

A Note on \mathcal{P} - vs. \mathcal{Q} -Expected Loss Portfolio Constraints

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Abstract

We consider portfolio optimization problems with expected loss constraints under the physical measure \mathcal{P} and the risk neutral measure \mathcal{Q} , respectively. Using Merton's portfolio as a benchmark portfolio, the optimal terminal wealth of the \mathcal{Q} -risk constraint problem can be easily replicated with the standard delta hedging strategy. Motivated by this, we consider the \mathcal{Q} -strategy fulfilling the \mathcal{P} -risk constraint and compare its solution with the true optimal solution of the \mathcal{P} -risk constraint problem. We show the existence and uniqueness of the optimal solution to the \mathcal{Q} -strategy fulfilling the \mathcal{P} -risk constraint, and provide a tractable evaluation method. The \mathcal{Q} -strategy fulfilling the \mathcal{P} -risk constraint is not only easier to implement with standard forwards and puts on a benchmark portfolio than the \mathcal{P} -risk constraint problem, but also easier to solve than either of the \mathcal{Q} - or \mathcal{P} -risk constraint problem. The numerical test shows that the difference of the values of the two strategies (the \mathcal{Q} -strategy fulfilling the \mathcal{P} -risk constraint and the optimal strategy solving the \mathcal{P} -risk constraint problem) is reasonably small.

Keywords: Optimal Portfolio, Expected Loss Constraint, Physical Measure \mathcal{P} , Risk-Neutral Measure \mathcal{Q} , \mathcal{Q} -Strategy Fulfilling \mathcal{P} -Risk Constraint.

1. Introduction

Maximizing expected utility, with the utility function formalized by a constant relative risk aversion, is a standard approach to portfolio optimization leading to the widely accepted constant proportion portfolio; see the pioneering work by Merton (1990). Both financial institutions and individuals often take such a strategy as starting point and

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then make appropriate deviations away from it. Such deviations can be motivated by e.g. liability considerations, liquidity needs, or tactical overlays. Liability considerations can be formalized as solvency constraints on the portfolio. Downward portfolio protection relative to a liability can be obtained, e.g., by the standard Constant Proportion Portfolio Insurance (CPPI) or Option-Based Portfolio Insurance (OBPI) strategies. They protect, at least for a continuous asset process, the portfolio fully against a loss relative to some hedgable liability benchmark. One can loosen the insurance to not protect fully but instead accept some extent of loss. The VaR constraint allows the investor to limit the probability of a loss. The expected loss constraint allows the investor to limit the expectation of the loss. We can motivate both by corresponding external solvency rules. In this note we focus exclusively on the expected loss constraint.

As a solvency rule, one expresses the expected loss constraint naturally under the objective probability measure. Nevertheless, the portfolio problem where the constraint is formulated under the risk-neutral measure is solved first, by Basak and Shapiro (2001), and has some appeal. The optimal portfolio turns out to be a combination of the following simple positions: a) A part of the assets in the optimal unconstrained portfolio; b) a long option position to protect that portfolio from a loss; and c) a short option position to allow for some loss after all. The options bought and sold are plain vanilla options. We recover in this paper the result by Basak and Shapiro (2001). We also derive the optimal portfolio when the expected loss is constrained under the physical measure. We find that one should replace the short plain vanilla position by a short position in a power option, i.e. an option that involves the power function of the underlying. We solve both problems by the martingale method, see Karatzas et al. (1987) and Cox and Huang (1989). One may also apply the dynamic programming principle to solve problems with constraints, see Kraft and Steffensen (2013).

The distinctly different solutions to the portfolio problems with different constraints call for a discussion on which problem and strategy is easier to implement and communicate. The combination of plain vanilla put options appears clearly as the simplest strategy both to implement and to communicate. However, the problem that it solves, based on the expected loss under the risk-neutral measure, appears more difficult to understand and communicate than the problem based on the expected loss under the physical measure. Note also that the risk-neutral expectation of the uncertain loss is equal to the financial value of that loss, except for a discount factor. Indeed, one can communicate that interpretation easily to people who think about risk in terms of the value of holding or selling it.

We highlight in this short note two key advantages of working with constraints formulated in terms of the risk-neutral measure instead of the objective measure: First, we can implement the optimal constrained portfolio easily by trading or hedging plain vanilla options, not non-marketed power options, which shall be appealing to all investors. Second,

since the risk-neutral expectation is essentially a representation of the value, constraining it is equivalent to constraining the financial value of the uncertain loss.

Two concerns about the specific model arise: First, the interpretation of constraining the value of the loss is true only for its theoretical value (arbitrage-free price within the model). However, a constraint under the physical measure is also a model-based theoretical constraint so that argument does not favor one measure over the other. Second, the optimal portfolios we find for given constraints, are optimal within the model only. Truly, both our qualitative and quantitative results hinge upon our market assumption and may not hold fully outside the model. However, the purpose of the note is to derive some structural insight and structural insight within a simple model is valuable in itself.

Even though the optimal portfolio constrained under the risk-neutral measure is easier to implement and communicate, one naturally formulates solvency rules as constraints under the physical measure, which leads to some important questions: 1) How do we construct the best portfolio, among the simple portfolios based on plain vanilla put options, such that the constraint under the physical measure is fulfilled? This is clearly not the optimal portfolio solving the constraint under the physical measure but it may be preferred due to its tractability. 2) If we, for simplicity, constrain the portfolio under the risk neutral measure instead of the physical measure, by the same upper level, what is the precision with which we actually fulfill the constraint under the physical measure? 3) What is the error we make? Does it, e.g., always have the same sign such that we know beforehand whether we are on the safe side, or unsafe side, if we simply replace a physical constraint by a risk-neutral constraint? Along with the derivation of the various strategies, these are the questions we address in this note.

The main contribution of this short paper is to clarify the relation between the optimal solutions with loss constraints under different measures. We show that the strategy solving the risk-neutral constraint is efficient in certain ways: a) it is easy and fast to implement; b) there exists a unique risk-neutrally constrained strategy that fulfills the objective constraint; c) the approximation is reasonably accurate by numerical tests.

The outline of the paper is as follows. In Section 2, we present the problems and derive the solutions. In Section 3, we find the risk-neutrally constrained strategy that fulfills the objective constraint and perform numerical tests. Section 4 concludes.

2. \mathcal{Q} - and \mathcal{P} -Expected Loss Constraints

We consider a continuous-time economy with finite horizon $[0, T]$, which is built on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathcal{P})$ on which a one-dimensional Brownian motion W is defined. Financial investment opportunities are given by an instantaneously risk-free money market account and a risky stock as in the Black-Scholes model. We suppose that the money market provides a constant interest rate r . The stock price S is represented

by a geometric Brownian motion (GBM) as

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

where the drift μ and the volatility σ are constants. We assume that a portfolio manager in this economy is endowed at time 0 with an initial wealth x_0 . He chooses a self-financing investment strategy π , where π_t denotes the proportion of wealth invested in the stock at time t . The wealth process X follows a controlled stochastic differential equation (SDE):

$$(2.1) \quad dX_t = rX_t dt + \pi_t X_t \sigma (\theta dt + dW_t), X_0 = x_0 > 0,$$

where $\theta = (\mu - r)/\sigma$ is the market price of risk and π is a progressively measurable, square integrable control process to be decided. The manager aims at maximizing his expected utility of the wealth at time T , i.e.,

$$\max_{\pi} E(U(X_T)),$$

where $U(x) = \frac{1}{\gamma} x^\gamma$, $0 < \gamma < 1$, with a risk constraint on its expected loss. Denote by \mathcal{P} and \mathcal{Q} the physical measure and the risk neutral measure, respectively, and E and $E^{\mathcal{Q}}$ the expectations under \mathcal{P} and \mathcal{Q} , respectively. The risk constraint can then be given by

$$\mathcal{P}\text{-expected loss: } E((K - X_T)^+) \leq \epsilon,$$

or

$$\mathcal{Q}\text{-expected loss: } E^{\mathcal{Q}}((K - X_T)^+) \leq \epsilon,$$

where K is a constant benchmark and ϵ is a positive constant.

Note that the GBM asset price process and the power utility are the simplest setting for utility maximization. For practical portfolio optimization, it is desirable to have more general asset price processes such as Lévy process and more general utility functions such as S-shaped utility, but then the problem is much more complicated and is very difficult to solve and any insights from the model may be hidden due to the technicality of the model. We choose the GBM asset price process and power utility in this paper, not only they are well known and commonly used, but also they provide clear insights and relations for two different risk measure constraints, which might not have been possible if general asset price processes and utility functions were used.

With a risk constraint on the \mathcal{Q} -expected loss, the optimization problem is solved in Basak & Shapiro (2001), see also Kraft & Steffensen (2013). The martingale method is used to transform the problem to a static optimization problem. Define the pricing kernel ξ_t by

$$\xi_t = \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right), 0 \leq t \leq T,$$

and the measure \mathcal{Q} by

$$\mathcal{Q}(A) = E(\xi_T 1_A), A \in \mathcal{F}_T,$$

where 1_A is the indicator that equals 1 if A happens and 0 otherwise. Then the discounted wealth process $(e^{-rt}X_t)_{0 \leq t \leq T}$ is a super-martingale under \mathcal{Q} . Since the market is complete, we may first find the optimal terminal wealth $X_T^{\mathcal{Q}}$ and then use the martingale representation theorem to derive the optimal control $\pi^{\mathcal{Q}}$. To find the optimal terminal wealth $X_T^{\mathcal{Q}}$, the static optimization problem becomes (note X_T is an \mathcal{F}_T measurable random variable)

$$\begin{aligned} & \max_{X_T} E[U(X_T)] \\ \text{subject to } & E^{\mathcal{Q}}(e^{-rT}X_T) \leq x_0 \text{ (budget constraint),} \\ & E^{\mathcal{Q}}((K - X_T)^+) \leq \epsilon \text{ (risk constraint),} \end{aligned}$$

which is equivalent to

$$(2.2) \quad \begin{aligned} & \max_{X_T} E[U(X_T)] \\ \text{subject to } & E(\xi_T X_T) \leq e^{rT}x_0, \\ & E(\xi_T(K - X_T)^+) \leq \epsilon. \end{aligned}$$

Note that there is no impact from the constraint on the formation of the pricing kernel ξ that reflects the risk attitude of a representative agent (different from the marginal investor we consider) in equilibrium. Given that pricing kernel, our marginal investor makes marginal decisions and introduces the \mathcal{Q} measure in the constraint only to express his preferences. There is not feedback effect from our marginal investor on the formation of \mathcal{Q} . It is beyond the scope of the paper to consider the fixed-point impact that arises if the representative agent specifies preferences in terms of the measure \mathcal{Q} he forms himself.

To avoid triviality, we make the following assumption:

Assumption 1 *The risk constraint in problem (2.2) is binding, i.e.,*

$$E[\xi_T(K - I(\lambda_0 \xi_T))^+] > \epsilon,$$

where I is the inverse function of U' , λ_0 is determined by

$$(2.3) \quad E(\xi_T I(\lambda_0 \xi_T)) = e^{rT}x_0.$$

Remark 2.1 *Assumption 1 is to remove the trivial case that the risk constraint is redundant. This can be seen as follows. If we solve problem (2.2) with only the budget constraint, then the optimal solution is $X_T^{\mathcal{Q}} = I(\lambda_0 \xi_T)$ and λ_0 is determined by (2.3). If $E[\xi_T(K - X_T^*)^+] \leq \epsilon$, then the risk constraint is automatically satisfied, and $X_T^{\mathcal{Q}}$ is the optimal solution to problem (2.2).*

The general scheme starts by taking the budget and risk constraints into account via Lagrange multipliers $\lambda, \omega \geq 0$. Consider the following problem:

$$(2.4) \quad \max_{X_T} E[U(X_T) - \lambda(\xi_T X_T - e^{rT}x_0) - \omega(\xi_T(K - X_T)^+ - \epsilon)].$$

Under Assumption 1, the optimal solution to Problem (2.2) is given by (see also Basak and Shapiro, 2001)

$$(2.5) \quad X_T^Q = \begin{cases} I(\lambda\xi_T), & \xi_T < U'(K)/\lambda, \\ K, & \xi_T \in [U'(K)/\lambda, U'(K)/(\lambda - \omega)], \\ I((\lambda - \omega)\xi_T), & \xi_T > U'(K)/(\lambda - \omega), \end{cases}$$

where $\lambda, \omega \geq 0$. Furthermore, we can argue that $\lambda > 0$ and $\omega > 0$. If $\lambda = 0$, then $X_T^Q = I(0) = \infty$ (since $u'(0) = \infty$), which violates the budget constraints, so $\lambda > 0$. If $\omega = 0$, then $X_T^Q = I(\lambda_0\xi_T)$ and λ_0 is determined from equation (2.3). Assumption 1 implies that the loss constraint $E(\xi_T(K - X_T^Q)) \leq \epsilon$ is not satisfied, so $\omega > 0$. We can determine $\lambda, \omega > 0$ by solving the following coupled nonlinear equations:

$$(2.6) \quad \begin{cases} E(\xi_T X_T^Q) = e^{rT} x_0, \\ E(\xi_T (K - X_T^Q)^+) = \epsilon. \end{cases}$$

Since the equality in the budget constraint holds, $e^{-rT}\xi_t X_t^Q$ is a martingale under \mathcal{P} and the optimal wealth process is given by

$$(2.7) \quad X_t^Q = E\left(X_T^Q \frac{\xi_T e^{-rT}}{\xi_t e^{-rt}} \mid \mathcal{F}_t\right), 0 \leq t \leq T.$$

Let $Y_T = I(\lambda\xi_T)$ and $Y_t = E\left(Y_T \frac{\xi_T e^{-rT}}{\xi_t e^{-rt}} \mid \mathcal{F}_t\right)$ for $0 \leq t \leq T$. Then Y satisfies the SDE

$$(2.8) \quad dY_t = Y_t \left(\left(r + \frac{\theta^2}{1 - \gamma} \right) dt + \frac{\theta}{1 - \gamma} dW_t \right),$$

which is a Merton's portfolio value process, with the initial value

$$Y_0 = \lambda^{\frac{1}{\gamma-1}} \exp \left[\left(\frac{1}{2} \frac{\gamma}{(\gamma-1)^2} \theta^2 - r \right) T \right].$$

The optimal terminal wealth X_T^Q can be represented as a piecewise linear function of Y_T by

$$X_T^Q = \begin{cases} Y_T, & Y_T > K, \\ K, & K_0 \leq Y_T \leq K, \\ \frac{K}{K_0} Y_T, & Y_T < K_0, \end{cases}$$

where $K_0 = I\left(\frac{\lambda U'(K)}{\lambda - \omega}\right)$. Figure 2.1 gives the one-to-one correspondence of X_T^Q and Y_T . Note that X_T^Q can be expressed as

$$(2.9) \quad X_T^Q = Y_T + (K - Y_T)^+ - \frac{K}{K_0} (K_0 - Y_T)^+.$$

Remark 2.2 *If Merton's portfolio Y is a benchmark portfolio (see Platen & Heath, 2009) and there exist European put options on Y_T with exercise prices K and K_0 at exercise time T , then we can replicate the optimal terminal wealth X_T^Q by holding a forward contract Y_T , a long put position with exercise price K and K/K_0 units of short put positions with exercise price K_0 .*

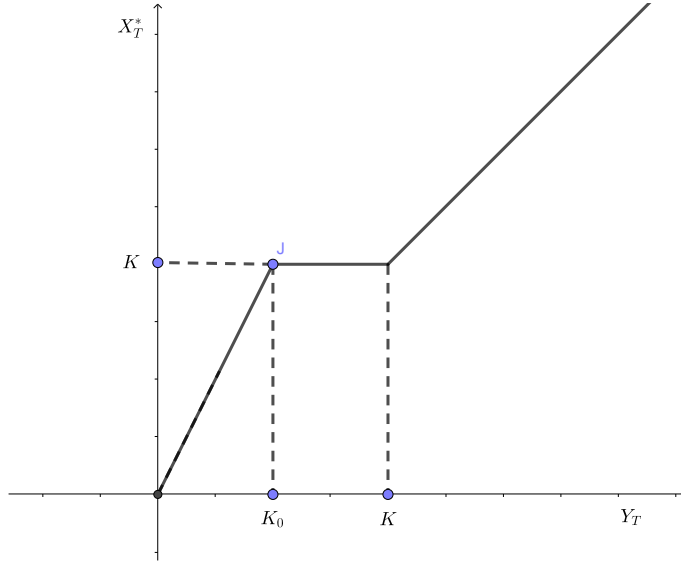


Figure 2.1: Final payoff X_T^Q

The value of the optimal wealth X_t^Q at any time $t \in [0, T]$ can easily be computed as

$$X_t^Q = Y_t + Put\left(Y_t, K, r, \frac{\theta}{1-\gamma}, T-t\right) - \frac{K}{K_0} Put\left(Y_t, K_0, r, \frac{\theta}{1-\gamma}, T-t\right),$$

where $Put(y, K, r, \sigma, \tau)$ represents the European put price with the current stock price y , strike price K , riskless interest rate r , stock volatility σ , and time to expiry τ . The optimal control X_t^Q can also be easily derived with the standard delta hedging strategy.

Now consider Problem (2.2) with the \mathcal{Q} -expected loss replaced by a \mathcal{P} -expected loss:

$$(2.10) \quad \begin{aligned} & \max_{X_T} E[U(X_T)] \\ & \text{subject to } E(\xi_T X_T) \leq e^{rT} x_0 \text{ (budget constraint),} \\ & E((K - X_T)^+) \leq \epsilon \text{ (risk constraint).} \end{aligned}$$

Again, to ensure the risk constraint is not redundant, we assume the following condition (see Remark 2.1):

Assumption 2 *The risk constraint in problem (2.10) is binding, i.e.,*

$$E[(K - I(\lambda_0 \xi_T))^+] > \epsilon,$$

where λ_0 is determined by (2.3).

Assumption 2 ensures that the risk constraint in (2.10) is not already fulfilled by the optimal solution to the unconstrained problem. From a practical point of view, consider the case of an asset manager choosing an asset strategy that violates a possible solvency rule expressed in terms of the risk constraint. The solvency rule is a VaR-based solvency rule where the asset manager has the obligation to pay K to a third party upon

termination but he is only solvent at time 0 if he can live up to his obligation with a certain probability. If the constraint is binding, this means that he has to adapt his strategy to obey the solvency rule (risk constraint), since the asset strategy he would use in a regime with no solvency rule, does not live up to the solvency rule in a regime where the solvency rule is VaR-based.

Under Assumption 2, the optimal solution to problem (2.10) is given by

$$(2.11) \quad X_T^P = \begin{cases} I(\lambda\xi_T), & \xi_T < U'(K)/\lambda, \\ K, & \xi_T \in [U'(K)/\lambda, (U'(K) + \omega)/\lambda], \\ I(\lambda\xi_T - \omega), & \xi_T > (U'(K) + \omega)/\lambda, \end{cases}$$

where $\lambda, \omega > 0$ are determined by

$$(2.12) \quad \begin{cases} E(\xi_T X_T^P) = e^{rT} x_0, \\ E((K - X_T^P)^+) = \epsilon. \end{cases}$$

The optimal wealth process $(X_t^P)_{0 \leq t \leq T}$ is given by inserting (2.11) in (2.7) and the optimal terminal wealth X_T^P can be written as

$$X_T^P = \begin{cases} Y_T, & Y_T > K, \\ K, & K_0 \leq Y_T \leq K, \\ I(U'(Y_T) - \omega), & Y_T < K_0, \end{cases}$$

where $K_0 = I(\omega + U'(K))$ and Y_T is Merton's portfolio at time T , see (2.8). Figure 2.2 gives the one-to-one correspondence of X_T^P and Y_T . In contrast to the case for the \mathcal{Q} -strategy, X_T^P cannot be replicated by simple forward and puts with the benchmark portfolio Y .

To compare the optimal solutions with \mathcal{P} - and \mathcal{Q} -risk constraint, respectively, we have the following result.

Proposition 2.1 *Let*

$$X_T^* = \arg \max \{ E[U(X_T)] : E(\xi_T X_T) \leq e^{rT} x_0, E^{\mathcal{Q}}((K - X_T)^+) \leq \epsilon \},$$

then we have $E((K - X_T^*)^+) \leq \epsilon$.

Proof: It is sufficient to prove

$$E((K - X_T^*)^+) \leq E^{\mathcal{Q}}((K - X_T^*)^+),$$

which is equivalent to

$$E((K_0 - Y_T)^+) \leq E^{\mathcal{Q}}((K_0 - Y_T)^+).$$

We have

$$E^{\mathcal{Q}}((K_0 - Y_T)^+) = e^{rT} \text{Put} \left(Y_0, K_0, r, \frac{\theta}{1 - \gamma}, T \right) = E((K_0 - \tilde{Y}_T)^+),$$

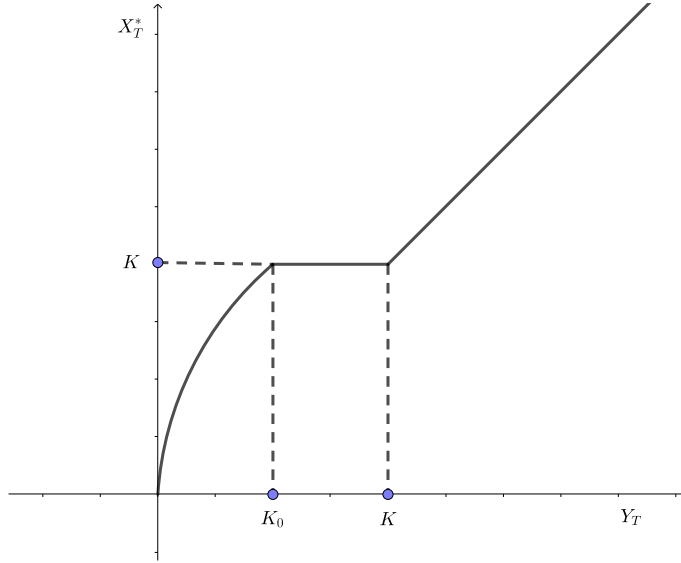


Figure 2.2: Final payoff X_T^P

where \tilde{Y} satisfies the SDE

$$d\tilde{Y}_t = \tilde{Y}_t \left(rdt + \frac{\theta}{1-\gamma} dW_t \right), \quad 0 \leq t \leq T,$$

with the initial condition $\tilde{Y}_0 = Y_0$. Since $Y_t \geq \tilde{Y}_t$ for $0 \leq t \leq T$, we have $E((K_0 - Y_T)^+) \leq E^Q((K_0 - Y_T)^+)$. \square

Proposition 2.1 says that the \mathcal{Q} -optimal solution is a \mathcal{P} -feasible solution. Therefore, the \mathcal{P} -optimal value is always greater than or equal to the \mathcal{Q} -optimal value.

3. Fulfilling the \mathcal{P} -Risk Constraint by the \mathcal{Q} -Strategy

In Section 2 we find that the \mathcal{Q} -strategy is piecewise linear in Merton's portfolio while the \mathcal{P} -strategy fails to admit this property. Hence the optimal terminal wealth of the \mathcal{Q} -strategy can easily be replicated with the standard delta hedging strategy while the \mathcal{P} -strategy cannot. This motivates us to work one step forward, that is, we aim at fulfilling the \mathcal{P} -risk constraint by utilizing the \mathcal{Q} -strategy.

There is another subtle but deep reason motivating us fulfilling the \mathcal{P} -risk constraint by the \mathcal{Q} -strategy as explained next. In solving the \mathcal{P} -risk constraint problem, we need to find Lagrange multipliers λ, ω which are positive solutions of Equation (2.12). The existence and uniqueness of the solution are not clear upfront. Bielecki et al. (2005) discuss a mean-variance problem with a nonnegative terminal wealth constraint. They show the Lagrange multipliers λ, ω for their problem must satisfy

$$(3.1) \quad E[(\lambda - \omega \xi_T)^+] = z \text{ and } E[(\lambda - \omega \xi_T)^+ \xi_T] = x_0$$

for some z and x_0 (see (5.1) in Bielecki et al. (2005)). To solve Equation (3.1), they define a new variable $\eta = \lambda/\omega$, divide the two equations in (3.1) to get

$$(3.2) \quad \frac{E[(\eta - \xi_T)^+ \xi_T]}{E[(\eta - \xi_T)^+]} = \frac{x_0}{z},$$

and then show that there exists a unique solution η^* to (3.2). The relation $\lambda = \eta^* \omega$ helps to find the Lagrange multipliers λ and ω . The success of that approach in reducing dimensionality crucially depends on the special structure of Equation (3.1). For our problem, with a power utility objective, Equation (2.12) is much more complicated and cannot be reduced into an equation with a scalar variable as in Bielecki et al. (2005) considering a mean-variance objective, so the existence and uniqueness of Lagrange multipliers λ , ω for our \mathcal{P} -risk constraint problem are not clear. We will now show that this is guaranteed if we use a \mathcal{Q} -strategy to fulfill the \mathcal{P} -risk constraint.

We work on the following sub-optimization problem (SOP),

$$(3.3) \quad \begin{aligned} & \max_{Y_0 \geq 0, 0 \leq K_0 \leq K} E[U(\bar{X}_T)] \\ & \text{subject to } E(\xi_T \bar{X}_T) = e^{rT} x_0 \text{ (budget constraint),} \\ & E((K - \bar{X}_T)^+) \leq \epsilon \text{ (risk constraint),} \end{aligned}$$

where

$$\bar{X}_T = \begin{cases} Y_T, & Y_T > K, \\ K, & K_0 \leq Y_T \leq K, \\ \frac{K}{K_0} Y_T, & Y_T < K_0, \end{cases}$$

and Y_T is the value of a Merton's portfolio at time T with Y satisfying the SDE (2.8).

This sub-optimization problem is also much easier to implement in practical asset management. This can be seen from the structure of \bar{X}_T and Remark 2.2. This sub-optimization problem takes the structure of the \mathcal{Q} -strategy by formalizing that strategy in terms of the pieces and slopes, via Y_0 and K_0 , of the claim one is to optimally hedge under the \mathcal{Q} -constraint. Y_0 defines the unit of the 1. axis in describing the optimal position as a contingent claim on Y_T . The flat piece goes from K_0 to K and that implicitly defines the slope for $Y < K_0$ to be K/K_0 . Thus the optimization problem is reduced to a static problem of optimizing the parameters of the claim inherited from the \mathcal{Q} -constraint. Doing that optimization under a \mathcal{P} -constraint in (3.3) is the very rationale of this section. This is a sub-optimization in the sense that we look for the optimum in a limited class of claims and we know from before that the optimal claim does not belong to that class. However, within that limited class we actually perform a true optimization. The purpose of the following Theorem 3.1 is now to show that this sub-optimization problem has a unique solution and to describe a tractable algorithm for finding that solution. The proof of Theorem 3.1 can be found in the Appendix.

Theorem 3.1 *Assume that the SOP admits a feasible solution. If the equation*

$$(3.4) \quad \epsilon = K \exp \left(\left(r + \frac{\theta^2}{1-\gamma} \right) T \right) Put \left(y, 1, r + \frac{\theta^2}{1-\gamma}, \frac{\theta}{1-\gamma}, T \right)$$

admits a nonnegative solution y , then let $C \geq 0$ be its unique solution; otherwise, let $C = 0$. We have the following assertions:

- *There exists a unique solution L^* to the equation*

$$(3.5) \quad \frac{x_0}{K} = L + Put \left(L, 1, r, \frac{\theta}{1-\gamma}, T \right) - Put \left(L_0, 1, r, \frac{\theta}{1-\gamma}, T \right)$$

with $L_0 = L_0^ := \max\{C, x_0/K\}$ and $0 \leq L^* \leq x_0/K$.*

- *The optimal solution to the SOP is given by*

$$Y_0^* = L^* K \text{ and } K_0^* = \frac{Y_0^*}{L_0^*}.$$

Remark 3.1 *If $\epsilon < K$ then there exists a unique $C > 0$ to equation (3.4) and if $\epsilon \geq K$ (the risk constraint is redundant) then $C = 0$. A sufficient condition for the existence of a feasible solution to the SOP is $x_0 \geq K e^{-rT}$, which makes $g(0) = -x_0/K + e^{-rT} - G(L_0^*) \leq 0$ and, together with $g(x_0/K) \geq 0$, there exists a solution $L^* \in [0, L_0^*]$ to the equation $g(L) = 0$. If we consider the SOP without a risk constraint, the solution is attained when $(L^*, L_0^*) = (\frac{x_0}{K}, \frac{x_0}{K})$, i.e., $K_0 = K$, $Y_0 = x_0$, which indeed gives Merton's portfolio.*

When considering the \mathcal{Q} - or \mathcal{P} -expected loss constrained problem, to obtain the optimal solution (2.5) or (2.11), we need to find parameters λ, ω (Lagrange multipliers), which are solutions of the two coupled nonlinear equations in (2.6) or (2.12), and solving them can be computationally expensive. In contrast, a \mathcal{Q} -strategy fulfilling the \mathcal{P} -risk constraint can be computationally efficient due to the linear structure of the optimal solution (2.5) and the reduced requirement of solving two decoupled one variable equations (3.4) and (3.5) only.

We now do a numerical test to investigate the difference between the two strategies. For fixed ϵ and K , we develop a pricing engine for the optimal terminal wealth of the \mathcal{P} -expected loss $P(x_0, \epsilon, K) = E(U(X_T^*))$ and that of the \mathcal{Q} -strategy fulfilling the \mathcal{P} -risk constraint $\bar{P}(\bar{x}_0, \epsilon, K) = E(U(\bar{X}_T^*))$. We are interested in finding the implicit relationship between x_0 and \bar{x}_0 if the same final utility is required, i.e., $P(x_0, \epsilon, K) = \bar{P}(\bar{x}_0, \epsilon, K)$. Fixing an initial wealth x_0 , we aim to compute \bar{x}_0 such that $P(x_0, \epsilon, K) = \bar{P}(\bar{x}_0, \epsilon, K)$. If ϵ is large enough, that is, the two constraints are not binding, see Assumptions 1 and 2, we know that $\bar{x}_0 = x_0$. If ϵ is smaller than a particular number $\epsilon_0 = K \exp \left(\left(r + \frac{\theta^2}{1-\gamma} \right) T \right) Put \left(\frac{x_0}{K}, 1, r + \frac{\theta^2}{1-\gamma}, \frac{\theta}{1-\gamma}, T \right)$, the two constraints are binding, as the \mathcal{Q} -strategy fulfilling the \mathcal{P} -risk constraint is a suboptimal solution to Problem

(2.10), we have $\bar{x}_0 > x_0$. We can find \bar{x}_0 with the bisection method with linear convergence.

We conduct numerical tests to find the pricing engines P and \bar{P} . To find P , we do 100,000 simulations for computing the expectations of the nonlinear dependence of Merton's portfolio in (2.12) and apply the iterated bisection method to solve two coupled nonlinear equations, which takes average computation time 2.5×10^4 seconds by MATLAB on a computer with an Intel 2.80 GHz CPU. To find \bar{P} , we apply the bisection method to solve two decoupled one variable equations (3.4) and (3.5), respectively, which takes average computation time 0.016 seconds. So computing under \bar{P} is more than 1.5×10^6 times faster than under P .

We set $\mu = 0.08, \sigma = 0.2, \gamma = 0.5, r = 0.04, K = 1, T = 1$ and $x_0 = 1$. Under this set of parameter, $\epsilon_0 = 7.52 \times 10^{-2}$ and for those ϵ is greater than ϵ_0 , diff value is zero. With varying ϵ , we present the value for $\text{diff} = \bar{x}_0 - x_0$ in Table 3.1. With a fixed $\epsilon = 0.001$, and varying γ and T , we report the diff value in Table 3.2. From the tables, we conclude that the diff value is reasonably small.

Table 3.1: Diff value with varying ϵ .

$\epsilon (\times 10^{-3})$	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
diff ($\times 10^{-3}$)	1.5	1.2	4.6	1.7	0.8	0.1	2.8	0.5	0.0	0.3	0.15

Table 3.2: Diff value with varying γ and T .

γ	0.3	0.5	0.8
T	diff ($\times 10^{-3}$)		
1	2.1	1.5	0.8
5	1.7	0.9	0.0
10	1.4	0.5	0.0

We have here numerically compared two strategies: The first exercise was the, complicated and slow to implement and complicated to understand, optimal strategy fulfilling the \mathcal{P} -constraint. The second exercise, simple and fast to implement and simple to understand, is that of fulfilling the \mathcal{P} -constraint via the structure of the solution to the \mathcal{Q} -constraint strategy. The latter is obviously performing inferior to the first. The tables show how much in terms of certainty equivalence. The quantity diff calculated how much extra initial capital is needed to obtain the same utility in the sub-optimization problems compared to the optimization problem. The tables show that, varying over the level of the constraint, epsilon, the risk aversion, and the time horizon, this extra capital is of the

order of magnitude 0-2 permille of the initial capital. For all practical purposes, this is a very small certainty equivalence. Taking model and risk aversion uncertainty in real life into account, we feel safe to say that the “price” of sub-optimizing instead of optimizing in terms of capital is almost zero, whereas the gain in terms of implementability and understanding is substantial.

4. Conclusions

By relating the optimal terminal wealth with Merton’s portfolio, the optimal terminal wealth under the \mathcal{Q} -expected loss constraint can be represented as a piecewise linear function of Merton’s portfolio, while the \mathcal{P} -optimal terminal wealth fails to admit this property. We consider the \mathcal{Q} -strategy fulfilling the \mathcal{P} -risk constraint and identify two major advantages: First, we have an efficient algorithm with linear convergence to find its solution, and second, we can replicate the optimal terminal wealth with the standard delta hedging strategy due to its piecewise linear dependence of Merton’s portfolio. We show in numerical examples that the approximation error is reasonably small. There remain many open questions. For example, do we still have a similar conclusion if we replace the GBM asset price process and power utility by general Lévy process and utility? What would be the impact on practical portfolio optimization and empirical analysis? We leave these and other questions for our future research.

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5. Appendix

5.1 Proof to Theorem 3.1

Assume \bar{X}_T is any feasible solution. We can rewrite $\bar{X}_T = Y_T + (K - Y_T)^+ - \frac{K}{K_0}(K_0 - Y_T)^+$. Combining this and the budget constraint $x_0 = e^{-rT} E(\xi_T \bar{X}_T)$, we have

$$x_0 = Y_0 + Put \left(Y_0, K, r, \frac{\theta}{1-\gamma}, T \right) - \frac{K}{K_0} Put \left(Y_0, K_0, r, \frac{\theta}{1-\gamma}, T \right).$$

Dividing both sides by K and denoting $L = \frac{Y_0}{K}$ and $L_0 = \frac{Y_0}{K_0}$, we have (3.5).

Noting that $0 \leq L \leq L_0$ and that the put price is a decreasing function of the current stock price, we have

$$Put \left(L, 1, r, \frac{\theta}{1-\gamma}, T \right) \geq Put \left(L_0, 1, r, \frac{\theta}{1-\gamma}, T \right),$$

which gives

$$(5.1) \quad 0 \leq L \leq \frac{x_0}{K}.$$

Equation (3.5) gives an implicit one-to-one correspondence between L_0 and L , and we may write

$$L_0 = f(L).$$

Denote by

$$(5.2) \quad G(y) := Put \left(y, 1, r, \frac{\theta}{1-\gamma}, T \right).$$

Equation (3.5) can be rewritten as

$$(5.3) \quad \frac{x_0}{K} = L + G(L) - G(L_0).$$

Taking derivatives with respect to L , bearing in mind L_0 is a function of L , we have

$$(5.4) \quad 0 = 1 + G'(L) - G'(L_0)f'(L).$$

Since the Delta of a put ranges from -1 to 0 , we have $G'(L), G'(L_0) \in (-1, 0)$, which implies from (5.4) that

$$(5.5) \quad f'(L) < 0.$$

If we set $L = \frac{x_0}{K}$ in equation (3.5), then we must have $L_0 = L = \frac{x_0}{K}$. In other words, $\frac{x_0}{K}$ is a fixed point of function f . Combining (5.1) and (5.5), we conclude that

$$(5.6) \quad L_0 = f(L) \geq f\left(\frac{x_0}{K}\right) = \frac{x_0}{K}.$$

Note that

$$\begin{aligned} E((K - \bar{X}_T)^+) &= KE\left(\left(1 - \frac{Y_T}{K_0}\right)^+\right) \\ &= K \exp\left(\left(r + \frac{\theta^2}{1-\gamma}\right)T\right) Put\left(L_0, 1, r + \frac{\theta^2}{1-\gamma}, \frac{\theta}{1-\gamma}, T\right), \end{aligned}$$

which is strictly decreasing in L_0 . If the budget constraint is not binding, then equation (3.4) does not admit a solution, and C is set to be 0 ; otherwise letting C be the unique solution to (3.4), we have that $E((K - \bar{X}_T)^+) \leq \epsilon$ is equivalent to $L_0 \geq C$. With the requirement that $L_0 \geq x_0/K$, we conclude $L_0 \geq L_0^* := \max\{C, x_0/K\}$.

Next we prove that there exists a unique solution $L^* \in [0, x_0/K]$ to equation (3.5) with $L_0 = L_0^*$. Since the SOP is assumed to have a feasible solution, there exist $L' \in [0, x_0/K]$ and $L'_0 \geq L_0^*$ such that

$$\frac{x_0}{K} = L' + G(L') - G(L'_0).$$

Define

$$g(L) := -\frac{x_0}{K} + L + G(L) - G(L_0^*).$$

Then g is a strictly increasing continuous function. Since G is a strictly decreasing function and $L_0^* \geq x_0/K$ and $L'_0 \geq L_0^*$, we have

$$g\left(\frac{x_0}{K}\right) = G\left(\frac{x_0}{K}\right) - G(L_0^*) \geq 0$$

and

$$g(L') = -\frac{x_0}{K} + L' + G(L') - G(L_0^*) = G(L'_0) - G(L_0^*) \leq 0.$$

The Intermediate Value Theorem tells us that there exists a unique $L^* \in [L', x_0/K] \subset [0, x_0/K]$ such that $g(L^*) = 0$. The first assertion is proved.

The proof of the second assertion is as follows. We first compute $E(U(\bar{X}_T))$. Since

$$U(\bar{X}_T) = \frac{K^\gamma}{\gamma} \left(\left(\frac{Y_T}{K} \right)^\gamma + \left(1 - \left(\frac{Y_T}{K} \right)^\gamma \right)^+ - \left(1 - \left(\frac{Y_T}{K_0} \right)^\gamma \right)^+ \right)$$

and

$$dY_t^\gamma = Y_t^\gamma \left[\gamma \left(r + \frac{1}{2} \frac{\theta^2}{1-\gamma} \right) dt + \frac{\theta\gamma}{1-\gamma} dW_t \right],$$

we have

$$\begin{aligned} E(U(\bar{X}_T)) &= \frac{\tilde{c}}{\gamma} \left[\left(\frac{Y_0}{K} \right)^\gamma + Put \left(\left(\frac{Y_0}{K} \right)^\gamma, 1, \gamma \left(r + \frac{1}{2} \frac{\theta^2}{1-\gamma} \right), \frac{\theta\gamma}{1-\gamma}, T \right) \right. \\ &\quad \left. - Put \left(\left(\frac{Y_0}{K_0} \right)^\gamma, 1, \gamma \left(r + \frac{1}{2} \frac{\theta^2}{1-\gamma} \right), \frac{\theta\gamma}{1-\gamma}, T \right) \right] \\ &= \frac{\tilde{c}}{\gamma} \left[L^\gamma + Put \left(L^\gamma, 1, \gamma \left(r + \frac{1}{2} \frac{\theta^2}{1-\gamma} \right), \frac{\theta\gamma}{1-\gamma}, T \right) \right. \\ &\quad \left. - Put \left(L_0^\gamma, 1, \gamma \left(r + \frac{1}{2} \frac{\theta^2}{1-\gamma} \right), \frac{\theta\gamma}{1-\gamma}, T \right) \right], \end{aligned}$$

where $\tilde{c} = K^\gamma \exp \left(\left(\gamma r + \frac{1}{2} \frac{\gamma}{1-\gamma} \theta^2 \right) T \right)$. Denote by

$$(5.7) \quad H(y) := Put \left(y^\gamma, 1, \gamma \left(r + \frac{1}{2} \frac{\theta^2}{1-\gamma} \right), \frac{\theta\gamma}{1-\gamma}, T \right).$$

Taking derivatives of G and H and doing some calculations, we have

$$(5.8) \quad H'(y) = G'(y) \gamma y^{\gamma-1}.$$

$E(U(\bar{X}_T))$ can also be rewritten as

$$E(U(\bar{X}_T)) = \frac{\tilde{c}}{\gamma} [L^\gamma + H(L) - H(L_0)].$$

Taking derivatives with respect to L , we obtain

$$\begin{aligned} \frac{\partial E(U(\bar{X}_T))}{\partial L} &= \frac{\tilde{c}}{\gamma} [\gamma L^{\gamma-1} + H'(L) - H'(L_0) f'(L)] \\ &= \frac{\tilde{c}}{\gamma} [\gamma L^{\gamma-1} + G'(L) \gamma L^{\gamma-1} - G'(L_0) f'(L) \gamma L_0^{\gamma-1}] \\ (5.9) \quad &= \tilde{c} [L^{\gamma-1} + G'(L) L^{\gamma-1} - (1 + G'(L)) L_0^{\gamma-1}] \\ &= \tilde{c} (L^{\gamma-1} - L_0^{\gamma-1}) (1 + G'(L)) \\ &\geq 0. \end{aligned}$$

Here the second equality holds due to (5.8), the third equality due to (5.4) and the last inequality due to $\tilde{c} > 0$, $G'(L) \in (-1, 0)$, $L \leq L_0$ and $\gamma < 1$. Combining (5.5) with (5.9), we also obtain

$$\frac{\partial E(U(\bar{X}_T))}{\partial L_0} \leq 0,$$

which implies that we need to choose L_0 as small as possible. Since $L_0 \geq L_0^*$, the smallest L_0 we may choose, is L_0^* . The maximum of $E(U(\bar{X}_T))$ is therefore attained at $L_0 = L_0^*$. When $C < \frac{x_0}{K}$, the solution to the SOP and the \mathcal{P} -expected loss constraint problem coincide, i.e., the risk constraint is not active. When $C \geq \frac{x_0}{K}$, the solution to the SOP attains the maximum when the risk constraint is binding.

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