition (i), there exists a sequence  
 23 of random variables  $\{\zeta^{(N)}\}_{N \in \mathbb{N}}$  in which  $\zeta^{(N)}$  has the same distribution as  $\xi^{(N)}$  such  
 24 that  $\{\zeta^{(N)}\}_{N \in \mathbb{N}}$  almost surely converges to a random variable  $\zeta^*$  which has the same  
 25 distribution as  $\xi^*$ . To simplify the notation, we denote

$$\tilde{X}^{(N)} := I\left(M^{(N)}, \gamma R^{(N)} + Y^{(N)}\zeta^{(N)}\right) \text{ and } \tilde{X}^* := I\left(M^*, \gamma R^* + Y^*\zeta^*\right). \quad (11)$$

26 The following proposition shows that the sequence of  $\{(Y^{(N)}, M^{(N)}, R^{(N)})\}_{N \in \mathbb{N}}$   
 27 converges to  $(Y^*, M^*, R^*)$ ; its technical proof is postponed to Appendix B.

28 **Proposition 3.2.**  $\lim_{N \rightarrow \infty} (Y^{(N)}, M^{(N)}, R^{(N)}) = (Y^*, M^*, R^*)$ .

29 By Proposition 3.2, the regularity of  $I$  ensures that  $\tilde{X}^{(N)} \rightarrow \tilde{X}^*$  a.s.; together  
 30 with the fact that  $\tilde{X}^{(N)}$  and  $\hat{X}^{(N)}$  (including  $N = *$ ) have the same distribution,  
 31  $\hat{X}^{(N)} \rightarrow \hat{X}^*$  in distribution. Since for each  $m$ ,  $I(m, y)$  is decreasing in  $y$  and so, for  
 32 each  $N$ ,

$$\tilde{X}^{(N)} = I\left(M^{(N)}, \gamma R^{(N)} + Y^{(N)}\zeta^{(N)}\right) \leq I\left(M^{(N)}, \gamma R^{(N)}\right).$$

33 By Proposition 3.2,  $I(M^{(N)}, \gamma R^{(N)}) \rightarrow I(M^*, \gamma R^*) < \infty$ , so we have  $\tilde{X}^{(N)} \leq$   
 34  $I(M^*, \gamma R^*) + 1$  a.s., for all large enough  $N$ . A simple application of the domi-  
 35 nated convergence theorem,  $\Psi(\tilde{X}^{(N)}) \rightarrow \Psi(\tilde{X}^*)$ . Together with the fact that  $\tilde{X}^{(N)}$   
 36 and  $\hat{X}^{(N)}$  (including  $N = *$ ) have the same distribution, the convergence of value  
 37 functions follows.  $\square$

38 Theorem 3.1 implies that the optimal solution for the utility-downside-risk prob-  
 39 lem is robust with respect to the change in pricing kernel. If the "change" in pricing  
 40 kernel is small enough, the optimal terminal wealth does not deviate much, and so  
 41 does the change in value function.

1 **4. Applications: convergence from discrete to continuous setting.** In this  
 2 section, we shall show that the sequence of terminal pricing kernels in the binomial  
 3 tree models satisfies the conditions stated in Theorem 3.1, with the limit of the  
 4 sequence of terminal pricing kernels being that under the Black-Scholes model. We  
 5 first recall the celebrated martingale central limit theorem, which is vital to our  
 6 main results in this section.

7 **Theorem 4.1** (Theorem 7.4.1 of [15]). *Assume that  $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$  are continuous  
 8 functions such that the following SDE*

$$dX(t) = b(X(t)) dt + \sigma(X(t)) dW(t), \quad X(0) = x_0,$$

9 *admits a unique weak solution  $X^*(\cdot)$ . Suppose that  $X^{(N)}(\cdot)$  is a sequence of Markov  
 10 processes with sample path in  $D[0, T]$ , which is the space of functions from  $[0, T]$   
 11 to  $\mathbb{R}$  that are right continuous with left limits under the Skorokhod's topology. If  
 12 there exist processes  $L^{(N)}(\cdot)$  and  $A^{(N)}(\cdot)$  with sample paths in  $D[0, T]$ , such that the  
 13 following conditions hold:*

- 14 1.  $\{X^{(N)}(0)\}_{N \in \mathbb{N}}$  converges to  $x_0$  in distribution;
- 15 2.  $A^{(N)}(t)$  is non-decreasing in  $t$  a.s.;
- 16 3.  $M^{(N)}(\cdot) := (X^{(N)} - L^{(N)})(\cdot)$  and  $((M^{(N)})^2 - A^{(N)})(\cdot)$  are martingales;
- 17 4. for all  $q > 0$ ,

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E} \left[ \sup_{t \leq \tau_q^{(N)}} \left| X^{(N)}(t) - X^{(N)}(t-) \right|^2 \right] &= 0; \\ \lim_{N \rightarrow \infty} \mathbb{E} \left[ \sup_{t \leq \tau_q^{(N)}} \left| L^{(N)}(t) - L^{(N)}(t-) \right|^2 \right] &= 0; \\ \lim_{N \rightarrow \infty} \mathbb{E} \left[ \sup_{t \leq \tau_q^{(N)}} \left| A^{(N)}(t) - A^{(N)}(t-) \right| \right] &= 0, \end{aligned}$$

18 where  $\tau_q^{(N)} := \inf \{t \leq T : |X^{(N)}(t)| \geq q \text{ or } |X^{(N)}(t-)| \geq q\} \wedge T$ ;

- 19 5. for all  $q > 0$ , the following two expressions:

- 20 i)  $\sup_{t \leq \tau_q^{(N)}} \left| L^{(N)}(t) - \int_0^t b(X^{(N)}(s)) ds \right|$ ,
- 21 ii)  $\sup_{t \leq \tau_q^{(N)}} \left| A^{(N)}(t) - \int_0^t (\sigma(X^{(N)}(s)))^2 ds \right|$
- 22 both converge to 0 in probability as  $N \rightarrow \infty$ .

23 Then the process  $\{X^{(N)}(\cdot)\}_{N \in \mathbb{N}}$  converges weakly to  $X^*(\cdot)$  in the sense that, for  
 24 any bounded continuous mapping  $h$  in  $D([0, T])$  to  $\mathbb{R}$ , we have  $\mathbb{E}[h(X^{(N)}(\cdot))] \rightarrow$   
 25  $\mathbb{E}[h(X^*(\cdot))]$  as  $N \rightarrow \infty$ .

26 To have the main result of this section, the next assumption stronger than As-  
 27 sumption 2.5, which has been adopted in Remark 4.4 in [21], is required:

28 **Assumption 4.2.** There exist  $\eta \in (0, 1)$  and  $\beta > 1$  such that for any  $y > 0$ ,  
 29  $U'(\beta y) \leq \eta U'(y)$ .

30 **Theorem 4.3.** *Given that Assumption 4.2 holds,  $U$  satisfies Property 2.1 and  $D$   
 31 is given by Definition 2.2, and  $\xi_n^{(N)}$  is given in (6). The sequence of the terminal  
 32 pricing kernels  $\{\xi_N^{(N)}\}_{N \in \mathbb{N}}$  satisfies the conditions stated in Theorem 3.1 with  $\xi^*(T)$   
 33 where  $\xi^*(t)$  is defined in (4). Hence, the optimal terminal wealths corresponding  
 34 to the sequence of binomial tree models converge in distribution to that under the  
 35 Black-Scholes model.*

1 Before we proceed to the proof, we first need the following two technical lemmas  
2 for verifying the conditions specified in Theorem 4.3.

3 **Lemma 4.4.** Define  $\bar{\xi}^*(t) := e^{-\frac{\alpha}{2\sigma^2}t - \frac{\alpha}{\sigma}W(t)}$  and  $\bar{\xi}^{(N)}(t) = \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)}$  for  $0 \leq t \leq T$ ,  
4 where  $\bar{\xi}_n^{(N)} := \prod_{k=1}^n (1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_k^{(N)})$ . The sequence of processes  $\{\bar{\xi}_{\lfloor \cdot / \Delta t \rfloor}^{(N)}\}_{N \in \mathbb{N}}$   
5 converges weakly to  $\bar{\xi}^*(\cdot)$ .

6 *Proof.* We first note that  $\bar{\xi}^*(\cdot)$  satisfies the following SDE:

$$d\bar{\xi}^*(t) = -\frac{\alpha}{\sigma} \bar{\xi}^*(t) dW(t), \quad \bar{\xi}^*(0) = 1;$$

7 and so,  $\bar{\xi}^*(0) = \bar{\xi}^{(N)}(0) = \bar{\xi}_0^{(N)} = 1$ , and hence Condition 1 is fulfilled.

8 By the very definition,  $\bar{\xi}^{(N)}(\cdot)$  is a martingale. We then set  $L^{(N)}(t) := 0$  and  
9  $A^{(N)}(t) := A_{\lfloor t/\Delta t \rfloor}^{(N)}$  where  $A_k^{(N)}$  is defined recursively via the relation:

$$A_{k+1}^{(N)} = A_k^{(N)} + \left(\bar{\xi}_k^{(N)}\right)^2 \frac{\alpha^2}{\sigma^2} \Delta t, \quad A_0^{(N)} = 0.$$

10 Clearly,  $A^{(N)}(t)$  is non-decreasing, and Condition 2 is satisfied. Besides,

$$\begin{aligned} & \mathbb{E} \left[ \left( \bar{\xi}_{k+1}^{(N)} \right)^2 - A_{k+1}^{(N)} \mid \bar{\xi}_k^{(N)}, A_k^{(N)} \right] \\ &= \left( \bar{\xi}_k^{(N)} \right)^2 \mathbb{E} \left[ \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_{k+1}^{(N)} \right)^2 \right] - A_k^{(N)} - \left( \bar{\xi}_k^{(N)} \right)^2 \frac{\alpha^2}{\sigma^2} \Delta t = \left( \bar{\xi}_k^{(N)} \right)^2 - A_k^{(N)}, \end{aligned}$$

11 which implies that the process  $((\bar{\xi}^{(N)})^2 - A^{(N)})(\cdot)$  is a martingale, and thus Condi-  
12 tion 3 is validated too.

13 For any  $N \in \mathbb{N}$  and  $q > 0$ , since  $(\bar{\xi}^{(N)}(t-))^2 \leq q^2(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t})^2 \leq 2q^2(1 + \frac{\alpha^2 T}{\sigma^2 N})$   
14 by the definitions of  $\bar{\xi}^{(N)}(t)$  and  $\tau_q^{(N)}$ , we have

$$\left| \bar{\xi}^{(N)}(t) - \bar{\xi}^{(N)}(t-) \right|^2 \leq \left| \bar{\xi}^{(N)}(t-) \frac{\alpha}{\sigma} \sqrt{\Delta t} \right|^2 \leq 2q^2 \left( 1 + \frac{\alpha^2 T}{\sigma^2 N} \right) \frac{\alpha^2 T}{\sigma^2 N},$$

15 for any  $t \leq \tau_q^{(N)}$ , thus

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ \sup_{t \leq \tau_q^{(N)}} \left| \bar{\xi}^{(N)}(t) - \bar{\xi}^{(N)}(t-) \right|^2 \right] \leq \lim_{N \rightarrow \infty} 2q^2 \left( 1 + \frac{\alpha^2 T}{\sigma^2 N} \right) \frac{\alpha^2 T}{\sigma^2 N} = 0.$$

16 Since  $|A^{(N)}(t) - A^{(N)}(t-)| \leq (\bar{\xi}^{(N)}(t-))^2 \frac{\alpha^2}{\sigma^2} \Delta t$ , similar derivations give  $\lim_{N \rightarrow \infty} \mathbb{E}[\sup_{t \leq \tau_q^{(N)}} |A^{(N)}(t) -$   
17  $A^{(N)}(t-)|] = 0$  which justify the validity of Condition 4. Finally, for any  $q > 0$ ,

$$\begin{aligned} \sup_{t \leq \tau_q^{(N)}} \left| A^{(N)}(t) - \int_0^t \left( \frac{\alpha}{\sigma} \bar{\xi}^{(N)}(s) \right)^2 ds \right| &= \sup_{t \leq \tau_q^{(N)}} \frac{\alpha^2}{\sigma^2} \left( \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right)^2 \left( t - \left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t \right) \\ &\leq \sup_{t \leq \tau_q^{(N)}} \frac{2\alpha^2}{\sigma^2} q^2 \left( 1 + \frac{\alpha^2 T}{\sigma^2 N} \right) \Delta t = \frac{2\alpha^2}{\sigma^2} q^2 \left( 1 + \frac{\alpha^2 T}{\sigma^2 N} \right) \frac{T}{N} \rightarrow 0, \end{aligned}$$

18 as  $N \rightarrow \infty$ . Thus, Condition 5 is also fulfilled.

19 Therefore, we can now apply Theorem 4.1 to conclude that  $\{\bar{\xi}^{(N)}(\cdot)\}_{N \in \mathbb{N}}$  converge  
20 weakly to  $\bar{\xi}^*(\cdot)$  as  $N \rightarrow \infty$ . □

21  
22 **Lemma 4.5.** (i)  $\mathbb{E}[(\xi_N^{(N)})^4] < e^{KT}$ , where  $K$  is a constant independent of  $N$ ;

1 (ii) For every  $n \in \mathbb{N}$ , there exists a finite constant  $M_n > 0$  such that  $\mathbb{E}[(\xi_N^{(N)})^{-n}] \leq$   
 2  $M_n$  for all  $N \in \mathbb{N}$ .

3 *Proof.* Define  $R_k^{(N)} := -\frac{\alpha}{\sigma}\sqrt{\Delta t}Z_k^{(N)}$ , we then rewrite  $\xi_N^{(N)} = \prod_{k=1}^N \frac{1+R_k^{(N)}}{1+r\Delta t}$ .  
 4 Thus, provided that  $N$  is large enough, for each  $k$ ,

$$\begin{aligned} \mathbb{E} \left[ \left(1 + R_k^{(N)}\right)^4 \right] &= \mathbb{E} \left[ 1 - 4\frac{\alpha}{\sigma}\sqrt{\Delta t}Z_k^{(N)} + 6\frac{\alpha^2}{\sigma^2}\Delta t - 4\frac{\alpha^3}{\sigma^3}(\Delta t)^{3/2} \left(Z_k^{(N)}\right)^3 + \frac{\alpha^4}{\sigma^4}(\Delta t)^2 \right] \\ &\leq 1 + \left(6\frac{\alpha^2}{\sigma^2} + \frac{\alpha^4}{\sigma^4}\right)\Delta t. \end{aligned}$$

5 Therefore,

$$\mathbb{E} \left[ \left(\xi_N^{(N)}\right)^4 \right] \leq \mathbb{E} \left[ \prod_{i=1}^N \left(1 + R_k^{(N)}\right)^4 \right] \leq \left(1 + \left(6\frac{\alpha^2}{\sigma^2} + \frac{\alpha^4}{\sigma^4}\right)\Delta t\right)^N \leq e^{(6\frac{\alpha^2}{\sigma^2} + \frac{\alpha^4}{\sigma^4})T},$$

6 for all  $N$ . Hence, the claim (i) follows. Fix  $n \in \mathbb{N}$ , by direct computations with an  
 7 application of Taylor series expansions, we have:

$$\begin{aligned} \mathbb{E} \left[ \left(1 + R_k^{(N)}\right)^{-n} \right] &= \frac{1}{2} \left( \left(1 - \frac{\alpha}{\sigma}\sqrt{\Delta t}\right)^{-n} + \left(1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}\right)^{-n} \right) \\ &= 1 + \frac{(n+1)n\alpha^2}{2\sigma^2}\Delta t + O(\Delta t^{\frac{3}{2}}). \end{aligned}$$

8 Then, there exists a constant  $K_n$  independent of  $N$  such that

$$\begin{aligned} \mathbb{E} \left[ \left(\xi_N^{(N)}\right)^{-n} \right] &= \mathbb{E} \left[ \prod_{i=1}^N \left(\frac{1 + R_k^{(N)}}{1 + r\Delta t}\right)^{-n} \right] = \left(1 + \left(rn + \frac{(n+1)n\alpha^2}{2\sigma^2}\right)\Delta t + O(\Delta t^{\frac{3}{2}})\right)^N \\ &\leq e^{K_n T}. \end{aligned}$$

9

□

10 *Proof of Theorem 4.3.*

11 By Lemma 4.4,  $\bar{\xi}^{(N)}(T) = \bar{\xi}_N^{(N)} \rightarrow \bar{\xi}^*(T)$  in distribution as  $N \rightarrow \infty$ . Since  
 12  $\lim_{N \rightarrow \infty} (1 + r\Delta t)^N = e^{rT}$ , we also have  $\xi_N^{(N)} = (1 + r\Delta t)^{-N} (\bar{\xi}_N^{(N)}) \rightarrow e^{-rT} \bar{\xi}^*(T) =$   
 13  $\xi^*(T)$  in distribution as  $N \rightarrow \infty$ . Hence, Condition (i) in Theorem 3.1 follows.

14 By Assumption 4.2, its inverse function  $(U')^{-1}$  satisfies the property:  $(U')^{-1}(\eta x) \leq$   
 15  $\beta(U')^{-1}(x)$  for all  $x > 0$ . Further,  $(U')^{-1}(\eta^y x) \leq \beta^{y+1}(U')^{-1}(x)$  for all  $x, y > 0$   
 16 and we also have  $(U')^{-1}(\eta^y x) \leq \beta^{y+1}(U')^{-1}(x) + (U')^{-1}(x)$  for all  $y \in \mathbb{R}, x > 0$ ;  
 17 therefore, we have

$$(U')^{-1}(z) \leq \beta z^{\frac{\ln \beta}{\ln \eta}} (U')^{-1}(1) + (U')^{-1}(1) \text{ for any } z > 0.$$

18 Then, by Minkowski inequality, we have

$$\left( \mathbb{E} \left[ \left( (U')^{-1} \left( \xi_N^{(N)} \right) \right)^4 \right] \right)^{\frac{1}{4}} \leq (U')^{-1}(1) \left( \beta^4 \mathbb{E} \left[ \left( \xi_N^{(N)} \right)^{4 \frac{\ln \beta}{\ln \eta}} \right] + 1 \right)^{\frac{1}{4}}.$$

19 According to Lemma 4.5(ii), we now have  $\mathbb{E}[\left((U')^{-1}(\xi_N^{(N)})\right)^4] < M_0$  for some constant  
 20  $M_0 \in (0, +\infty)$  independent of  $N$ . Consider that, for any  $K > 0$ ,

$$\mathbb{E} \left[ \left( \xi_N^{(N)} \left( (U')^{-1} \left( \xi_N^{(N)} \right) + K \right) \right)^2 \right] \leq \mathbb{E} \left[ \left( \frac{1}{2} \left( \left( \xi_N^{(N)} \right)^2 + \left( (U')^{-1} \left( \xi_N^{(N)} \right) + K \right)^2 \right) \right)^2 \right]$$

$$\leq \mathbb{E} \left[ \left( \frac{1}{2} \left( \left( \xi_N^{(N)} \right)^2 + 2 \left( (U')^{-1} \left( \xi_N^{(N)} \right) \right)^2 + 2K^2 \right) \right)^2 \right].$$

1 Then, by a simple application of Minkowski inequality together with Lemma 4.5(i),  
 2 we can see that  $\mathbb{E}[(\xi^{(N)}((U')^{-1}(\xi_N^{(N)}) + K))^2] < M$  for some constant  $M \in (0, +\infty)$   
 3 independent of  $N$ . Hence, Condition (ii) in Theorem 3.1 follows by page 127 to 128  
 4 in [38]. By Theorem 3.1, the desired convergence result for the optimal terminal  
 5 wealth follows.

6

□

7 Furthermore, a similar approximation result for the optimal value function can  
 8 be obtained:

9 **Corollary 4.6.** *Under Assumption 4.2, the optimal value functions for the utility-*  
 10 *downside-risk problem corresponding to the sequence of binomial tree models as in*  
 11 *Example 2.7 converges to that under the continuous-time Black-Scholes model as in*  
 12 *Example 2.6 as the time interval mesh size decreases to 0 (i.e.  $N \rightarrow \infty$ ).*

#### 13 4.1. Weak convergence of optimal wealth and investment processes.

14 In this subsection, we establish the weak convergence of the optimal wealth processes  
 15 and the corresponding investment strategies of a sequence of binomial tree models  
 16 to those under the Black-Scholes model in (2.6), provided that some regularity  
 17 conditions are satisfied. More precisely, if we consider the optimal wealth at  $t$ ,  
 18  $\hat{X}^*(t)$ , under the Black-Scholes model to be a function in terms of both the current  
 19 time  $t$  and the time- $t$  pricing kernel  $\xi^*(t)$ , and such a function is a third-order  
 20 continuously differentiable one, and all derivatives are of polynomial growth, then  
 21 the weak convergence of both wealth processes and optimal investment strategies  
 22 follows.

23 Indeed, under the Black-Scholes model in Example 2.6, the optimal terminal  
 24 wealth is given by  $\hat{X}^* = I(M^*, \gamma R^* + Y^* \xi^*(T))$ , where  $I$  is given in (7), and  
 25  $(Y^*, M^*, R^*)$  is the solution of the system of nonlinear equations of (8)-(10) under  
 26  $\xi = \xi^*(T)$ . Then, the corresponding wealth process  $\hat{X}^*(\cdot)$  that attains  $\hat{X}^*$  at  $T$  is  
 27 given by

$$\begin{aligned} \hat{X}^*(t) &= (\xi^*(t))^{-1} \mathbb{E}[\xi^*(T) I(M^*, \gamma R^* + Y^* \xi^*(T)) | \xi^*(t)] \\ &= e^{rt} \left( \bar{\xi}^*(t) \right)^{-1} \mathbb{E} \left[ e^{-rT} \bar{\xi}^*(T) I \left( M^*, \gamma R^* + Y^* e^{-rT} \bar{\xi}^*(T) \right) \middle| \bar{\xi}^*(t) \right] \\ &= h^{(Y^*, M^*, R^*)} \left( t, \bar{\xi}^*(t) \right), \end{aligned} \quad (12)$$

28 where recall  $\bar{\xi}^*(t) := e^{-\frac{\alpha^2}{2\sigma^2}t - \frac{\alpha}{\sigma}W(t)}$  and

$$h^{(Y, M, R)}(t, \bar{\xi}) := e^{rt} \bar{\xi}^{-1} \mathbb{E} \left[ e^{-rT} \bar{\xi}^*(T) I \left( M, \gamma R + Y e^{-rT} \bar{\xi}^*(T) \right) \middle| \bar{\xi}^*(t) = \bar{\xi} \right]. \quad (13)$$

29 By an application of Ito's Lemma, we can obtain the investment strategy:

$$\pi^*(t) = -\frac{\alpha}{\sigma^2} \bar{\xi}^*(t) \frac{\partial}{\partial \bar{\xi}} h^{(Y^*, M^*, R^*)} \left( t, \bar{\xi}^*(t) \right). \quad (14)$$

30 Similarly, under the binomial tree model specified in Example 2.7, the optimal  
 31 wealth process is given by

$$\hat{X}^{(N)}(t)$$

$$\begin{aligned}
&= \left( \xi^{(N)}(t) \right)^{-1} \mathbb{E} \left[ \xi^{(N)}(T) I \left( M^{(N)}, \gamma R^{(N)} + Y^{(N)} \xi^{(N)}(T) \right) \mid \xi^{(N)}(t) \right] \\
&= (1 + r\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor} \left( \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \right)^{-1} \mathbb{E} \left[ (1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left( M^{(N)}, \gamma R^{(N)} + Y^{(N)} e^{-rT} \bar{\xi}_N^{(N)} \right) \mid \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \right] \\
&= h^{(N, Y^{(N)}, M^{(N)}, R^{(N)})} \left( t, \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \right), \tag{15}
\end{aligned}$$

- 1 where  $(Y^{(N)}, M^{(N)}, R^{(N)})$  is the solution of (8)-(10) under  $\xi = \xi^{(N)}(T)$ ,  $\bar{\xi}_{n+1}^{(N)} :=$   
2  $(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_{n+1}^{(N)}) \bar{\xi}_n^{(N)}$ ,  $\bar{\xi}_0^{(N)} := 1$ , and

$$h^{(N, Y, M, R)}(t, \bar{\xi}) := (1 + r\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor} \bar{\xi}^{-1} \mathbb{E} \left[ (1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left( M, \gamma R + Y e^{-rT} \bar{\xi}_N^{(N)} \right) \mid \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} = \bar{\xi} \right].$$

- 3 For simplicity, we denote  $h^{(N, Y^{(N)}, M^{(N)}, R^{(N)})}$  by  $h^{(N)}$  unless otherwise specified. By  
4 considering the dynamics (5), the investment strategy for the binomial tree model  
5 is given by

$$\begin{aligned}
\pi^{(N)}(t) &= \frac{1}{2\sigma\sqrt{\Delta t}} \left( h^{(N)} \left( t + \Delta t, \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right) \right. \\
&\quad \left. - h^{(N)} \left( t + \Delta t, \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right) \right). \tag{16}
\end{aligned}$$

- 6 **Theorem 4.7.** *Suppose that the optimal wealth process  $X^*(\cdot)$  and portfolio  $\pi^*(\cdot)$*   
7 *under the Black-Scholes model are given by (12) and (14) respectively, while the*  
8 *optimal wealth process  $X^{(N)}(\cdot)$  and portfolio  $\pi^{(N)}(\cdot)$  under the binomial tree model*  
9 *are given by (15) and (16) respectively. If  $h^{(Y, M, R)} \in C_p^{3,3}([0, T] \times \mathbb{R}_+)$  as defined*  
10 *in (13), i.e.  $h$  is differentiable with respect to both arguments up to the third order,*  
11 *and all of these derivatives up to the third order are continuous and satisfy the*  
12 *following polynomial growth condition:*

- 13 *There exist  $A > 0$  and  $q > 1$  such that*

$$\left| h^{(Y, M, R)}(t, \bar{\xi}) \right| \leq A(1 + |\bar{\xi}|^q), \quad \forall (t, \bar{\xi}, Y, M, R) \in [0, T] \times \mathbb{R}_+^4.$$

- 14 *Further suppose that  $h_{\bar{\xi}\bar{\xi}\bar{\xi}}$  is Lipschitz continuous with respect to  $\bar{\xi}$ . Then the processes*  
15  *$\{X^{(N)}(\cdot)\}_{N \in \mathbb{N}}$  and  $\{\pi^{(N)}(\cdot)\}_{N \in \mathbb{N}}$  converge weakly to  $X^*(\cdot)$  and  $\pi^*(\cdot)$  as  $N \rightarrow \infty$ .*

- 16 *Proof.* See Appendix C. □

- 17 The polynomial growth condition can be guaranteed if  $I$  in (7) is differentiable  
18 with respect to the second argument up to the sixth order, and all of these deriva-  
19 tives are continuous and satisfy the polynomial growth condition; indeed, by this  
20 time,  $h$  is continuously differentiable with respect to the argument  $\bar{\xi}$  up to the sixth  
21 order by Theorem 5.5 in [17]. Then, as  $h$  satisfies the following boundary value  
22 problem by Kolmogorov's forward equation:

$$\begin{cases} rh = \frac{\partial h}{\partial t} + \frac{\alpha^2}{\sigma^2} \bar{\xi} \frac{\partial h}{\partial \bar{\xi}} + \frac{1}{2} \frac{\alpha^2}{\sigma^2} \bar{\xi}^2 \frac{\partial^2 h}{\partial \bar{\xi}^2} \\ h^{(Y, M, R)}(T, \bar{\xi}) = I(M, \gamma R + Y \bar{\xi}), \end{cases} \tag{17}$$

- 23 we can then deduce the continuous differentiability with respect to the argument  $t$   
24 up to the third order by expressing  $\frac{\partial h}{\partial t}$ ,  $\frac{\partial^2 h}{\partial t^2}$ , and  $\frac{\partial^3 h}{\partial t^3}$  in terms of the special partial  
25 derivatives of  $h$  up to the sixth order. The idea of proof of the weak convergence in  
26 Theorem 4.7 is based on the arguments proposed in [19]. We first consider:

$$X^{(N)}(t)$$

$N$	$Y^{(N)}$	$M^{(N)}$	$R^{(N)}$	$\pi^{(N)}(0)$	$U^{(N)}$	$SV^{(N)}$	$J^{(N)}$	Change
25	1.2547	3.9077	1.5571	2.3042	3.4072	2.2446	3.182717	
50	1.2561	3.8816	1.5456	2.2517	3.3959	2.2093	3.174965	-0.007752
100	1.2570	3.8666	1.5428	2.2262	3.3902	2.1905	3.171126	-0.003839
200	1.2573	3.8607	1.5394	2.2135	3.3874	2.1823	3.169219	-0.001908
400	1.2575	3.8573	1.5383	2.2073	3.3861	2.1779	3.168274	-0.000945
800	1.2575	3.8557	1.5375	2.2041	3.3854	2.1758	3.167795	-0.000479
1600	1.2576	3.8550	1.5371	2.2025	3.3850	2.1748	3.167557	-0.000238
3200	1.2576	3.8546	1.5369	2.2018	3.3849	2.1742	3.167439	-0.000118
6400	1.2576	3.8544	1.5368	2.2014	3.3848	2.1740	3.167379	-0.000060

TABLE 1. Convergence of optimal solutions under the case of utility-semivariance.

$$\begin{aligned}
&= h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left( \left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t, \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \right) \\
&\quad - \left[ h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left( \left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t, \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \right) - h^{(N)} \left( \left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t, \bar{\xi}_{\lfloor \frac{t}{\Delta t} \rfloor}^{(N)} \right) \right].
\end{aligned} \tag{18}$$

1 The first term in (18) can be shown to converge weakly to  $X^*(\cdot)$  by establishing that  
2 the sequence of process  $\{\bar{\xi}^{(N)}(\cdot)\}_{N \in \mathbb{N}}$  converges weakly to  $\bar{\xi}^*(\cdot)$  and then applying  
3 the continuous mapping theorem. The second term in (18) converges weakly to the  
4 zero process via an application of the martingale central limit theorem (Theorem  
5 4.1).

6 **5. Numerical example.** In this section, we shall numerically illustrate Corollary  
7 4.6 that the sequence of optimal utility risk terminal wealths and value functions  
8 under binomial tree models converge as the number of period  $N$  increases to infinity,  
9 or equivalently,  $\Delta = \frac{T}{N} \rightarrow 0$ . To be realistic in any modest economic condition,  
10 we set the parameters to be:  $r = 0.03, \alpha = 0.07, \sigma = 0.2, T = 8, x_0 = 1$  and  
11  $\gamma = 0.1$ . Besides, we set the utility function to be a power function:  $U(x) =$   
12  $2x^{\frac{1}{2}}$ , and so  $U'(x) = x^{-\frac{1}{2}}$ , and the risk function adopted is semivariance ( $D(x) =$   
13  $\frac{x^2}{2}$ ). We compute each solution of the nonlinear system in (8)-(10) ( $Y^{(N)}, M^{(N)},$   
14  $R^{(N)}$ ) for every  $N$ , optimal control at  $t = 0$  ( $\pi^{(N)}(0)$ ), and then determine the  
15 optimal expected utility ( $U^{(N)}$ ), the semivariance ( $SV^{(N)}$ ), the optimal objective  
16 value ( $J^{(N)}$ ), and the sequential change in optimal objective value for different  
17 values of  $N$ .

18 According to the numerical results as quoted in Table 1, whenever we double the  
19 number of periods  $N$ , the change in value function is approximately halved. This  
20 result suggests that the convergence of the value function is in the linear order of  
21 the time interval of the binomial model. Following this trend pattern, the result  
22  $N = 50$  could serve as a good approximation to the optimal solution for the utility  
23 risk problem under the continuous-time Black-Scholes model since the possible error  
24 is less than 0.25% in terms of the optimal objective value.

25 **Appendix A. Outline of Proof of Theorem 2.10.** We outline the main idea  
26 of the proof, all the claims in this section can be established using the same idea in  
27 [39].

1 First, we recall Theorem 4.2 in [39], which derive a static problem, called *Non-*  
 2 *linear Moment Problem*, and the corresponding verification theorem to characterize  
 3 the optimal solution for Problem 2:

4 **Theorem A.1.** *Suppose that there exists  $\hat{X} \in \mathcal{X}$  satisfying  $U'(\hat{X}) \in \mathcal{L}^2$  and*  
 5  *$D'(\mathbb{E}[\hat{X}] - \hat{X}) \in \mathcal{L}^2$  and there exist constants  $Y, M, R \in \mathbb{R}$  so that the quadruple*  
 6  *$(\hat{X}, Y, M, R)$  solves for the Nonlinear Moment Problem,  $NMP(\xi(T))$ , involving*

$$Y\xi(T) = U'(\hat{X}) - \gamma R + \gamma D'(M - \hat{X}) \quad a.s., \quad (19)$$

7 *subject to the nonlinear moment constraints*

$$\mathbb{E}[\xi(T)\hat{X}] = x_0, \quad (20)$$

$$\mathbb{E}[\hat{X}] = M, \quad (21)$$

$$\mathbb{E}\left[D'(M - \hat{X})\right] = R. \quad (22)$$

8 *Then,  $\hat{X}$  is the unique optimal solution for Problem 2, and it is also the unique*  
 9 *optimal terminal wealth of Problem 1.*

10 Note that we denote (19)-(22) as  $NMP(\xi(T))$  since its solution depends on the  
 11 choice of the terminal random variable of the pricing kernel,  $\xi(T)$ . With Theorem  
 12 A.1, it is sufficient for us to solve NMP. To this end, we eliminate (19) with the  
 13 following result:

14 **Proposition A.2.** *There exists an implicit function  $I : \mathbb{R} \times (0, \infty) \rightarrow (0, \infty)$  sat-*  
 15 *isfying (7). Moreover, this function  $I$  possesses the following regularities:*

- 16 (a) For each  $y \in (0, \infty)$ ,
- 17 (i)  $I(m, y) < m$  for all  $m \in \{m \in \mathbb{R} : y > U'(m)\}$
- 18 (ii)  $I(m, y) = (U')^{-1}(y) \in (0, \infty)$  for all  $m \in \{m \in \mathbb{R} : y \leq U'(m)\}$
- 19 (b) (i) For each  $m \in \mathbb{R}$ ,  $I(m, y)$  is strictly decreasing in  $y$  on  $(0, \infty)$ ;
- 20 (ii) For each  $y \in (0, \infty)$ ,  $I(m, y)$  is strictly increasing in  $m$  on  $\{m \in \mathbb{R} | y \geq$   
 21  $U'(m)\}$ .
- 22 (c)  $I(m, y)$  is jointly continuous in  $(m, y)$  on  $\mathbb{R} \times (0, \infty)$ .

23 *Proof.* We only show the assertion (a)(i) as other assertions have been established  
 24 in Proposition 4.5 in [39]. Assume a contrary that the assertion (a)(i) does not  
 25 hold, i.e.  $I(m, y) \geq m$ . We have

$$U'(I(m, y)) \leq U'(I(m, y)) + \gamma D'(m - I(m, y)) = y < U'(m) \leq U'(I(m, y)),$$

26 which is absurd.  $\square$

27 With the implicit function  $I$ , Nonlinear Moment Problem (19)-(22) is further  
 28 reduced into the nonlinear programming problem (8)-(10).

29 Define  $\kappa := \text{esssup } \xi(T)$ . Now, we shall solve for roots  $Y, M$  and  $R$  for (8)-(10) one  
 30 by one via applying the intermediate value theorem successively as in Proposition  
 31 4.6 in [39]:

32 **Lemma A.3.** *Given  $Y \in (0, \infty)$ , we consider two exclusive cases: (i)  $M \in$*   
 33  *$((U')^{-1}(Y\kappa), \infty)$  and (ii)  $M \in (0, (U')^{-1}(Y\kappa))$ .*

34 (i) *there exists a unique  $R = R_{Y, M} \in (0, D'(M))$  satisfying*

$$\mathbb{E}[D'(M - I(M, \gamma R + Y\xi(T)))] = R; \quad (23)$$

1 or equivalently by (7):

$$\mathbb{E}[U'(I(M, \gamma R + Y\xi(T)))] = Y\mathbb{E}[\xi(T)]. \quad (24)$$

2 Furthermore, for every fixed  $Y$ ,  $R_{Y,M}$  is strictly increasing in  $M$ , and it is  
3 also strictly increasing in  $Y$  for each fixed  $M$ .

4 (ii)  $R = R_{Y,M} = 0$  uniquely solves (23).

5 **Lemma A.4.** Given  $Y \in (0, \infty)$  and  $R_{Y,M}$  as specified for each  $M \in (0, \infty)$  in  
6 Lemma A.3, there exists a unique  $M = M_Y \in ((U')^{-1}(Y\kappa), \infty)$  such that

$$\mathbb{E}[I(M, \gamma R_{Y,M} + Y\xi(T))] = M; \quad (25)$$

7 furthermore,  $M_Y$  is strictly decreasing in  $Y$ .

8 **Lemma A.5.** Given  $R_{Y,M}$  and  $M_Y$  as specified in Lemmas A.3 and A.4 respectively  
9 for each  $Y, M \in (0, \infty)$ , there exists a  $Y^* \in (0, \infty)$  such that

$$\mathbb{E}[\xi(T)I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}} + Y^*\xi(T))] = x_0. \quad (26)$$

10 According to lemmas A.3, A.4 and A.5, the triple  $(Y^*, M_{Y^*}, R_{Y^*, M_{Y^*}})$  solves the  
11 system of nonlinear equations in (8)-(10). Thus, the quadruple,  $(\hat{X}, Y^*, M_{Y^*}, R_{Y^*, M_{Y^*}})$   
12 with

$$\hat{X} := I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}} + Y^*\xi(T))$$

13 solves the Nonlinear Moment Problem (19)-(22). Next, it is clear that

$$0 \leq \hat{X} = I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}} + Y^*\xi(T)) \leq I(M_{Y^*}, \gamma R_{Y^*, M_{Y^*}}) < \infty. \quad (27)$$

14 With the uniform boundedness of  $\hat{X}$  in (27), we can verify  $U'(\hat{X}), D'(\mathbb{E}(\hat{X}) - \hat{X}) \in$   
15  $\mathcal{L}^2$  and  $\hat{X} \in \mathcal{X}$ . Then, by Theorem A.1,  $\hat{X}$  is the unique optimal solution for  
16 Problem 2.

17 Since  $\hat{X}$  is the unique solution and  $Y = \mathbb{E}[U'(\hat{X})]/\mathbb{E}[\xi(T)]$  by (24),  $Y$  obtained in  
18 Lemma A.5 is unique. Hence, only a unique set of triple  $(Y, M, R)$  solves (8)-(10).  
19 Finally, by Theorem 2.9,  $\hat{X}$  is an optimal terminal wealth of utility-downside-risk  
20 problem.

21 **Appendix B. Proof of Proposition 3.2.** Clearly, for each  $N$ ,  $\zeta^{(N)}$  has the same  
22 distribution as  $\xi^{(N)}$ , the respective systems of nonlinear equations in (8)-(10) with  
23  $\xi(T) = \zeta^{(N)}$  and  $\xi(T) = \xi^{(N)}$  share the same set of solution  $(Y^{(N)}, M^{(N)}, R^{(N)})$ ,  
24 i.e.  $(Y^{(N)}, M^{(N)}, R^{(N)})$  satisfy:

$$\mathbb{E}\left[D'\left(M^{(N)} - I\left(M^{(N)}, \gamma R^{(N)} + Y^{(N)}\zeta^{(N)}\right)\right)\right] = R^{(N)}, \quad (28)$$

$$\mathbb{E}\left[I\left(M^{(N)}, \gamma R^{(N)} + Y^{(N)}\zeta^{(N)}\right)\right] = M^{(N)}, \quad (29)$$

$$\mathbb{E}\left[\zeta^{(N)}I\left(M^{(N)}, \gamma R^{(N)} + Y^{(N)}\zeta^{(N)}\right)\right] = x_0. \quad (30)$$

25 Given the terminal pricing kernel  $\xi(T) = \xi^*$ , we can define  $R_{Y^*, M}^*$  and  $M_{Y^*}^*$  from  
26 Lemmas A.3 and A.4. With the fact that  $\xi^*$  and  $\zeta^*$  have the same distribution, We  
27 know that  $(Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^*)$  solves the system of nonlinear equations of (8)-(10)  
28 with  $\xi(T) = \xi^*$  and  $\xi(T) = \zeta^*$ . Since the solution of the system is unique, we have  
29  $(Y^*, M^*, R^*) = (Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^*)$ .

30 Note that the following simplified notation in (11) applies in this section:

$$\tilde{X}^{(N)} := I\left(M^{(N)}, \gamma R^{(N)} + Y^{(N)}\zeta^{(N)}\right). \quad (31)$$

1 We first consider the next three lemmas which establish the convergences of  $R^{(N)}$ ,  $M^{(N)}$   
2 and  $Y^{(N)}$ .

3 **Lemma B.1.** *Suppose there is a subsequence  $\{(M^{(N_k)}, Y^{(N_k)})\}_{k \in \mathbb{N}}$  and  $M_0, Y_0 \in$   
4  $(0, \infty)$  such that  $M_0 = \lim_{k \rightarrow \infty} M^{(N_k)}$  and  $Y_0 = \lim_{k \rightarrow \infty} Y^{(N_k)}$ , then  $\lim_{k \rightarrow \infty} R^{(N_k)} =$   
5  $R_{Y_0, M_0}^*$  where  $R_{Y, M}^*$  is defined in Lemma A.3 corresponding to  $\xi(T) = \zeta^*$  and  
6  $(Y^{(N)}, M^{(N)}, R^{(N)})$  satisfy system of equations (28)-(30).*

7 *Proof.* By Bolzano-Weierstrass theorem, there exists a subsequence  $\{N_{k_i}\}_{i \in \mathbb{N}} \subset$   
8  $\{N_k\}_{k \in \mathbb{N}}$  such that  $R_0 := \lim_{i \rightarrow \infty} R^{(N_{k_i})} \in [0, D'(M_0)]$  exists. To facilitate the  
9 proof, without loss of generality, we just assume  $\{N_i\}_{i \in \mathbb{N}}$  to be the notation of a  
10 particular chosen subsequence of  $\{N_k\}_{k \in \mathbb{N}}$ . Since  $I$  and  $D'$  are continuous, thus

$$\lim_{i \rightarrow \infty} D' \left( M^{(N_i)} - \tilde{X}^{(N_i)} \right) = D' (M_0 - I(M_0, \gamma R_0 + Y_0 \zeta^*)) \text{ a.s.}$$

11 Then, by the dominated convergence theorem and (28),

$$\mathbb{E} [D' (M_0 - I(M_0, \gamma R_0 + Y_0 \zeta^*))] = \lim_{i \rightarrow \infty} \mathbb{E} \left[ D' \left( M^{(N_i)} - \tilde{X}^{(N_i)} \right) \right] = \lim_{i \rightarrow \infty} R^{(N_i)} = R_0$$

12 Since Lemma A.3 with  $\xi(T) = \zeta^*$  is uniquely solved by  $R_{Y_0, M_0}^*$ , we have  $R_0 =$   
13  $R_{Y_0, M_0}^*$ . Since every subsequential limit of each subsequence  $\{R^{(N_k)}\}_{k \in \mathbb{N}}$  agrees  
14 with  $R_{Y_0, M_0}^*$ , the result follows. □

16 **Lemma B.2.** *Suppose that there is a subsequence  $\{Y^{(N_k)}\}_{k \in \mathbb{N}}$  and  $Y_0 \in (0, \infty)$   
17 such that  $Y_0 = \lim_{k \rightarrow \infty} Y^{(N_k)}$ , then  $\lim_{k \rightarrow \infty} M^{(N_k)} = M_{Y_0}^*$  where  $M_Y^*$  is defined in  
18 Lemma A.4 corresponding to  $\xi(T) = \zeta^*$  and  $(Y^{(N)}, M^{(N)}, R^{(N)})$  satisfy system of  
19 equations (28)-(30).*

20 *Proof.* By Bolzano-Weierstrass theorem, there is a subsequence  $\{N_{k_i}\} \subset \{N_k\}$  such  
21 that  $M_0 := \lim_{i \rightarrow \infty} M^{(N_{k_i})} \in [0, \infty]$  exists. To facilitate the proof, without loss  
22 of generality, we just assume  $\{N_i\}_{i \in \mathbb{N}}$  to be the notation of a particular chosen  
23 subsequence of  $\{N_k\}_{k \in \mathbb{N}}$ .

24 We first show that  $M_0 \in (0, \infty)$  by following the next 4 steps:

- Step 1: Prove if  $M_0 = 0$ , then  $\lim_{i \rightarrow \infty} \tilde{X}^{(N_i)} > 0$  a.s.; note that  $\tilde{X}^{(N)}$  is given by (31);
- Step 2: Check that  $M_0 > 0$ ;
- Step 3: Show that if  $M_0 = \infty$ , then  $\lim_{i \rightarrow \infty} R^{(N_i)} = \infty$ ;
- Step 4: Establish that  $M_0 < \infty$ .

25 • Step 1: Assume the contrary that there exists a sample  $\omega \in \{\omega \in \Omega : \zeta^*(\omega) <$   
26  $\text{esssup } \zeta^*\}$  and a subsequence  $\{N_{i_j}\}_{j \in \mathbb{N}} \subset \{N_i\}_{i \in \mathbb{N}}$  such that  $\lim_{j \rightarrow \infty} \tilde{X}^{(N_{i_j})}(\omega) =$   
27  $0$ . To facilitate the proof, without loss of generality, we just assume  $\{N_j\}_{j \in \mathbb{N}}$   
28 to be the notation of a particular chosen subsequence of  $\{N_i\}_{i \in \mathbb{N}}$ . Then  
29  $\lim_{j \rightarrow \infty} D'(M^{(N_j)} - \tilde{X}^{(N_j)}(\omega)) = D'(M_0) = D'(0)$  and  $\lim_{j \rightarrow \infty} U'(\tilde{X}^{(N_j)}(\omega)) =$   
30  $\infty$ . But by the fact that  $I$  satisfies (7) with  $m = M^{(N_j)}$  and  $y = \gamma R^{(N_j)} +$   
31  $Y^{(N_j)} \zeta^{(N_j)}(\omega)$  and the fact that  $R^{(N_j)} \leq D'(M^{(N_j)})$  from (28), we have:

$$\infty = \lim_{j \rightarrow \infty} U' \left( \tilde{X}^{(N_j)}(\omega) \right)$$

$$\begin{aligned}
&= \lim_{j \rightarrow \infty} Y^{(N_j)} \zeta^{(N_j)}(\omega) + \gamma \lim_{j \rightarrow \infty} \left( R^{(N_j)} - D' \left( M^{(N_j)} - \tilde{X}^{(N_j)}(\omega) \right) \right) \\
&\leq \zeta^*(\omega) Y_0 + \gamma \lim_{j \rightarrow \infty} \left( D'(M^{(N_j)}) - D' \left( M^{(N_j)} - \tilde{X}^{(N_j)}(\omega) \right) \right) \\
&\leq \zeta^*(\omega) Y_0 + \gamma (D'(0) - D'(0)) < \infty,
\end{aligned}$$

1 which leads to a contradiction. Hence,  $\lim_{i \rightarrow \infty} \tilde{X}^{(N_i)} > 0$  a.s.

2 • Step 2: Assume a contrary that  $M_0 = 0$ . By Step 1, we have  $\lim_{i \rightarrow \infty} \tilde{X}^{(N_i)} > 0$   
3 a.s. By the Fatou's lemma and (29), we have

$$0 = M_0 = \liminf_{i \rightarrow \infty} M^{(N_i)} = \liminf_{i \rightarrow \infty} \mathbb{E} \left[ \tilde{X}^{(N_i)} \right] \geq \mathbb{E} \left[ \liminf_{i \rightarrow \infty} \tilde{X}^{(N_i)} \right] > 0,$$

4 which leads to a contradiction. Hence  $M_0 > 0$ .

5 • Step 3: By (7) with  $m = M^{(N_i)}$  and  $y = \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)}$ , we first have

$$U' \left( \tilde{X}^{(N_i)} \right) - Y^{(N_i)} \zeta^{(N_i)} = \gamma R^{(N_i)} - \gamma D' \left( M^{(N_i)} - \tilde{X}^{(N_i)} \right). \quad (32)$$

6 When the left hand side of (32) is positive infinite as taking limit infimum  
7 with  $i \rightarrow \infty$ , we have  $\lim_{i \rightarrow \infty} R^{(N_i)} = \infty$  since the second term of the right  
8 hand side is always negative, therefore, the claim follows immediately. On  
9 the other hand, when the left hand side of (32) is almost surely finite as  
10 taking limit infimum with  $i \rightarrow \infty$ , we further assume a contrary that there  
11 exists a subsequence  $\{N_j\}_{j \in \mathbb{N}} \subset \{N_i\}_{i \in \mathbb{N}}$  such that  $\lim_{j \rightarrow \infty} R^{(N_j)} < \infty$ , then,  
12 due to  $M_0 = \lim_{j \rightarrow \infty} M^{(N_j)} = \infty$ , we have  $\lim_{j \rightarrow \infty} \tilde{X}^{(N_j)} = \infty$  a.s. Thus,  
13  $\lim_{j \rightarrow \infty} (U'(\tilde{X}^{(N_j)}) - Y^{(N_j)} \zeta^{(N_j)}) = -Y_0 \zeta^*$  a.s.

14 Then, by (28), the reverse Fatou's lemma, and (32), we further obtain:

$$\begin{aligned}
0 &> -Y_0 \mathbb{E}[\zeta^*] = \mathbb{E} \left[ \limsup_{j \rightarrow \infty} \left( U' \left( \tilde{X}^{(N_j)} \right) - Y^{(N_j)} \zeta^{(N_j)} \right) \right] \\
&\geq \limsup_{j \rightarrow \infty} \mathbb{E} \left[ U' \left( \tilde{X}^{(N_j)} \right) - Y^{(N_j)} \zeta^{(N_j)} \right] \\
&= \limsup_{j \rightarrow \infty} \mathbb{E} \left[ \gamma R^{(N_j)} - \gamma D' \left( M^{(N_j)} - \tilde{X}^{(N_j)} \right) \right] = 0,
\end{aligned}$$

15 which is a contradiction, therefore,  $\lim_{i \rightarrow \infty} R^{(N_i)} = \infty$ .

16 • Step 4: Assume that  $M_0 = \infty$ , we aim to show that

$$\lim_{i \rightarrow \infty} \left( M^{(N_i)} - \tilde{X}^{(N_i)} \right) = \infty, \text{ a.s.} \quad (33)$$

17 Indeed, given a sample  $\omega \in \Omega$ , assume the contrary that there exists a sub-  
18 sequence  $\{N_{i_j}\}_{j \in \mathbb{N}} \subset \{N_i\}_{i \in \mathbb{N}}$  such that  $\lim_{j \rightarrow \infty} (M^{(N_{i_j})} - \tilde{X}^{(N_{i_j})}(\omega)) < \infty$ .  
19 To facilitate the proof, without loss of generality, we just assume  $\{N_j\}_{j \in \mathbb{N}}$  to  
20 be the notation of a particular chosen subsequence of  $\{N_i\}_{i \in \mathbb{N}}$ . Then clearly  
21  $\lim_{j \rightarrow \infty} D'(M^{(N_j)} - \tilde{X}^{(N_j)}(\omega)) < \infty$ . Since  $\lim_{j \rightarrow \infty} R^{(N_j)} = \infty$  by Step 3 and  
22  $I$  satisfies (7), we have  $\lim_{j \rightarrow \infty} U'(\tilde{X}^{(N_j)}(\omega)) = \infty$ , thus  $\lim_{j \rightarrow \infty} \tilde{X}^{(N_j)}(\omega) =$   
23  $0$ , and hence  $\lim_{j \rightarrow \infty} (M^{(N_j)} - \tilde{X}^{(N_j)}(\omega)) = \infty$ , which contradicts to the as-  
24 sumption on its finiteness a priori. Hence, (33) holds.

25 By (29), the Fatou's lemma, and then (33), we now have

$$0 = \liminf_{i \rightarrow \infty} \mathbb{E} \left[ M^{(N_i)} - \tilde{X}^{(N_i)} \right] \geq \mathbb{E} \left[ \liminf_{i \rightarrow \infty} \left( M^{(N_i)} - \tilde{X}^{(N_i)} \right) \right] = \infty, \quad (34)$$

26 which leads to a contradiction.

1 In conclusion, by Steps 2 and 4, we have  $M_0 \in (0, \infty)$ . By Lemma B.1,  $\lim_{i \rightarrow \infty} R^{(N_i)} =$   
 2  $R_{Y_0, M_0}^*$ , thus

$$\lim_{i \rightarrow \infty} I \left( M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} \right) = I \left( M_0, \gamma R_{Y_0, M_0}^* + Y_0 \zeta^* \right).$$

3 By the dominated convergence theorem and (29),

$$\begin{aligned} \mathbb{E} \left[ I \left( M_0, \gamma R_{Y_0, M_0}^* + Y_0 \zeta^* \right) \right] &= \lim_{i \rightarrow \infty} \mathbb{E} \left[ I \left( M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)} \right) \right] \\ &= \lim_{i \rightarrow \infty} M^{(N_i)} = M_0. \end{aligned}$$

4 By the uniqueness of Lemma A.4 with  $\xi(T) = \zeta^*$ , we have  $M_0 = M_{Y_0}^*$ . Since  
 5 every subsequential limit of  $\{M_{N_k}\}_{k \in \mathbb{N}}$  agrees with  $M_0 = \lim_{i \rightarrow \infty} M^{(N_i)} = M_{Y_0}^*$ ,  
 6 the claim follows.  $\square$

7 **Lemma B.3.**  $\lim_{N \rightarrow \infty} Y^{(N)} = Y^*$ .

8 *Proof.* By Bolzano-Weierstrass theorem, there exists a subsequence  $\{N_k\}_{k \in \mathbb{N}}$  such  
 9 that  $Y_0 := \lim_{k \rightarrow \infty} Y^{(N_k)} \in [0, \infty]$  exists.

10 We will prove  $Y_0 \in (0, \infty)$  by using following 4 steps:

- Step 1: Prove if  $Y_0 = 0$ ,  $\mathbb{P}[\lim_{k \rightarrow \infty} \tilde{X}^{(N_k)} = \infty] > 0$ , note that  $\tilde{X}^{(N)}$  is given by (31);
- Step 2: Check that  $Y_0 > 0$ .
- Step 3: Show that if  $Y_0 = \infty$ ,  $\lim_{k \rightarrow \infty} M^{(N_k)} < \infty$ .
- Step 4: Establish that  $Y_0 < \infty$ .

11 • Step 1: Assume the contrary that there exists an arbitrary a sample  $\omega_0 \in$   
 12  $\{\omega \in \Omega : \zeta^*(\omega) < \text{esssup } \zeta^*\}$  and a sequence  $\{N_{k_i}\}_{i \in \mathbb{N}} \subset \{N_k\}_{k \in \mathbb{N}}$  such that  
 13  $\lim_{i \rightarrow \infty} \tilde{X}^{(N_{k_i})}(\omega_0) < \infty$ . To facilitate the proof, without loss of generality,  
 14 we just assume  $\{N_i\}_{i \in \mathbb{N}}$  to be the notation of a particular chosen subsequence  
 15 of  $\{N_k\}_{k \in \mathbb{N}}$ . For any sample  $\omega \in \{\omega \in \Omega : \zeta^*(\omega) > \zeta^*(\omega_0)\}$ , we have  
 16  $\zeta^{(N_i)}(\omega) > \zeta^{(N_i)}(\omega_0)$  for large enough  $i$ , hence, by the fact that  $\tilde{X}^{(N_i)} =$   
 17  $I(M^{(N_i)}, \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)})$  is strictly decreasing in  $\zeta^{(N_i)}$  according to Prop  
 18 A.2 (b)(i), we have  $U'(\tilde{X}^{(N_i)}(\omega)) > U'(\tilde{X}^{(N_i)}(\omega_0))$ .

19 Therefore, for any  $\omega \in \{\omega \in \Omega : \zeta^*(\omega) > \zeta^*(\omega_0)\}$ ,

$$\liminf_{i \rightarrow \infty} U'(\tilde{X}^{(N_i)}(\omega)) \geq \lim_{i \rightarrow \infty} U'(\tilde{X}^{(N_i)}(\omega_0)) > 0.$$

20 Fulfilling Condition (ii) in Theorem 3.1 implies that  $\mathbb{E}[\zeta^{(N_i)}]$  is uniformly  
 21 bounded from above, then, by (7) with  $m = M^{(N_i)}$  and  $y = \gamma R^{(N_i)} +$   
 22  $Y^{(N_i)} \zeta^{(N_i)}$ , (28), and the Fatou's lemma,

$$\begin{aligned} 0 &= \liminf_{i \rightarrow \infty} Y^{(N_i)} \mathbb{E}[\zeta^{(N_i)}] = \liminf_{i \rightarrow \infty} \mathbb{E} \left[ U' \left( \tilde{X}^{(N_i)} \right) \right] \geq \mathbb{E} \left[ \liminf_{i \rightarrow \infty} U' \left( \tilde{X}^{(N_i)} \right) \right] \\ &\geq \lim_{i \rightarrow \infty} U' \left( \tilde{X}^{(N_i)}(\omega_0) \right) \mathbb{P}[\zeta^* > \zeta^*(\omega_0)] > 0. \end{aligned}$$

23 which results in a contradiction, hence  $\lim_{k \rightarrow \infty} \tilde{X}^{(N_k)}(\omega_0) = \infty$  for all  $\omega_0 \in$   
 24  $\{\omega \in \Omega : \zeta^*(\omega) < \text{esssup } \zeta^*\}$ .

- 1 • Step 2: Assume the contrary that  $Y_0 = 0$ , by Step 1, we further have  
 2  $\mathbb{P}[\lim_{k \rightarrow \infty} (\zeta^{(N_k)} \tilde{X}^{(N_k)}) = \infty] > 0$ . By (30) and the Fatou's lemma, we have

$$x_0 = \liminf_{k \rightarrow \infty} \mathbb{E} \left[ \zeta^{(N_k)} \tilde{X}^{(N_k)} \right] \geq \mathbb{E} \left[ \liminf_{k \rightarrow \infty} \zeta^{(N_k)} \tilde{X}^{(N_k)} \right] = \infty,$$

- 3 which is impossible and thus  $Y_0 > 0$ .  
 4 • Step 3: Assume that  $\lim_{k \rightarrow \infty} M^{(N_k)} = \infty$ . Following the similar argument as  
 5 Step 4 of the proof of Lemma B.2, we can derive the same absurd inequality  
 6 as (34).  
 7 • Step 4: Assume  $Y_0 = \infty$ . We claim that  $\lim_{k \rightarrow \infty} \tilde{X}^{(N_k)} = 0$  a.s. Assume  
 8 the contrary that there exists a sample  $\omega_0 \in \Omega$  and a sequence  $\{N_{k_i}\}_{i \in \mathbb{N}} \subset$   
 9  $\{N_k\}_{k \in \mathbb{N}}$  with  $Y^{(N_{k_i})} \rightarrow \infty$  such that  $\lim_{i \rightarrow \infty} \tilde{X}^{(N_{k_i})}(\omega_0) > 0$ . To facilitate  
 10 the proof, without loss of generality, we just assume  $\{N_i\}_{i \in \mathbb{N}}$  to be the notation  
 11 of a particular chosen subsequence of  $\{N_k\}_{k \in \mathbb{N}}$ . By (7) with  $m = M^{(N_i)}$  and  
 12  $y = \gamma R^{(N_i)} + Y^{(N_i)} \zeta^{(N_i)}(\omega_0)$ , we first have

$$\begin{aligned} & U' \left( \tilde{X}^{(N_i)}(\omega_0) \right) - Y^{(N_i)} \zeta^{(N_i)}(\omega_0) \\ &= \gamma R^{(N_i)} - \gamma D' \left( M^{(N_i)} - \tilde{X}^{(N_i)}(\omega_0) \right) \geq -\gamma D' \left( M^{(N_i)} \right). \end{aligned}$$

- 13 Taking limit to the both sides of above with the fact that  $\lim_{k \rightarrow \infty} M^{(N_k)} < \infty$   
 14 from Step 3, we have

$$-\infty = \lim_{i \rightarrow +\infty} \left( U' \left( \tilde{X}^{(N_i)}(\omega_0) \right) - Y^{(N_i)} \zeta^{(N_i)}(\omega_0) \right) \geq -\gamma \lim_{i \rightarrow +\infty} D' \left( M^{(N_i)} \right) > -\infty,$$

- 15 which is absurd. Therefore, we have

$$\lim_{k \rightarrow \infty} \tilde{X}^{(N_k)} = 0 \text{ a.s.} \quad (35)$$

- 16 By Prop A.2 (a), we have

$$\begin{aligned} \tilde{X}^{(N_k)} &= I \left( M^{(N_k)}, Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)} \right) \leq M^{(N_k)} + (U')^{-1} \left( Y^{(N_k)} \zeta^{(N_k)} + \gamma R^{(N_k)} \right) \\ &\leq 2 \lim_{k \rightarrow \infty} M^{(N_k)} + (U')^{-1}(\zeta^{(N_k)}), \end{aligned}$$

- 17 for large enough  $k$ . Due to Condition (ii) in Theorem 3.1,  $\{\zeta^{(N_k)} \tilde{X}^{(N_k)}\}_{k \in \mathbb{N}}$   
 18 is uniformly integrable. By the dominated convergence theorem under the  
 19 uniform integrability assumption, (30), and (35),

$$x_0 = \lim_{k \rightarrow \infty} \mathbb{E} \left[ \zeta^{(N_k)} \tilde{X}^{(N_k)} \right] = \mathbb{E} \left[ \lim_{k \rightarrow \infty} \zeta^{(N_k)} \tilde{X}^{(N_k)} \right] = 0,$$

- 20 which is absurd, and hence  $Y_0 < \infty$ .

- 21 In conclusion, by Steps 2 and 4, we have  $Y_0 \in (0, \infty)$ . By Lemmas B.1 and B.2,  
 22  $\lim_{k \rightarrow \infty} (Y^{(N_k)}, M^{(N_k)}, R^{(N_k)}) = (Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^*)$ , therefore, we have

$$\lim_{k \rightarrow \infty} \zeta^{(N_k)} \tilde{X}^{(N_k)} = \zeta^* I \left( M_{Y_0}^*, \gamma R_{Y_0, M_{Y_0}^*}^* + Y_0 \zeta^* \right).$$

- 23 Due to Condition (ii) in Theorem 3.1, we have  $\{\zeta^{(N_k)} \tilde{X}^{(N_k)}\}_{k \in \mathbb{N}}$  uniformly inte-  
 24 grable, and by dominated convergence theorem and (30),

$$x_0 = \lim_{k \rightarrow \infty} \mathbb{E} \left[ \zeta^{(N_k)} \tilde{X}^{(N_k)} \right] = \mathbb{E} \left[ \zeta^* I \left( M_{Y_0}^*, \gamma R_{Y_0, M_{Y_0}^*}^* + Y_0 \zeta^* \right) \right]$$

1 Together with the fact that  $R_{Y,M}^*$  and  $M_Y^*$  solve (23) and (25) with  $\xi = \zeta^*$ ,  
 2  $(Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^*)$  solving for the system of equations (8)-(10) with  $\xi = \zeta^*$ . Since  
 3 the solution of the system is unique, so we have  $(Y_0, M_{Y_0}^*, R_{Y_0, M_{Y_0}^*}^*) = (Y^*, M^*, R^*)$ .  
 4 Since every subsequential limit of  $\{Y^{(N)}\}_{N \in \mathbb{N}}$  agrees with  $Y_0 = \lim_{k \rightarrow \infty} Y^{(N_k)} = Y^*$ ,  
 5 therefore,  $Y^* = \lim_{N \rightarrow \infty} Y^{(N)}$ .  $\square$

6 By Lemmas B.1 and B.2, we have

$$\lim_{N \rightarrow \infty} (Y^{(N)}, M^{(N)}, R^{(N)}) = (Y^*, M_{Y^*}^*, R_{Y^*, M_{Y^*}^*}^*) = (Y^*, M^*, R^*),$$

7 consequently, the proposition follows.

8 **Appendix C. Proof of Theorem 4.7.** To facilitate the proof, different constants  
 9  $C$  which appear throughout the following proof may be different line by line. First  
 10 recall that the following version of Continuous Mapping Theorem will be used to  
 11 prove this theorem:

12 **Proposition C.1** (Lemma 1 of [19]). *Assume that the sequence of processes in*  
 13  $\mathbb{R}^n$ ,  $\{X^{(N)}(\cdot)\}_{N \in \mathbb{N}}$ , *converges weakly to*  $X^*(\cdot)$ . *Suppose that*  $h(x)$  *is a continuous*  
 14 *mapping from*  $\mathbb{R}^n$  *to*  $\mathbb{R}$ . *Then*  $\{h(X^{(N)}(\cdot))\}_{N \in \mathbb{N}}$  *converges weakly to*  $h(X^*(\cdot))$ .

15 First, we telescope the following expression:

$$X^{(N)}(t) = h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left( \left\lfloor \frac{t}{\Delta t} \right\rfloor \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \right) - \varepsilon^{(N)}(t), \quad (36)$$

16 where we define such truncation errors:  $\varepsilon^{(N)}(t) := \varepsilon_{\lfloor t/\Delta t \rfloor}^{(N)}$  and

$$\begin{aligned} \varepsilon_k^{(N)} &:= h^{(Y^{(N)}, M^{(N)}, R^{(N)})} \left( k\Delta t, \bar{\xi}_k^{(N)} \right) - h^{(N, Y^{(N)}, M^{(N)}, R^{(N)})} \left( k\Delta t, \bar{\xi}_k^{(N)} \right) \\ &=: h \left( k\Delta t, \bar{\xi}_k^{(N)} \right) - h^{(N)} \left( k\Delta t, \bar{\xi}_k^{(N)} \right), \end{aligned}$$

17 where for simplicity, we denote  $h^{(Y^{(N)}, M^{(N)}, R^{(N)})}$  and  $h^{(N, Y^{(N)}, M^{(N)}, R^{(N)})}$  respec-  
 18 tively by  $h$  and  $h^{(N)}$ ; same type of simplification will be adopted for their deriva-  
 19 tives.

20 By Lemma 4.4, the sequence of processes  $\{\bar{\xi}_{\lfloor \cdot / \Delta t \rfloor}^{(N)}\}_{N \in \mathbb{N}}$  converges weakly to  $\bar{\xi}^*(\cdot)$ ,  
 21 thus the sequence of processes  $\{(\lfloor \frac{\cdot}{\Delta t} \rfloor \Delta t, \bar{\xi}_{\lfloor \cdot / \Delta t \rfloor}^{(N)}, Y^{(N)}, M^{(N)}, R^{(N)})\}_{N \in \mathbb{N}}$  in  $\mathbb{R}^5$  con-  
 22 verges weakly to  $(\cdot, \bar{\xi}^*(\cdot), Y^*, M^*, R^*)$ . Then, by Proposition C.1,  $h^{(Y^{(N)}, M^{(N)}, R^{(N)})}(\lfloor \frac{\cdot}{\Delta t} \rfloor \Delta t, \bar{\xi}_{\lfloor \cdot / \Delta t \rfloor}^{(N)})$   
 23 converges weakly to  $h^{(Y^*, M^*, R^*)}(\cdot, \bar{\xi}^*(\cdot))$ . Thus, we only need to prove that the pro-  
 24 cess  $\{\varepsilon^{(N)}(\cdot)\}_{N \in \mathbb{N}}$  converges weakly to a zero process as  $N$  goes to  $\infty$ . To this end,  
 25 we need the following lemmas regarding the boundedness of  $\varepsilon_k^{(N)}$  (their proofs will  
 26 be postponed to Subsections C.1 and C.2):

27 **Lemma C.2.** *There exist*  $C > 0$  *and a positive integer*  $p$  *such that for any*  $k$  *and*  $N$ ,  
 28 *there are*  $\delta$  *and*  $M$ , *which are independent of*  $k$  *and*  $N$ , *such that*  $|O(\Delta t)| \leq M\Delta t$   
 29 *when*  $0 < \Delta t < \delta$ , *and we also have*

$$|\varepsilon_k^{(N)}| \leq C \left( 1 + |\bar{\xi}_k^{(N)}|^p \right) \left( \sqrt{\Delta t} + O(\Delta t) \right).$$

30 **Lemma C.3.** *For any*  $p \in \mathbb{N}$ , *there exists a*  $M_p > 0$  *such that*  $\mathbb{E}[\sup_{1 \leq n \leq N} |\bar{\xi}_k^{(N)}|^p] \leq$   
 31  $M_p$  *for all*  $N \in \mathbb{N}$ .

1 Now, we are going to show that  $\{\varepsilon^{(N)}(\cdot)\}_{N \in \mathbb{N}}$  converges weakly to 0 by applying  
 2 martingale central limit theorem as stated in Theorem 4.1.

3 By Lemma C.2, there exist  $C > 0$  and a positive integer  $p$  such that

$$\left| \varepsilon^{(N)}(0) \right| = \left| \varepsilon_0^{(N)} \right| \leq C \left( 1 + \left| \bar{\xi}_0^{(N)} \right|^p \right) \left( \sqrt{\Delta t} + O(\Delta t) \right) = 2C \left( \sqrt{\Delta t} + O(\Delta t) \right) \rightarrow 0,$$

4 as  $N \rightarrow \infty$ , and so Condition 1 in Theorem 4.1 is fulfilled.

5 By setting  $L^{(N)}(t) := \varepsilon^{(N)}(t)$ ,  $A^{(N)}(t) := 0$ , and  $M^{(N)}(t) := \varepsilon^{(N)}(t) - L^{(N)}(t) =$   
 6 0, and obviously, Conditions 2 and 3 in Theorem 4.1 are both satisfied.

7 By Lemma C.2, for any  $k$ , we have

$$\left| \varepsilon_k^{(N)} \right|^2 \leq \frac{C}{2} \left( 1 + 2 \left| \bar{\xi}_k^{(N)} \right|^p + \left| \bar{\xi}_k^{(N)} \right|^{2p} \right) \left( \Delta t + O(\Delta t^{\frac{3}{2}}) + O(\Delta t^2) \right) \leq C \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^{2p} \right) \left( \Delta t + O(\Delta t^{\frac{3}{2}}) \right),$$

8 for some  $C > 0$ . By Lemma C.3, we have

$$\mathbb{E} \left[ \sup_{1 \leq k \leq N} \left| \varepsilon_k^{(N)} \right|^2 \right] \leq C \left( 1 + \mathbb{E} \left[ \sup_{1 \leq k \leq N} \left| \bar{\xi}_k^{(N)} \right|^{2p} \right] \right) \left( \Delta t + O(\Delta t^{\frac{3}{2}}) \right) \leq M \left( \Delta t + O(\Delta t^{\frac{3}{2}}) \right).$$

9 Therefore, for any  $t$ , we have

$$\begin{aligned} \mathbb{E} \left[ \sup_t \left| L^{(N)}(t) - L^{(N)}(t-) \right|^2 \right] &\leq 2 \mathbb{E} \left[ \sup_t \left( \left| \varepsilon^{(N)}(t) \right|^2 + \left| \varepsilon^{(N)}(t-) \right|^2 \right) \right] \\ &\leq 4 \mathbb{E} \left[ \sup_{1 \leq k \leq N} \left| \varepsilon_k^{(N)} \right|^2 \right] \leq 4M \left( \Delta t + O(\Delta t^{\frac{3}{2}}) \right) \rightarrow 0, \end{aligned}$$

10 as  $N \rightarrow \infty$ . Thus, Condition 4 in Theorem 4.1 is satisfied.

11 By Lemmas C.2 and C.3, we have

$$\begin{aligned} \mathbb{E} \left[ \sup_t \left| L^{(N)}(t) \right| \right] &\leq \mathbb{E} \left[ \sup_{1 \leq k \leq N} \left| \varepsilon_k^{(N)} \right| \right] \leq C \left( 1 + \mathbb{E} \left[ \sup_{1 \leq k \leq N} \left| \bar{\xi}_k^{(N)} \right|^p \right] \right) \left( \sqrt{\Delta t} + O(\Delta t) \right) \\ &\leq M \left( \sqrt{\Delta t} + O(\Delta t) \right) \rightarrow 0, \end{aligned}$$

12 as  $N \rightarrow \infty$ . Thus,  $\sup_t \left| L^{(N)}(t) \right| \rightarrow 0$  in probability as  $N \rightarrow \infty$ . Hence, Condition  
 13 5 in Theorem 4.1 is again fulfilled. Hence, by Theorem 4.1, we have  $\{\varepsilon^{(N)}(\cdot)\}_{N \in \mathbb{N}}$   
 14 converges weakly to zero. Now, we have already shown that the first term in the  
 15 right hand side of (36) converges weakly to the process  $h^{(Y^*, M^*, R^*)}(\cdot, \bar{\xi}^*(\cdot))$ , which  
 16 is exactly the  $X^*(\cdot)$ , and the second term on the right hand side of (36) converges  
 17 weakly to zero, and hence  $X^{(N)}(\cdot)$  converges weakly to  $X^*(\cdot)$ .

18 For the convergence of investment policy, by (16), we have

$$\begin{aligned} \pi^{(N)}(t) &= \frac{1}{2\sigma\sqrt{\Delta t}} \left( h^{(N)} \left( t + \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right) \right. \\ &\quad \left. - h^{(N)} \left( t + \Delta t, \bar{\xi}_{\lfloor t/\Delta t \rfloor}^{(N)} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right) \right) \\ &=: \frac{1}{2\sigma\sqrt{\Delta t}} \left( f_{\lfloor t/\Delta t \rfloor}^{(N)+}(1) - f_{\lfloor t/\Delta t \rfloor}^{(N)-}(1) - \varepsilon_{\lfloor t/\Delta t \rfloor+1}^{(N)+} + \varepsilon_{\lfloor t/\Delta t \rfloor+1}^{(N)-} \right), \quad (37) \end{aligned}$$

19 where we set

$$\begin{aligned} \bar{\xi}_{k+1}^{(N)+} &:= \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \bar{\xi}_k^{(N)}, & \bar{\xi}_{k+1}^{(N)-} &:= \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \bar{\xi}_k^{(N)}, \\ f_k^{(N)+}(\tau) &:= h \left( (k + \tau)\Delta t, \bar{\xi}_k^{(N)} + \tau \left( \bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \right), \\ f_k^{(N)-}(\tau) &:= h \left( (k + \tau)\Delta t, \bar{\xi}_k^{(N)} + \tau \left( \bar{\xi}_{k+1}^{(N)-} - \bar{\xi}_k^{(N)} \right) \right), \end{aligned}$$

$$\begin{aligned}\varepsilon_k^{(N)+} &:= h\left(k\Delta t, \bar{\xi}_k^{(N)+}\right) - h^{(N)}\left(k\Delta t, \bar{\xi}_k^{(N)+}\right), \\ \varepsilon_k^{(N)-} &:= h\left(k\Delta t, \bar{\xi}_k^{(N)-}\right) - h^{(N)}\left(k\Delta t, \bar{\xi}_k^{(N)-}\right).\end{aligned}$$

1 Since  $h$  is continuously differentiable in both  $t$  and  $\bar{\xi}$  up to the third order,  $f_k^{(N)+}$   
2 and  $f_k^{(N)-}$  admit the following Taylor expansions:

$$f_k^{(N)+}(1) = f_k^{(N)+}(0) + \frac{\partial}{\partial \tau} f_k^{(N)+}(0) + \frac{1}{2} \frac{\partial^2}{\partial \tau^2} f_k^{(N)+}(0) + P_k^{(N)}, \quad (38)$$

$$f_k^{(N)-}(1) = f_k^{(N)-}(0) + \frac{\partial}{\partial \tau} f_k^{(N)-}(0) + \frac{1}{2} \frac{\partial^2}{\partial \tau^2} f_k^{(N)-}(0) + Q_k^{(N)}, \quad (39)$$

3 where

$$P_k^{(N)} := \frac{1}{6} \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} f_k^{(N)+}(s) ds, \quad (40)$$

$$Q_k^{(N)} := \frac{1}{6} \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} f_k^{(N)-}(s) ds. \quad (41)$$

4 A direct calculation through an application of the law of total derivatives gives the  
5 following:

$$\frac{\partial}{\partial \tau} f_k^{(N)+}(0) = h_t \Delta t + \left( \bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)+} \right) h_{\bar{\xi}} = h_t \Delta t - \frac{\alpha \bar{\xi}_k^{(N)+}}{\sigma} h_{\bar{\xi}} \sqrt{\Delta t}, \quad (42)$$

$$\frac{\partial}{\partial \tau} f_k^{(N)-}(0) = h_t \Delta t + \left( \bar{\xi}_{k+1}^{(N)-} - \bar{\xi}_k^{(N)-} \right) h_{\bar{\xi}} = h_t \Delta t + \frac{\alpha \bar{\xi}_k^{(N)-}}{\sigma} h_{\bar{\xi}} \sqrt{\Delta t}, \quad (43)$$

$$\frac{\partial^2}{\partial \tau^2} f_k^{(N)+}(0) = h_{tt} (\Delta t)^2 - 2 \frac{\alpha \bar{\xi}_k^{(N)+}}{\sigma} h_{t\bar{\xi}} \Delta t^{\frac{3}{2}} + \frac{\alpha^2}{\sigma^2} \left( \bar{\xi}_k^{(N)+} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t, \quad (44)$$

$$\frac{\partial^2}{\partial \tau^2} f_k^{(N)-}(0) = h_{tt} (\Delta t)^2 + 2 \frac{\alpha \bar{\xi}_k^{(N)-}}{\sigma} h_{t\bar{\xi}} \Delta t^{\frac{3}{2}} + \frac{\alpha^2}{\sigma^2} \left( \bar{\xi}_k^{(N)-} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t, \quad (45)$$

6 where all partial derivatives of  $h$  are evaluated at the point  $(k\Delta t, \bar{\xi}_k^{(N)})$  while all  
7 of these functions and derivatives are indexed with superscripts  $(Y^{(N)}, M^{(N)}, R^{(N)})$   
8 and we suppress these notations for the sake of convenience. Using these expressions  
9 from (38) to (45), we obtain:

$$\begin{aligned}& \frac{1}{2\sigma\sqrt{\Delta t}} \left( f_{[t/\Delta t]}^{(N)+}(1) - f_{[t/\Delta t]}^{(N)-}(1) \right) \\ &= -\frac{\alpha \bar{\xi}_{[t/\Delta t]}^{(N)+}}{\sigma^2} h_{\bar{\xi}} - \frac{\alpha \bar{\xi}_{[t/\Delta t]}^{(N)-}}{\sigma^2} h_{t\bar{\xi}} \Delta t + \frac{1}{6} \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} \left( f_{[t/\Delta t]}^{(N)+}(s) - f_{[t/\Delta t]}^{(N)-}(s) \right) ds.\end{aligned}$$

10 Thus, (37) can be rewritten as

$$\pi^{(N)}(t) = -\frac{\alpha \bar{\xi}_{[t/\Delta t]}^{(N)}}{\sigma^2} h_{\bar{\xi}} \left( [t/\Delta t] \Delta t, \bar{\xi}_{[t/\Delta t]}^{(N)} \right) + e_{[t/\Delta t]}^{(N)}, \quad (46)$$

11 where

$$\begin{aligned}e_k^{(N)} &:= -\frac{\alpha \bar{\xi}_k^{(N)}}{\sigma^2} h_{t\bar{\xi}} \Delta t + \frac{1}{6} \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} \left( f_k^{(N)+}(s) - f_k^{(N)-}(s) \right) ds \\ &\quad - \frac{1}{2\sigma\sqrt{\Delta t}} \left( \varepsilon_{k+1}^{(N)+} - \varepsilon_{k+1}^{(N)-} \right).\end{aligned} \quad (47)$$

12 By Theorem C.1, the first term in (46) converges weakly to the process  $(-\frac{\alpha}{\sigma^2} \bar{\xi}^*(\cdot) h_{\bar{\xi}}^{(Y^*, M^*, R^*)}(\cdot, \bar{\xi}^*(\cdot)))$ ,  
13 which is exactly the optimal investment process  $\pi^*(\cdot)$ . Hence, we only need to show  
14 that the process  $\{e_{[t/\Delta t]}^{(N)}\}_{N \in \mathbb{N}}$  converges weakly to zero. To this end, we need the

1 following lemma regarding the boundedness of  $e_k^{(N)}$  (its proof will be postponed to  
2 the next subsection [C.3](#)):

3 **Lemma C.4.** *There exist  $C > 0$  and a positive integer  $p$  such that for any  $k$  and  $N$ ,  
4 there are  $\delta$  and  $M$ , which are independent of  $k$  and  $N$ , such that  $|O(\Delta t)| \leq M\Delta t$   
5 when  $0 < \Delta t < \delta$ , and we also have*

$$\left| e_k^{(N)} \right| \leq C \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left( \sqrt{\Delta t} + O(\Delta t) \right).$$

6 Then, with an application of the martingale central limit theorem (Theorem [4.1](#))  
7 with the same procedure as in the case of  $\varepsilon^{(N)}(\cdot)$ , which solely depends on the bound-  
8 edness property but not the expression, one can deduce the process  $\{e_{\lfloor \cdot / \Delta t \rfloor}^{(N)}\}_{N \in \mathbb{N}}$   
9 converges weakly to the process of identically zero; and so the weak convergence of  
10 optimal investment process also follows.

11 **C.1. Proof of Lemma C.2.** We proceed the proof by splitting into 3 steps as  
12 follows:

- Step 1: To establish that

$$\begin{aligned} & \varepsilon_k^{(N)} \\ = & \mathbb{E}^{(N)} \left[ \left( \sum_{n=k}^{N-1} \beta(n\Delta t, \bar{\xi}_n^{(N)}) \frac{1}{(1+r\Delta t)^{n-k}} (\Delta t)^2 + \gamma_n^{(N)} \frac{1}{(1+r\Delta t)^{n-k}} \right) \middle| \bar{\xi}_k^{(N)} \right], \end{aligned} \quad (48)$$

where  $\mathbb{E}^{(N)}$  is the expectation with respect to the probability measure so  
that  $\mathbb{P}[Z_n^{(N)} = 1] = \frac{1-\frac{\alpha}{\sigma}\sqrt{\Delta t}}{2}$  and  $\mathbb{P}[Z_n^{(N)} = -1] = \frac{1+\frac{\alpha}{\sigma}\sqrt{\Delta t}}{2}$ ;  $\beta(t, \xi) :=$   
 $-\frac{(h_{tt}(t, \xi) + 2\frac{\alpha^2}{\sigma^2}\xi h_{t\xi}(t, \xi))}{2(1+r\Delta t)}$  and  $\gamma_k^{(N)} := -\frac{(1-\frac{\alpha}{\sigma}\sqrt{\Delta t})P_k^{(N)} + (1+\frac{\alpha}{\sigma}\sqrt{\Delta t})Q_k^{(N)}}{2(1+r\Delta t)}$ .

- Step 2: Then we aim to show that

$$\sum_{n=k}^{N-1} \mathbb{E}^{(N)} \left[ \left| \beta(n\Delta t, \bar{\xi}_n^{(N)}) \right| \left| \bar{\xi}_k^{(N)} \right| (\Delta t)^2 \leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \Delta t, \quad (49)$$

for some  $A > 0$ , which is independent of  $k$  and  $N$ , and a positive integer  
 $p$ .

- Step 3: Finally, as stated in the lemma, there are  $\delta$  and  $M$ , which are  
independent of  $k$  and  $N$ , such that  $|O(\Delta t)| \leq M\Delta t$  when  $0 < \Delta t < \delta$ , we  
need to have

$$\sum_{n=k}^{N-1} \mathbb{E}^{(N)} \left[ \left| \gamma_n^{(N)} \right| \left| \bar{\xi}_k^{(N)} \right| \right] \leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left( \sqrt{\Delta t} + O(\Delta t) \right)$$

for some  $A > 0$  and a positive integer  $p$ .

- 13 • Step 1: In order to derive the expression of  $\varepsilon_k^{(N)}$ , we should first calculate two  
14 items, namely,  $\mathbb{E}^{(N)}[h((k+1)\Delta t, \bar{\xi}_{k+1}^{(N)}) \left| \bar{\xi}_k^{(N)} \right|]$  and  $h^{(N)}(k\Delta t, \bar{\xi}_k^{(N)})$ . Using the  
15 expressions as obtained in [\(38\)](#) – [\(45\)](#) with  $f_k^{(N)\pm}(0) = h(k\Delta t, \bar{\xi}_k^{(N)})$ , which is  
16 clear from the definitions, we first have

$$\mathbb{E}^{(N)} \left[ \frac{1}{1+r\Delta t} h \left( (k+1)\Delta t, \bar{\xi}_{k+1}^{(N)} \right) \middle| \bar{\xi}_k^{(N)} \right]$$

$$\begin{aligned}
&= \frac{1}{2} \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{1 + r\Delta t} f_k^{(N)+}(1) + \frac{1}{2} \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{1 + r\Delta t} f_k^{(N)-}(1) \\
&= \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1 + r\Delta t)} \\
&\quad \cdot \left( h + h_t \Delta t - \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{\bar{\xi}} \sqrt{\Delta t} + \frac{1}{2} h_{tt} (\Delta t)^2 - \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} (\Delta t)^{\frac{3}{2}} + \frac{1}{2} \frac{\alpha^2}{\sigma^2} \left( \bar{\xi}_k^{(N)} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t + P_k^{(N)} \right) \\
&\quad + \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1 + r\Delta t)} \\
&\quad \cdot \left( h + h_t \Delta t + \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{\bar{\xi}} \sqrt{\Delta t} + \frac{1}{2} h_{tt} (\Delta t)^2 + \frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} (\Delta t)^{\frac{3}{2}} + \frac{1}{2} \frac{\alpha^2}{\sigma^2} \left( \bar{\xi}_k^{(N)} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t + Q_k^{(N)} \right) \\
&= \frac{1}{1 + r\Delta t} \left( h + h_t \Delta t + \frac{\alpha^2}{\sigma^2} \bar{\xi}_k^{(N)} h_{\bar{\xi}} \Delta t + \frac{1}{2} h_{tt} (\Delta t)^2 + \frac{\alpha^2}{\sigma^2} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} (\Delta t)^2 + \frac{1}{2} \frac{\alpha^2}{\sigma^2} \left( \bar{\xi}_k^{(N)} \right)^2 h_{\bar{\xi}\bar{\xi}} \Delta t \right) \\
&\quad + \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1 + r\Delta t)} P_k^{(N)} + \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1 + r\Delta t)} Q_k^{(N)} \\
&= h \left( k\Delta t, \bar{\xi}_k^{(N)} \right) + \frac{h_{tt} + 2 \frac{\alpha^2}{\sigma^2} \bar{\xi}_k^{(N)} h_{t\bar{\xi}} (\Delta t)^2}{2(1 + r\Delta t)} + \frac{\left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) P_k^{(N)} + \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) Q_k^{(N)}}{2(1 + r\Delta t)}, \quad (50)
\end{aligned}$$

1 where we use the PDE (17) satisfied by  $h$ .

2 Next, consider the recursive relation for  $X_k^{(N)}$ , by Tower property, we have:

$$\begin{aligned}
&h^{(N)} \left( k\Delta t, \bar{\xi}_k^{(N)} \right) \\
&= (1 + r\Delta t)^k \left( \bar{\xi}_k^{(N)} \right)^{-1} \mathbb{E} \left[ (1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left( M, \gamma R + Y e^{-rT} \bar{\xi}_N^{(N)} \right) \middle| \bar{\xi}_k^{(N)} \right] \\
&= \left( \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1 + r\Delta t)} (1 + r\Delta t)^{k+1} \left( \bar{\xi}_{k+1}^{(N)+} \right)^{-1} \mathbb{E} \left[ (1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left( M, \gamma R + Y e^{-rT} \bar{\xi}_N^{(N)} \right) \middle| \bar{\xi}_{k+1}^{(N)+} \right] \right. \\
&\quad \left. + \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1 + r\Delta t)} (1 + r\Delta t)^{k+1} \left( \bar{\xi}_{k+1}^{(N)-} \right)^{-1} \mathbb{E} \left[ (1 + r\Delta t)^{-N} \bar{\xi}_N^{(N)} I \left( M, \gamma R + Y e^{-rT} \bar{\xi}_N^{(N)} \right) \middle| \bar{\xi}_{k+1}^{(N)-} \right] \right) \\
&= \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1 + r\Delta t)} h^{(N)} \left( (k+1)\Delta t, \bar{\xi}_{k+1}^{(N)+} \right) + \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1 + r\Delta t)} h^{(N)} \left( (k+1)\Delta t, \bar{\xi}_{k+1}^{(N)-} \right) \\
&= \mathbb{E}^{(N)} \left[ \frac{1}{1 + r\Delta t} h^{(N)} \left( (k+1)\Delta t, \bar{\xi}_{k+1}^{(N)} \right) \middle| \bar{\xi}_k^{(N)} \right]. \quad (51)
\end{aligned}$$

3 Note that  $\varepsilon_N^{(N)} = 0$ , combining (50) and (51), we have

$$\begin{aligned}
\varepsilon_k^{(N)} &:= h \left( k\Delta t, \bar{\xi}_k^{(N)} \right) - h^{(N)} \left( k\Delta t, \bar{\xi}_k^{(N)} \right) \\
&= \mathbb{E}^{(N)} \left[ \frac{1}{1 + r\Delta t} h \left( (k+1)\Delta t, \bar{\xi}_{k+1}^{(N)} \right) \middle| \bar{\xi}_k^{(N)} \right] + \beta \left( k\Delta t, \bar{\xi}_k^{(N)} \right) (\Delta t)^2 + \gamma_k^{(N)} \\
&\quad - \mathbb{E}^{(N)} \left[ \frac{1}{1 + r\Delta t} h^{(N)} \left( (k+1)\Delta t, \bar{\xi}_{k+1}^{(N)} \right) \middle| \bar{\xi}_k^{(N)} \right] \\
&= \mathbb{E}^{(N)} \left[ \frac{1}{1 + r\Delta t} \varepsilon_{k+1}^{(N)} \middle| \bar{\xi}_k^{(N)} \right] + \beta \left( k\Delta t, \bar{\xi}_k^{(N)} \right) (\Delta t)^2 + \gamma_k^{(N)}.
\end{aligned}$$

4 Then we can prove (48) by backward induction.

5 • Step 2: Since, for any  $q > 1$ ,

$$\begin{aligned}
\mathbb{E}^{(N)} \left[ \left| \bar{\xi}_{k+1}^{(N)} \right|^q \middle| \bar{\xi}_k^{(N)} \right] &= \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^q \left| \bar{\xi}_k^{(N)} \right|^q + \frac{1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^q \left| \bar{\xi}_k^{(N)} \right|^q \\
&= \left( 1 + \frac{\alpha^2}{\sigma^2} \frac{(q+1)q}{2} \Delta t + O \left( \Delta t^{\frac{3}{2}} \right) \right) \left| \bar{\xi}_k^{(N)} \right|^q,
\end{aligned}$$

1 by Tower property, for any  $n > k$ , we have, by a simple application of Markovian property,

$$\begin{aligned} \mathbb{E}^{(N)} \left[ \left| \bar{\xi}_n^{(N)} \right|^q \left| \bar{\xi}_k^{(N)} \right|^q \right] &= \left( 1 + \frac{\alpha^2 (q+1)q}{\sigma^2} \Delta t + O(\Delta t^{\frac{3}{2}}) \right)^{n-k} \left| \bar{\xi}_k^{(N)} \right|^q \\ &\leq e^{KT} \left| \bar{\xi}_k^{(N)} \right|^q, \end{aligned} \quad (52)$$

2 for some  $K > 0$ . Since  $h_{tt}$  and  $h_{t\bar{\xi}}$  satisfy the polynomial growth condition,  $\beta$  also satisfies  
3 the polynomial growth condition of the same power as well, and thus we have, for any  
4  $p \in \mathbb{Z}^+$ ,

$$\mathbb{E}^{(N)} \left[ \left| \beta \left( n\Delta t, \bar{\xi}_n^{(N)} \right) \right| \left| \bar{\xi}_k^{(N)} \right|^p \right] \leq A \left( 1 + \mathbb{E}^{(N)} \left[ \left| \bar{\xi}_n^{(N)} \right|^p \left| \bar{\xi}_k^{(N)} \right|^p \right] \right) \leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right)$$

5 for some  $A > 0$ , which confirms the claim as in Step 2.

6 • Step 3: First, differentiate  $f_k^{(N)+}(\tau)$  up to the third order, we have, for  $s \in (0, 1)$ ,

$$\begin{aligned} \frac{\partial^3}{\partial \tau^3} f_k^{(N)+}(s) &= -\frac{\alpha^3}{\sigma^3} \left( \bar{\xi}_k^{(N)} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (k+s)\Delta t, \bar{\xi}_k^{(N)} \right) + s \left( \bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \Delta t^{\frac{3}{2}} \\ &\quad + 3\frac{\alpha^2}{\sigma^2} \left( \bar{\xi}_k^{(N)} \right)^2 h_{t\bar{\xi}\bar{\xi}} \left( (k+s)\Delta t, \bar{\xi}_k^{(N)} \right) + s \left( \bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \Delta t^2 \\ &\quad - 3\frac{\alpha}{\sigma} \bar{\xi}_k^{(N)} h_{tt\bar{\xi}} \left( (k+s)\Delta t, \bar{\xi}_k^{(N)} \right) + s \left( \bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \Delta t^{\frac{5}{2}} \\ &\quad + h_{ttt} \left( (k+s)\Delta t, \bar{\xi}_k^{(N)} \right) + s \left( \bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \Delta t^3. \end{aligned}$$

7 Under the polynomial growth condition on the partial derivatives of  $h$  up to the third  
8 order, we have, for  $s \in (0, 1)$ ,

$$\begin{aligned} \left| \frac{\partial^3}{\partial \tau^3} f_k^{(N)+}(s) \right| &\leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^3 \right) \left( 1 + \left| \bar{\xi}_k^{(N)} \right| + s \left( \bar{\xi}_{k+1}^{(N)+} - \bar{\xi}_k^{(N)} \right) \right)^q \left( \Delta t^{\frac{3}{2}} + O(\Delta t^2) \right) \\ &= A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^3 \right) \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^q \left( 1 - \frac{\alpha}{\sigma} s\sqrt{\Delta t} \right)^q \right) \left( \Delta t^{\frac{3}{2}} + O(\Delta t^2) \right) \\ &\leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left( \Delta t^{\frac{3}{2}} + O(\Delta t^2) \right), \end{aligned} \quad (53)$$

9 for some positive integer  $p$ . As a result, for any  $N$  and  $k$ , we have

$$\left| P_k^{(N)} \right| \leq \frac{1}{6} \int_0^1 (1-s)^3 \left| \frac{\partial^3}{\partial \tau^3} f_k^{(N)+}(s) \right| ds \leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left( \Delta t^{\frac{3}{2}} + O(\Delta t^2) \right).$$

10 Similar estimate for  $|Q_k^{(N)}|$  can be obtained via the same approach, and thus we have, for  
11 any  $p \in \mathbb{Z}^+$ ,  $|\gamma_k^{(N)}| \leq A(1 + |\bar{\xi}_k^{(N)}|^p)(\Delta t^{\frac{3}{2}} + O(\Delta t^2))$ , for some  $A > 0$ . Then, by using the  
12 estimate (52), we attain

$$\mathbb{E}^{(N)} \left[ \left| \gamma_n^{(N)} \right| \left| \bar{\xi}_k^{(N)} \right|^p \right] \leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left( \Delta t^{\frac{3}{2}} + O(\Delta t^2) \right),$$

13 which confirms the validity of Step 3.

14 And hence we conclude with the claim of the lemma.

15 **C.2. Proof of Lemma C.3.** We first define  $S_n^{(N)} := -\sum_{k=1}^n Z_k^{(N)}$  and  $S_*^{(N)} :=$   
16  $\max_{1 \leq n \leq N} S_n^{(N)}$ , and aim to establish the following lemma beforehand:

17 **Lemma C.5.** *For any  $N \in \mathbb{N}$ , either one of the followings is true:*

- 18 1. *if  $N$  is odd,  $\mathbb{P}(2k - \frac{1}{2} < S_*^{(N)} < 2k + \frac{3}{2}) = 2\mathbb{P}(S_N^{(N)} = 2k + 1)$  for any*  
19 *non-negative integer  $k$ ;*
- 20 2. *if  $N$  is even,  $\mathbb{P}(2k - \frac{3}{2} < S_*^{(N)} < 2k + \frac{1}{2}) = 2\mathbb{P}(S_N^{(N)} = 2k)$  for any non-negative*  
21 *integer  $k$ .*

22 *Proof.* Define a stopping time  $\tau_a = \inf\{n \in \mathbb{N} \mid S_n^{(N)} \geq a\}$ . We here only show that  
23 the case of odd value of  $N$ ; the case for even value of  $N$  is similar, and we therefore  
24 omit the details. For any non-negative integer  $k$ ,

$$\mathbb{P} \left( S_N^{(N)} > 2k - \frac{1}{2} \right) = \mathbb{P} \left( S_N^{(N)} \geq 2k + 1 \right) = \mathbb{P} \left( S_N^{(N)} \geq 2k + 1, \tau_{2k} < N \right)$$

$$\begin{aligned}
&= \mathbb{P}\left(S_N^{(N)} - S_{\tau_{2k}}^{(N)} \geq 1, \tau_{2k} < N\right) = \mathbb{P}\left(S_N^{(N)} - S_{\tau_{2k}}^{(N)} \geq 1 \mid \tau_{2k} < N\right) \mathbb{P}(\tau_{2k} < N) \\
&= \frac{1}{2} \mathbb{P}\left(S_*^{(N)} > 2k - \frac{1}{2}\right)
\end{aligned}$$

1 where the first equality follows because  $S_N^{(N)}$  can only be equal to an odd number,  
2 the third equality holds because  $S_{\tau_{2k}}^{(N)} = 2k$  by the definition of  $\tau_{2k}$ . For the last  
3 equality, note that  $\mathbb{P}(S_N^{(N)} - S_{\tau_{2k}}^{(N)} \geq 1 \mid \tau_{2k} < N)$  and  $\mathbb{P}(S_N^{(N)} - S_{\tau_{2k}}^{(N)} \geq 0 \mid \tau_{2k} < N)$   
4 are equivalent since  $S_N^{(N)}$  is odd and  $S_{\tau_{2k}}^{(N)}$  is even, while  $\mathbb{P}(S_N^{(N)} \geq S_{\tau_{2k}}^{(N)} \mid \tau_{2k} <$   
5  $N) = \frac{1}{2}$  because of the future independent increments after  $\tau_{2k}$  and the symmetry  
6 of binomial tree model.

7 Besides, for any odd number  $N$ ,

$$\begin{aligned}
&\mathbb{P}\left(S_N^{(N)} = 2k + 1\right) = \mathbb{P}\left(S_N^{(N)} > 2k - \frac{1}{2}\right) - \mathbb{P}\left(S_N^{(N)} > 2k + \frac{3}{2}\right) \\
&= \frac{1}{2} \mathbb{P}\left(S_*^{(N)} > 2k - \frac{1}{2}\right) - \frac{1}{2} \mathbb{P}\left(S_*^{(N)} > 2k + \frac{3}{2}\right) = \frac{1}{2} \mathbb{P}\left(2k - \frac{1}{2} < S_*^{(N)} < 2k + \frac{3}{2}\right),
\end{aligned}$$

8 for every non-negative integer  $k$ , therefore the claim follows.  $\square$

9 Upon the event  $\{S_*^{(N)} = k\}$ , for any integer  $1 \leq n \leq N$

$$\begin{aligned}
\bar{\xi}_n^{(N)} &= \prod_{k=1}^n \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} Z_k^{(N)}\right) \\
&= \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^k \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^a \left(1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^b \text{ for some integers } a \leq b \\
&\leq \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^k
\end{aligned}$$

10 Thus,  $\sup_{1 \leq n \leq N} (\bar{\xi}_n^{(N)})^p \leq \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{pk}$  upon the event  $\{S_*^{(N)} = k\}$ . Then, we  
11 have

$$\mathbb{E} \left[ \sup_{1 \leq n \leq N} (\bar{\xi}_n^{(N)})^p \right] \leq \sum_{k=0}^N \mathbb{P}\left(S_*^{(N)} = k\right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{pk}.$$

12 i) For if  $N$  is odd, there is an integer  $m$  such that  $N = 2m + 1$ , then, by Lemma  
13 **C.5**, we have

$$\begin{aligned}
&\sum_{k=0}^N \mathbb{P}\left(S_*^{(N)} = k\right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{pk} \\
&= \sum_{k=0}^m \mathbb{P}\left(S_*^{(2m+1)} = 2k\right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{p(2k)} + \sum_{k=0}^m \mathbb{P}\left(S_*^{(2m+1)} = 2k + 1\right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{p(2k+1)} \\
&\leq \sum_{k=0}^m 2\mathbb{P}\left(S_{2m+1}^{(2m+1)} = 2k + 1\right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{p(2k)} + \sum_{k=0}^m 2\mathbb{P}\left(S_{2m+1}^{(2m+1)} = 2k + 1\right) \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{p(2k+1)} \\
&\leq 4 \sum_{k=0}^m \frac{C_k^{2m+1}}{2^{2m+1}} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{p(2k+1)} = \frac{4}{2^{2m+1}} \sum_{k=0}^m C_k^{2m+1} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{p(2m+1-2k)} \\
&\leq \frac{4}{2^N} \sum_{k=0}^N C_k^N \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{p(N-k)} \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^{-pk} = \frac{4}{2^N} \left( \left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^p + \frac{1}{\left(1 + \frac{\alpha}{\sigma} \sqrt{\Delta t}\right)^p} \right)^N \\
&= 4 \left(1 + p^2 \frac{\alpha^2}{\sigma^2} \Delta t + O\left(\Delta t^{\frac{3}{2}}\right)\right)^N \leq 4e^{KT},
\end{aligned}$$

14 for some  $K > 0$ .

1 ii) The case for even value of  $N$  can be resolved similarly.

2 **C.3. Proof of Lemma C.4.** On the one hand, the polynomial growth condition  
3 on  $h_{t\bar{\xi}}$  implies that for any  $p \in \mathbb{Z}^+$ ,

$$\left| \bar{\xi}_k^{(N)} h_{\bar{\xi}\bar{\xi}} \cdot \Delta t \right| \leq C \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \Delta t, \quad (54)$$

4 for some  $C > 0$ . On the other hand, by (53), we have

$$\begin{aligned} & \int_0^1 (1-s)^3 \frac{\partial^3}{\partial \tau^3} \left( f_k^{(N)+}(s) - f_k^{(N)-}(s) \right) ds \\ & \leq C \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left( \Delta t^{\frac{3}{2}} + O(\Delta t^2) \right), \end{aligned} \quad (55)$$

5 for some  $C > 0$  and positive integer  $p$ . According to the definition of  $e_k^{(N)}$ , we have  
6 showed the boundness of the first two terms in (47). Therefore, the proof of this  
7 lemma can be accomplished if we can show that

$$\left| \frac{1}{2\sigma\sqrt{\Delta t}} \left( \varepsilon_{k+1}^{(N)+} - \varepsilon_{k+1}^{(N)-} \right) \right| \leq C \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left( \sqrt{\Delta t} + O(\Delta t) \right),$$

8 for some  $C > 0$  and positive integer  $p$ . To this end, we use the expression (48), and  
9 then we have

$$\begin{aligned} & \left| \frac{1}{2\sigma\sqrt{\Delta t}} \left( \varepsilon_{k+1}^{(N)+} - \varepsilon_{k+1}^{(N)-} \right) \right| \\ & = \sum_{n=k+1}^{N-1} \frac{1}{2\sigma\sqrt{\Delta t}} \left( \left| \mathbb{E}^{(N)} \left[ \beta \left( n\Delta t, \bar{\xi}_n^{(N)} \right) \middle| \bar{\xi}_{k+1}^{(N)+} \right] - \mathbb{E}^{(N)} \left[ \beta \left( n\Delta t, \bar{\xi}_n^{(N)} \right) \middle| \bar{\xi}_{k+1}^{(N)-} \right] \right| \frac{1}{(1+r\Delta t)^{m-k-1}} (\Delta t)^2 \right. \\ & \quad \left. + \left| \mathbb{E}^{(N)} \left[ \gamma_n^{(N)} \middle| \bar{\xi}_{k+1}^{(N)+} \right] - \mathbb{E}^{(N)} \left[ \gamma_n^{(N)} \middle| \bar{\xi}_{k+1}^{(N)-} \right] \right| \frac{1}{(1+r\Delta t)^{m-k-1}} \right), \end{aligned} \quad (56)$$

10 where  $\beta(t, \xi) := -\frac{(h_{tt}(t, \xi) + 2\frac{\alpha}{\sigma^2} \xi h_{t\xi}(t, \xi))}{2(1+r\Delta t)}$  and  $\gamma_k^{(N)} := -\frac{(1-\frac{\alpha}{\sigma}\sqrt{\Delta t})P_k^{(N)} + (1+\frac{\alpha}{\sigma}\sqrt{\Delta t})Q_k^{(N)}}{2(1+r\Delta t)}$ .

11 To bound the first term on the right hand side of (56), by (49), for any  $p \in \mathbb{Z}^+$ ,

$$\begin{aligned} & \sum_{n=k+1}^{N-1} \frac{1}{2\sigma\sqrt{\Delta t}} \left( \left| \mathbb{E}^{(N)} \left[ \beta \left( n\Delta t, \bar{\xi}_n^{(N)} \right) \middle| \bar{\xi}_{k+1}^{(N)+} \right] - \mathbb{E}^{(N)} \left[ \beta \left( n\Delta t, \bar{\xi}_n^{(N)} \right) \middle| \bar{\xi}_{k+1}^{(N)-} \right] \right| \frac{1}{(1+r\Delta t)^{m-k-1}} (\Delta t)^2 \right) \\ & \leq \sum_{n=k+1}^{N-1} \frac{1}{2\sigma\sqrt{\Delta t}} \left( \left| \mathbb{E}^{(N)} \left[ \beta \left( n\Delta t, \bar{\xi}_n^{(N)} \right) \middle| \bar{\xi}_{k+1}^{(N)+} \right] \right| + \left| \mathbb{E}^{(N)} \left[ \beta \left( n\Delta t, \bar{\xi}_n^{(N)} \right) \middle| \bar{\xi}_{k+1}^{(N)-} \right] \right| \right) (\Delta t)^2 \\ & \leq \frac{1}{2\sigma\sqrt{\Delta t}} \left( A \left( 1 + \left| \bar{\xi}_{k+1}^{(N)+} \right|^p \right) \Delta t + A \left( 1 + \left| \bar{\xi}_{k+1}^{(N)-} \right|^p \right) \Delta t \right) \\ & \leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^p + \left| \bar{\xi}_k^{(N)} \right|^p \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right)^p \right) \sqrt{\Delta t} \\ & \leq A \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^q \right) \left( \sqrt{\Delta t} + O(\Delta t) \right) \end{aligned} \quad (57)$$

12 for some  $A > 0$ .

13 To bound the second term on the right hand side of (56), we first fix any  $n$  and  $k$  with  $n > k$ ,  
14 at the same, for the sake of notation, we suppress the indices by taking  $\bar{\xi}_k^{(N)} = \bar{\xi} > 0$ . For any  
15  $s \in (0, 1)$ , by the Lipschitz continuity and the polynomial growth of  $h_{\bar{\xi}\bar{\xi}\bar{\xi}}$ , for any  $p \in \mathbb{Z}^+$ ,

$$\begin{aligned} & \left| \mathbb{E}^{(N)} \left[ \left( \bar{\xi}_n^{(N)} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (s+n)\Delta t, \bar{\xi}_n^{(N)} + s \left( \bar{\xi}_{n+1}^{(N)+} - \bar{\xi}_n^{(N)} \right) \right) \middle| \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\ & \quad \left. - \mathbb{E}^{(N)} \left[ \left( \bar{\xi}_n^{(N)} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (s+n)\Delta t, \bar{\xi}_n^{(N)} + s \left( \bar{\xi}_{n+1}^{(N)+} - \bar{\xi}_n^{(N)} \right) \right) \middle| \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left( \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (s+n)\Delta t, \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left( 1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right) \right] \right. \\
&\quad \left. - \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left( \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (s+n)\Delta t, \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left( 1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right) \right] \right| \\
&+ \left| \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left( \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (s+n)\Delta t, \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left( 1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right) \right] \right. \\
&\quad \left. - \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left( \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (s+n)\Delta t, \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left( 1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right) \right] \right| \\
&\leq \mathbb{E}^{(N)} \left[ L \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left( \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right)^3 2\bar{\xi} \frac{\alpha}{\sigma} \sqrt{\Delta t} \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left( 1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right] \\
&\quad + \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \bar{\xi}^3 \left( 6 \frac{\alpha}{\sigma} \sqrt{\Delta t} + O(\Delta t^{\frac{3}{2}}) \right) A \left( 1 + \left| \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \left( 1 - \frac{\alpha}{\sigma} s \sqrt{\Delta t} \right) \right|^q \right) \right] \\
&\leq C \bar{\xi}^4 \left( \sqrt{\Delta t} + O(\Delta t) \right) \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^4 \right] + C \bar{\xi}^3 \left( \sqrt{\Delta t} + O(\Delta t) \right) \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 \left( 1 + \bar{\xi}^p \left| \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right|^q \right) \right] \\
&\leq C \left( 1 + \bar{\xi}^p \right) \left( \sqrt{\Delta t} + O(\Delta t) \right),
\end{aligned}$$

- 1 for some  $C > 0$ , where the last inequality follows since  $\mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^q \right] \leq M_q$  for some  $M_q > 0$  by
- 2 (52). Note that similar approaches can be applied to other terms in  $\frac{\partial^3}{\partial \tau^3} f_n^{(N)+}$ , therefore we have

$$\begin{aligned}
&\left| \mathbb{E}^{(N)} \left[ \frac{\partial^3}{\partial \tau^3} f_n^{(N)+}(s) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] - \mathbb{E}^{(N)} \left[ \frac{\partial^3}{\partial \tau^3} f_n^{(N)+}(s) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
&\leq \frac{\alpha^3}{\sigma^3} (\Delta t)^{\frac{3}{2}} \left| \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (s+n)\Delta t, \bar{\xi}_n^{(N)} + s \left( \bar{\xi}_{n+1}^{(N+)} - \bar{\xi}_n^{(N)} \right) \right) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
&\quad \left. - \mathbb{E}^{(N)} \left[ \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^3 h_{\bar{\xi}\bar{\xi}\bar{\xi}} \left( (s+n)\Delta t, \bar{\xi}_n^{(N)} + s \left( \bar{\xi}_{n+1}^{(N+)} - \bar{\xi}_n^{(N)} \right) \right) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
&+ (\Delta t)^2 \left| \mathbb{E}^{(N)} \left[ 3 \frac{\alpha^2}{\sigma^2} \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^2 h_{t\bar{\xi}\bar{\xi}} - 3 \frac{\alpha}{\sigma} \bar{\xi}_n^{(N)} h_{tt\bar{\xi}} \sqrt{\Delta t} + h_{ttt} \Delta t \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
&\quad \left. - \mathbb{E}^{(N)} \left[ 3 \frac{\alpha^2}{\sigma^2} \left( \frac{\bar{\xi}_n^{(N)}}{\bar{\xi}_{k+1}^{(N)}} \right)^2 h_{t\bar{\xi}\bar{\xi}} - 3 \frac{\alpha}{\sigma} \bar{\xi}_n^{(N)} h_{tt\bar{\xi}} \sqrt{\Delta t} + h_{ttt} \Delta t \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
&\leq A \left( 1 + \bar{\xi}^p \right) \left( \Delta t^2 + O \left( \Delta t^{\frac{5}{2}} \right) \right),
\end{aligned}$$

- 3 for some  $A > 0$ . Then,

$$\begin{aligned}
&\left| \mathbb{E}^{(N)} \left[ P_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] - \mathbb{E}^{(N)} \left[ P_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
&\leq \frac{1}{6} \int_0^1 (1-s)^3 \left| \mathbb{E}^{(N)} \left[ \frac{\partial^3}{\partial \tau^3} f_n^{(N)+}(s) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right. \\
&\quad \left. - \mathbb{E}^{(N)} \left[ \frac{\partial^3}{\partial \tau^3} f_n^{(N)+}(s) \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| ds \\
&\leq \frac{1}{6} \int_0^1 (1-s)^3 A \left( 1 + \bar{\xi}^p \right) \left( \Delta t^2 + O \left( \Delta t^{\frac{5}{2}} \right) \right) ds \leq A \left( 1 + \bar{\xi}^p \right) \left( \Delta t^2 + O \left( \Delta t^{\frac{5}{2}} \right) \right).
\end{aligned}$$

- 4 The same approach can be applied to  $Q_n^{(N)}$ , and thus we finally have

$$\begin{aligned}
&\left| \mathbb{E}^{(N)} \left[ \gamma_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] - \mathbb{E}^{(N)} \left[ \gamma_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
&\leq \frac{1 - \frac{\alpha}{\sigma} \sqrt{\Delta t}}{2(1+r\Delta t)} \left| \mathbb{E}^{(N)} \left[ P_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] - \mathbb{E}^{(N)} \left[ P_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right|
\end{aligned}$$

$$\begin{aligned}
& + \frac{1 + \frac{\alpha}{\sigma}\sqrt{\Delta t}}{2(1+r\Delta t)} \left| \mathbb{E}^{(N)} \left[ Q_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] - \mathbb{E}^{(N)} \left[ Q_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)} = \bar{\xi} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \right] \right| \\
& \leq A \left( 1 + \bar{\xi}^p \right) \left( (\Delta t)^2 + O \left( (\Delta t)^{\frac{5}{2}} \right) \right).
\end{aligned}$$

1 Therefore,

$$\begin{aligned}
& \sum_{n=k+1}^{N-1} \frac{1}{2\sigma\sqrt{\Delta t}} \left( \left| \mathbb{E}^{(N)} \left[ \gamma_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)+} \right] - \mathbb{E}^{(N)} \left[ \gamma_n^{(N)} \mid \bar{\xi}_{k+1}^{(N)-} \right] \right| \frac{1}{(1+r\Delta t)^{m-k-1}} \right) \\
& \leq A \left( 1 + \left( \bar{\xi}_k^{(N)} \right)^p \right) \left( \sqrt{\Delta t} + O(\Delta t) \right), \tag{58}
\end{aligned}$$

2 and so substituting (57) and (58) into (56), we obtain that for any  $p \in \mathbb{Z}^+$ ,

$$\left| \frac{1}{2\sigma\sqrt{\Delta t}} \left( \varepsilon_{k+1}^{(N)+} - \varepsilon_{k+1}^{(N)-} \right) \right| \leq C \left( 1 + \left| \bar{\xi}_k^{(N)} \right|^p \right) \left( \sqrt{\Delta t} + O(\Delta t) \right),$$

3 for some  $C > 0$ . Together with (54) and (55), the claim of this lemma is concluded.

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