Risk and Decision Analysis 00 (20xx) 1–13
DOI 10.3233/RDA-170131
IOS Press

On infectious model for dependent defaults

Wai-Ki Ching [a,b,c], Jia-Wen Gu [d,*], Xiaoyue Li [e], Tak-Kuen Siu [f] and Harry Zheng [g]

[a] *Advanced Modeling and Applied Computing Laboratory, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong*
E-mail: wching@hku.hk
[b] *Hughes Hall, University of Cambridge, Wollaston Road, Cambridge, UK*
[c] *School of Economics and Management, Beijing University of Chemical Technology, North Third Ring Road, Beijing, China*
[d] *Department of Mathematical Sciences, Southern University of Science and Technology, Shenzhen, China*
E-mail: jwgu.hku@gmail.com
[e] *Department of Physics, National University of Defense Technology, Changsha, China*
E-mail: lixiaoyuemxy@163.com
[f] *Department of Actuarial Studies, and Center for Financial Risk, Faculty of Business and Economics, Macquarie University, Sydney, NSW 2109, Australia*
E-mail: ken.siu@mq.edu.au
[g] *Department of Mathematics, Imperial College, London, SW7 2AZ, UK*
E-mail: h.zheng@imperial.ac.uk

Abstract. In this paper, we propose a general framework for modeling discrete-time default risk where default processes for all the entities are governed by predictable interacting default probabilities. We give a general formula for the joint distribution of two important random variables featuring the severity of the crisis: duration of a crisis (T) and severity of the defaults (W_T). In particular, we present a two-sector Markovian infectious model, where the default probability is switching over time and depends on the current number of defaults of both sectors. The central idea of this model is that the causality of defaults of two sectors is in both directions, which enrich dynamics of the dependent default risk. The Bayesian Information Criterion (BIC) is adopted to compare the proposed model with the two-sector model in credit literature using real data. Numerical experiments are given to demonstrate that our proposed model is statistically better.

Keywords: Infectious models, correlated defaults, crisis duration

1. Introduction

Modeling dependent default risk has been a key issue in credit risk modeling. There are two important approaches to model the dependent default risk. The structural firm model has its origin in Merton [19] and Black and Scholes [3], which models the relationship between the firm's asset value and the defaults. The reduced-form intensity-based, proposed by Jarrow and Turnbull [14], employ the Poisson jump processes to model the default event.

Copula function has been a very popular tool for modeling dependent risk. The idea of Copula is to transform the marginal variables to uniform variables by a simple transformation. After this is done, a n-dimensional function is used to model the dependence of the uniform variables, which is so called a Copula function. The Copula function enables one to deal with a multivariate distribution of uniform variables, without consideration of the original marginal variables. There are many useful Copula functions in finance, e.g. the Gaussian Copula, introduced by Li [17], is widely used in risk modeling and financial assessment.

In addition, conditional independence model is also a commonly used model in credit risk modeling. Conditional on the systematical common factor, the loss random variables are independent. For example, the Bernoulli mixture model is adopted by *CreditMetrics* and *KMV*-model, while the Poisson mixture model is adopted by *CreditRisk*$^+$. In a recession, the default of a company is triggered by the underlying common risk factor and also by the related company's defaults. The

*Corresponding author. E-mail: jwgu.hku@gmail.com.

contagion model is used to describe how the credit event of one company affects the other companies. Davis and Lo [9] introduce an infectious default model, where in a portfolio, a bond may be infected by defaults of other bonds or default directly. Jarrow and Yu [15] propose a reduced-form model to describe the defaultable bonds of different company, where the concept of counterparty risk is first introduced to the credit literature. Dong and Wang [10] show the impact of dependent jumps of the firm value and the default threshold on the default probabilities.

Ching et al. [6] introduce an infectious default model based on the idea of Greenwood's model considered in Daley and Gani [8]. This model aims at modeling the impact of default of a bond on the likelihood of defaults of other bonds. The original version of Greenwood's model is a one-sector model. It is then extended to a two-sector model in Ching et al. [5]. Besides, the joint probability distribution function for the duration of a default crisis, (i.e., the default cycle), and the severity of defaults during the crisis period was also derived. Two concepts, namely, Crisis Value-at-Risk (CRVaR) and Crisis Expected Shortfall (CRES), are also introduced and applied to assess the impact of a default crisis. The Greenwood's model is also extended to a network of sectors in [5]. Gu et al. [12] propose a Markovian infectious model to describe the dependent relationship of default processes of credit securities based on [5,6], where the central idea is the concept of common shocks which is one of the major approaches to describe insurance risk. In recent years, Markov model is widely used in credit risk assessment. Although the Markov model does not use all the historical data, it can be seen from the literature [1,2,16,18] that it gives substantially good results. For example, in the literature [2], they consider a bottom-up Markovian copula model of portfolio credit risk.

If the number of defaults is small, other models in [20,21] and the theory in the book [22] can be applied. In literature, Mitra [21] proposed a new risk management framework and method which allows one to assess the risk of pension funds in terms of their value and provides a risk management framework for decision-making. It was proposed to modeling and managing pensions as European call options. If the correlation of default changes over time, one can refer to the method in [13]. They established a link between the dynamics of house price changes and the dynamics of default rates in the Gaussian copula framework by specifying a time series model for a common risk factor.

In this paper, we propose a general framework for modeling discrete-time default risk where default processes for all the entities are governed by predictable default probabilities. Existing literature [5,6,12] serve as our special cases. We give a general formula for the joint distribution of two important random variables featuring the severity of the crisis, i.e., the duration of the crisis and the severity of the defaults. In particular, we present a two-sector Markovian infectious model, where the default probability is switching over time and depends on the current number of defaults of both sectors. This model is a special case of our general framework and compared with the existing work, this can capture the causality of defaults from both direction. We adopt the maximum likelihood method to estimate the parameters and the Bayesian Information Criterion (BIC) to compare the proposed model with two-sector model considered in Ching et al. [6]. Experimental results show that our proposed model is statistically better[1] (i.e., has a lower value of the BIC).

In this paper, the default is modeled as an absorbing state. There are many research works that regard default as an absorbing state [4,7,23]. For example, they employ Copula theory to model the dependence across default rates in a credit card portfolio of a large UK bank and to estimate the likelihood of joint high default rates in [7]. And in [23], they focus on the predictability of sovereign debt crisis and propose a two-step procedure centered on the idea of a multidimensional distance-to-collapse point.

The remainder of the paper is structured as follows. Section 2 presents our general model framework. We derive a general formula for the joint probability distribution for the default cycle and the number of defaults during the crisis. We also discuss the limiting case. Section 3, we present a special case of our general model, namely the two-sector Markovian model and derive a recursive formula for the probability law of the two variables. We also outline the parameter estimation procedure. Section 4 presents the ideas of the CRVaR and the CRES. In Section 5, we present the results of empirical analysis using our proposed model. Finally, Section 6 concludes the paper.

2. The general model framework

Let $\mathcal{T}$ be the time index set $\{1, 2, \ldots\}$ of our model. To model the uncertainty, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a

[1] "Statistically better" means our purposed model has lower Bayesian information criterion value.

complete filtered probability space, where P is a real-world probability and $\{\mathcal{F}_t\}_{t\geqslant 0}$ is a filtration satisfying the usual conditions (the right-continuity and P-completeness). We consider n credit entities, where each entity may default and the entity will stay at the default state once it happens. For each $i = 1, 2, \ldots, n$, let τ_i be the default time of name i, which is a stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Write $N_i(t) = 1_{\{\tau_i \leqslant t\}}$ the default indicator process and $\{\mathcal{F}_t^i\}_{t\geqslant 0}$ is the P-complete, natural filtration generated by N_i. For each $t \geqslant 0$, we write

$$\mathcal{F}_t = \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^n, \tag{1}$$

where $\mathcal{F}_t$ is the minimal σ-algebra containing information about the processes $\{N_i\}_{i=1}^n$ up to and including time t. That is, $\mathcal{F}_t$ contains information about the common factor process and the defaults of the n credit entities up to time t. It represents the observed market information up to time t.

We assume that for each $i = 1, 2, \ldots, n$, N_i possesses a nonnegative, $\{\mathcal{F}_t\}_{t\geqslant 0}$-predictable process p_i[2] satisfying

$$E\big[N_i(t) \mid \mathcal{F}_{t-1}\big] = p_i(t), \quad t \geqslant 0.$$

To determine the impact of a default crisis, we define the duration of the default crisis (T), namely, the default cycle, and the severity of the defaults (W_T) during the crisis period. We give a precise definition of the default cycle as a stopping time:

$$T := \inf\{t \in \mathcal{T} \mid W_t = W_{t-1}\}, \tag{2}$$

where W_t represents the number of defaults over the time duration $[1, t]$.

We let

$$\mathcal{I}(t) = \big(N_1(t), N_2(t), \ldots, N_n(t)\big).$$

It can be verified that $\mathcal{I}(t)$ is a Markov chain with state space $\mathbb{S}$ of size 2^n. We let $Q(t)$ denote the transition matrix of Markov chain $\mathcal{I}$ from time t to $t+1$ and $Q^*(t)$ the matrix that results from replacing the diagonal entries by 0 in $Q(t)$.

Proposition 1. *The joint distribution of* (T, W_T) *is given by*

$$P\big((T, W_T) = (t, w)\big) = \sum_{\boldsymbol{x}\in\mathbb{S}, \|\boldsymbol{x}\|=w} \bar{Q}(t-2)(\boldsymbol{0}, \boldsymbol{x}) \cdot Q(t-1)(\boldsymbol{x}, \boldsymbol{x})$$

for $t \in \mathcal{T}$, $w \in \mathbb{N}$, *where*

$$\bar{Q}(t-2) = \prod_{s=0}^{t-2} Q^*(s),$$

$\|\boldsymbol{x}\| = \boldsymbol{x}^T\boldsymbol{x}$ *and* $\boldsymbol{0} = (0, \ldots, 0)$.

The main idea of the proof is to sum up all the possible paths of the chain to stop at time t with w defaults. However, the computation cost can be huge when n becomes large as the matrix size grows very quickly. In Section 3, we shall consider a special case of practical value where the default probability of each name is time-homogeneous and is assigned by some rules.

In what follows, we consider the simplest case that the default probability for each name is a constant, i.e., $p_i(t) = p \in (0, 1)$. The process W_t then becomes a Markov chain, with transition probability matrix P where $P(i, j) = 0$ if $i > j$ and

$$P(i, j) = \binom{n-i}{j-i} p^{j-i}(1-p)^{n-j}, \quad \text{if } i \leqslant j.$$

We let P^* denote the matrix that results from replacing the diagonal entries by 0 of P. We can obtain the probability law of (T, W_T) by summing up all the possible paths for the chain to stop at time t with w defaults.

Proposition 2. *The joint distribution of* (T, W_T) *is given by*

$$P\big((T, W_T) = (t, w)\big) = \bar{P}(0, w) \cdot P(w, w)$$

for $t \in \mathcal{T}$, $w \in \mathbb{N}$, *where* $\bar{P} = (P^*)^{t-1}$.

3. The two-sector model

In this section, we assume all the names are divided into two sectors, namely Sector A and Sector B. To apply the concepts of default cycle and the severity of the defaults to our proposed two-sector model, we write $W_{t_1}^1$ and $W_{t_2}^2$ to represent the number of defaults in Sector A and Sector B, respectively, in $(0, t_1]$ and $(0, t_2]$. We denote

[2] For a $\{\mathcal{F}_t\}_{t\geqslant 0}$-predictable process p_i, we have $p_i(t)$ is $\mathcal{F}_{t-1}$-measurable.

$$T_1 := \inf\{t \in \mathcal{T} \mid W_t^1 = W_{t-1}^1\} \quad \text{and}$$
$$T_2 := \inf\{t \in \mathcal{T} \mid W_t^2 = W_{t-1}^2\}.$$

To model the default probability, we define

$$X := \{X_t\}_{t\in\mathcal{T}} \quad \text{and} \quad Y := \{Y_t\}_{t\in\mathcal{T}}$$

to denote two stochastic processes on $(\Omega, \mathcal{F}, P)$, where $X_t = (X_t^1, X_t^2)$ represent the numbers of surviving bonds at $t \in T$ in Sector A and Sector B, respectively, while $Y_t = (Y_t^1, Y_t^2)$ represent the numbers of defaulted bonds at $t \in T$ in Sector A and Sector B, respectively, e.g., $Y_t^i = W_t^i$, $i = 1, 2$. We assume that the initial conditions are given as follow:

$$X_0 = (x_0^1, x_0^2), \qquad Y_0 = (y_0^1, y_0^2)$$

and

$$x_0^1 + y_0^1 = n_1, \qquad x_0^2 + y_0^2 = n_2,$$

where n_1, n_2 represent the number of names in Sector A and Sector B, respectively. Note that for each $t \in \mathcal{T}$, the sum of the numbers of the defaulted bonds and the surviving bonds at the time epoch $t+1$ must equal the number of surviving bonds at time t in every sector, i.e.,

$$\begin{aligned} &X_{t+1}^1 + Y_{t+1}^1 = X_t^1 \quad \text{and} \\ &X_{t+1}^2 + Y_{t+1}^2 = X_t^2. \end{aligned} \tag{3}$$

For each $t \in \mathcal{T}$, let α_t and β_t be the probabilities that the default of a surviving bond is infected by the defaulted bonds at time t in Sector A and Sector B, respectively. The joint probability distribution of $\{X_{t+1}, Y_{t+1}\}$ given $\{X_t, Y_t\}$ is given by the following Binomial probability:

$$\begin{aligned} &p_{(x_t,y_t)}(x_{t+1}, y_{t+1}) \\ &\quad = P\{(X_{t+1}, Y_{t+1}) = (x_{t+1}, y_{t+1}) \mid \\ &\qquad (X_t, Y_t) = (x_t, y_t)\} \\ &\quad = \begin{pmatrix} x_t^1 \\ y_{t+1}^1 \end{pmatrix} (\alpha_t)^{y_{t+1}^1} (1-\alpha_t)^{x_{t+1}^1} \\ &\qquad \times \begin{pmatrix} x_t^2 \\ y_{t+1}^2 \end{pmatrix} (\beta_t)^{y_{t+1}^2} (1-\beta_t)^{x_{t+1}^2}. \end{aligned} \tag{4}$$

We consider here the situation that the joint future default probability depends on the current number of defaulted bonds in both industrial sectors. We assume that

$$\begin{aligned} \alpha_t &= a(y_t) \\ &= \begin{cases} a_0 & \text{if } y_t^1 = y_t^2 = 0, \\ a_1 & \text{if } y_t^1 > 0, y_t^2 = 0, \\ a_2 & \text{if } y_t^1 = 0, y_t^2 > 0, \\ a_3 & \text{if } y_t^1 > 0, y_t^2 > 0 \end{cases} \\ &= a_0 h_0(y_t^1, y_t^2) + a_1 h_1(y_t^1, y_t^2) \\ &\quad + a_2 h_2(y_t^1, y_t^2) + a_3 h_3(y_t^1, y_t^2) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \beta_t &= b(y_t) \\ &= \begin{cases} b_0 & \text{if } y_t^1 = y_t^2 = 0, \\ b_1 & \text{if } y_t^1 = 0, y_t^2 > 0, \\ b_2 & \text{if } y_t^1 > 0, y_t^2 = 0, \\ b_3 & \text{if } y_t^1 > 0, y_t^2 > 0 \end{cases} \\ &= b_0 h_0(y_t^2, y_t^1) + b_1 h_1(y_t^2, y_t^1) \\ &\quad + b_2 h_2(y_t^2, y_t^1) + b_3 h_3(y_t^2, y_t^1), \end{aligned} \tag{6}$$

where

$$h_0(x, y) = \begin{cases} 1 & \text{if } x = y = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$h_1(x, y) = \begin{cases} 1 & \text{if } x > 0, y = 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$h_2(x, y) = \begin{cases} 1 & \text{if } x = 0, y > 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$h_3(x, y) = \begin{cases} 1 & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise}. \end{cases}$$

As it is shown in Eq. (3) and Eq. (4), one can see that $\{X_t, t = 0, 1, 2, \ldots\}$ is a second-order Markov chain process. We remark that this two-sector model provides a novel and flexible dependent structure for correlated defaults of two different industrial sectors. First, an infectious default within one time period is modeled by a Binomial distribution, which has been widely used in modeling the spread of epidemics whose situation seems similar to that of a financial crisis. The causality of the infection is supposed to be in both direction, i.e., a “looping default”. Second, the process (X_t, Y_t) has the Markov property, where the probabilistic structure of future states only depends on the current state. Third, conditioning on the current state (X_t, Y_t), the future state of two sectors (X^1_{t+1}, Y^1_{t+1}) and (X^2_{t+1}, Y^2_{t+1}) are stochastically independent. The step functions $h_i(x, y)$ are used to describe the dependence of the default probabilities on the state of previous time epoch. This method provides a tractable and analytic solution for parameter estimation from empirical data.

3.1. Default cycle and severity

In this subsection, we proceed to derive the joint probability distribution function for the duration of the default crisis (T), namely, the default cycle, and the severity of the defaults (W_T) during the crisis period. These two concepts are essential in determining the impact of a default crisis [6]. Under the two-sector Markovian model, we obtain

$$T_1 := \inf\{t \in \mathcal{T} \mid Y^1_t = 0\} \quad \text{and}$$
$$T_2 := \inf\{t \in \mathcal{T} \mid Y^2_t = 0\}.$$

To obtain the joint distribution of $(W^i_{T_i}, T_i)$ for $i = 1, 2$, we assume that $(X_0, Y_0) = (x_0, y_0)$ with $y^1_0 > 0$, $y^2_0 > 0$. Let

$$P_n(x_1, x_2, h) = P\{T_1 \geqslant n+1, X^1_n = x_1, X^2_n = x_2, I_{\{Y^2_n > 0\}} = h\}.$$

The following lemma gives a recursive formulas for $P_n(x_1, x_2, h)$ and the proof can be found in the Appendix.

Lemma 1.

$$\begin{aligned} &P_n(x_1, x_2, 0) \\ &= \sum_{s_1 > x_1} \binom{s_1}{x_1} \big[P_{n-1}(s_1, x_2, 0) \\ &\quad \times (a_1)^{s_1 - x_1}(1-a_1)^{x_1}(1-b_2)^{x_2} \\ &\quad + P_{n-1}(s_1, x_2, 1) \\ &\quad \times (a_3)^{s_1 - x_1}(1-a_3)^{x_1}(1-b_3)^{x_2}\big] \\ &\quad \times P_n(x_1, x_2, 1) \\ &= \sum_{s_1 > x_1} \sum_{s_2 > x_2} \binom{s_1}{x_1}\binom{s_2}{x_2} \big[P_{n-1}(s_1, s_2, 0) \\ &\quad \times (a_1)^{s_1 - x_1}(1-a_1)^{x_1}(b_2)^{s_2 - x_2}(1-b_2)^{x_2} \\ &\quad + P_{n-1}(s_1, s_2, 1) \\ &\quad \times (a_3)^{s_1 - x_1}(1-a_3)^{x_1}(b_3)^{s_2 - x_2}(1-b_3)^{x_2}\big], \end{aligned}$$

where the initial condition is given by

$$P_0(x_1, x_2, h) = \begin{cases} 1, & (x_1, x_2, h) = (x^1_0, x^2_0, 1), \\ 0, & \text{otherwise}. \end{cases}$$

By Lemma 1, we obtain the following proposition and its proof can be found in the Appendix.

Proposition 3. *The joint distribution of $(T_1, W^1_{T_1})$ is given by*

$$\begin{aligned} &P\{(T_1, W^1_{T_1}) = (n, x)\} \\ &\quad = \sum_{x_2} P_{n-1}(x^1_0 - x, x_2, 0)(1-a_1)^{x^1_0 - x} \\ &\qquad + \sum_{x_2} P_{n-1}(x^1_0 - x, x_2, 1)(1-a_3)^{x^1_0 - x}. \end{aligned}$$

We remark that due to the symmetric property of the two sectors, the joint distribution $(W^2_{T_2}, T_2)$ shares a similar form of $(W^1_{T_1}, T_1)$.

3.2. Parameter estimation

In the two-sector model, there are eight parameters: a_0, a_1, a_2, a_3 and b_0, b_1, b_2 and b_3. We employ the maximum likelihood method to estimate the parameters. Given the total bonds n_1, n_2 and the observations of the number of defaulted bonds $y^1_0, y^1_1, \ldots, y^1_N$ and $y^2_0, y^2_1, \ldots, y^2_N$, where N denotes the period of observation time, the number of surviving bonds $x^1_0, x^1_1, \ldots, x^1_N$ and $x^2_0, x^2_1, \ldots, x^2_N$ are deterministic.

The following proposition gives analytical expressions for the maximum likelihood estimates of the model parameters.

Proposition 4. *For $i = 0, 1, 2, 3$,*

$$\hat{a}_i = \frac{\sum_{t=0}^{N-1} y^1_{t+1} h_i(y^1_t, y^2_t)}{\sum_{t=0}^{N-1} x^1_t h_i(y^1_t, y^2_t)} \quad \text{and}$$

$$\hat{b}_i = \frac{\sum_{t=0}^{N-1} y^2_{t+1} h_i(y^2_t, y^1_t)}{\sum_{t=0}^{N-1} x^2_t h_i(y^2_t, y^1_t)}.$$

Proof. We prove the expression for $\hat{a}_0$ here and the proof for the others are similar. The likelihood function $L(a, b \mid x_0, x_1, \dots, x_N, y_0, y_1, \dots, y_N)$ is then the joint probability density function $f(x_0, x_1, \dots, x_N, y_0, y_1, \dots, y_N \mid a, b)$:

$$\begin{aligned}
&L(a, b \mid x_0, x_1, \dots, x_N, y_0, y_1, \dots, y_N) \\
&\quad = f(x_0, x_1, \dots, x_N, y_0, y_1, \dots, y_N \mid a, b) \\
&\quad = \binom{x_0^1}{x_1^1} \big(1 - a(y_0)\big)^{x_1^1} a(y_0)^{y_1^1} \\
&\qquad \times \binom{x_0^2}{x_1^2} \big(1 - b(y_0)\big)^{x_1^2} b(y_0)^{y_1^2} \\
&\qquad \times \binom{x_1^1}{x_2^1} \big(1 - a(y_1)\big)^{x_2^1} a(y_1)^{y_2^1} \\
&\qquad \times \binom{x_1^2}{x_2^2} \big(1 - b(y_1)\big)^{x_2^2} b(y_1)^{y_2^2} \times \cdots \\
&\qquad \times \binom{x_{N-1}^1}{x_N^1} \big(1 - a(y_{N-1})\big)^{x_N^1} a(y_{N-1})^{y_N^1} \\
&\qquad \times \binom{x_{N-1}^2}{x_N^2} \big(1 - b(y_{N-1})\big)^{x_N^2} b(y_{N-1})^{y_N^2}.
\end{aligned}$$

Then by solving

$$\frac{\partial \ln L(a, b \mid x_0, x_1, \dots, x_N, y_0, y_1, \dots, y_N)}{\partial a_0} = 0,$$

we have

$$-\sum_{t=0}^{N-1} \frac{x_{t+1}^1 h_0(y_t^1, y_t^2)}{1 - a(y_t)} + \sum_{t=0}^{N-1} \frac{y_{t+1}^1 h_0(y_t^1, y_t^2)}{a(y_t)} = 0.$$

Since for any t,

$$\frac{1}{1 - a(y_t)} = \sum_{i=0}^{3} \frac{h_i(y_t^1, y_t^2)}{1 - a_i} \quad \text{and} \quad \frac{1}{a(y_t)} = \sum_{i=0}^{3} \frac{h_i(y_t^1, y_t^2)}{a_i}$$

we have

$$\begin{aligned}
0 &= -\sum_{t=0}^{N-1} \sum_{i=0}^{3} \frac{x_{t+1}^1 h_0(y_t^1, y_t^2) h_i(y_t^1, y_t^2)}{1 - a_i} \\
&\quad + \sum_{t=0}^{N-1} \sum_{i=0}^{3} \frac{y_{t+1}^1 h_0(y_t^1, y_t^2) h_i(y_t^1, y_t^2)}{a_i} \\
&= -\sum_{t=0}^{N-1} \frac{x_{t+1}^1 h_0(y_t^1, y_t^2)}{1 - a_0} \\
&\quad + \sum_{t=0}^{N-1} \frac{y_{t+1}^1 h_0(y_t^1, y_t^2)}{a_0}.
\end{aligned}$$

Thus we obtain

$$\hat{a}_0 = \frac{\sum_{t=0}^{N-1} y_{t+1}^1 h_0(y_t^1, y_t^2)}{\sum_{t=0}^{N-1} x_t^1 h_0(y_t^1, y_t^2)}. \qquad \square$$

4. Crisis VaR and crisis ES

In this section, we give a brief introduction to the concepts of the CRVaR and the CRES in Ching et al. [5,6]. Then we present the evaluation of the CRVaR and the CRES using the proposed models. The CRVaR and the CRES are measures for the duration and the severity of a default crisis. Let

$$L(\cdot, \cdot)(\omega) : \mathcal{T} \times \mathbb{R} \times \Omega \to \mathbb{R}$$

be a real-valued function $L(T, W_T)(\omega)$ of T and W_T. We then suppose that for a fixed $\omega \in \Omega$,

$$\begin{aligned}
&T(\omega) = t, \qquad W_t(\omega) = w, \quad \text{and} \\
&L(t, w)(\omega) = l(t, w) \in \mathbb{R}.
\end{aligned}$$

That is, the loss from the default crisis is $l(t, w)$ when the duration of default crisis $T = t$ and the number of defaulted bonds in the crisis $W_t = w$. We write $\mathcal{L}(T, W_T)$ for the space of all loss functions $L(T, W_T)(\omega)$ generated by T and W_T.

The CRVaR with probability level β under P is then defined as a functional $V_\beta(\cdot) : \mathcal{L}(T, W_T) \to \mathbb{R}$ such that for each $L(T, W_T) \in \mathcal{L}(T, W_T)$,

$$\begin{aligned}
&V_\beta\big(L(T, W_T)\big) \\
&\quad := \inf\big\{l \in \mathbb{R} \big| P\big(L(T, W_T) > l\big) \leqslant \beta\big\}. \qquad (7)
\end{aligned}$$

In the language of statistics, $V_\beta(L(T, W_T))$ is the generalized β-quantile of the distribution of the loss variable $L(T, W_T)$ under P. Since the loss from the default crisis $L(T, W_T)$ is completely determined when T and W_T are given, $P(L(T, W_T) > l)$ is completely determined by the joint p.d.f. of W_T and T.

The CRES with probability level β under P is also defined as a functional $E_\beta(\cdot) : \mathcal{L}(T, W_T) \to R$ such that for each $L(T, W_T) \in L(T, W_T)$,

$$E_\beta\big(L(T, W_T)\big) := E_P\big[L(T, W_T) \mid L(T, W_T) \geqslant V_\beta\big(L(T, W_T)\big)\big]. \tag{8}$$

In other words, $E_\beta(L(T, W_T))$ is the average of the loss from the default crisis when the loss exceeds the CRVaR of the default crisis with probability level β under P.

5. Numerical experiments

In this section, we present the empirical results of the proposed two-sector model using real default data extracted from the figures in Giampieri et al. [11], where we adopt the estimation methods and techniques presented in the previous section.

The default data comes from four different sectors. They include consumer/service sector, energy and natural resources sector, leisure time/media sector and transportation sector. Table 1 shows the default data taken from Giampieri et al. [11]. From the table, the proportions of defaults for Consumer, Energy, Media and Transport are 24.1%, 16.9%, 20.5% and 21.0%, respectively. The default probabilities of all four sectors are significantly greater than zero. This means that the default risk of each of the four sectors is substantial.

We then construct the infectious disease model using these real data. The asterisk "*" in the table indicates the pair of sectors which has the largest correlation. From Table 2, we see that all correlations are positive. This provides some preliminary evidence for supporting the use of the two-sector model from the perspective of descriptive statistical analysis. We shall provide more empirical evidence for supporting the use of the proposed infectious model by the results of BIC later in this section. To build the infectious model, for each row (Sector A), we may find a partner (Sector B) by searching the one with the largest correlation in magnitude (i.e., the one with the asterisk "*"). Figure 1 gives the partner relations among the sectors using correlation. Later in this section, we shall give the results for BIC to support the matched pair presented in Fig. 1. The estimation results for proposed infectious model and two-sector model studied in Ching et al. [5] are presented in Table 3.

Table 1

The default data (taken from Giampieri et al. [11])

Sectors	Total	Defaults
Consumer	1041	251
Energy	420	71
Media	650	133
Transport	281	59

Table 2

Correlations of the sectors

	Consumer	Energy	Media	Transport
Consumer	–	0.0224	0.6013*	0.3487
Energy	0.0224	–	0.1258*	0.1045
Media	0.6013*	0.1258	–	0.3708
Transport	0.3487	0.1045	0.3708*	–

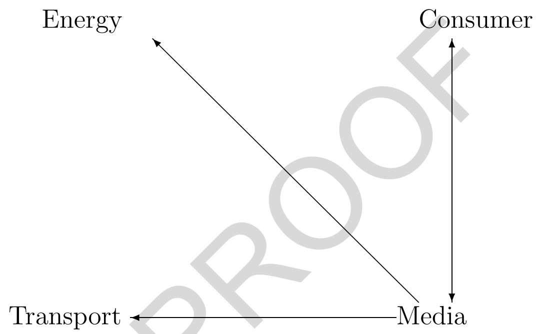

Fig. 1. The partner relations among the sectors using correlation.

To compare the proposed infectious model with the two-sector model in Ching et al. [5], we consider the Bayesian information criterion(BIC), which is also named as Schwarz criterion. The formula for the BIC is given by

$$\mathrm{BIC} = -2\log(L) + k\log(m),$$

where m is the number of observation data, k is the number of free parameters to be estimated, and L is the maximized value of the likelihood function for the estimated model. Given any two estimated models, the smaller the value of BIC is, the better the model will be. Table 4 presents the value of the BIC for the proposed model and the two-sector model in Ching et al. [5]. We remark that for all the four sectors, the proposed model with lower value of BIC is statistically better.

To compare the matched pairs in Fig. 1 with other matched pairs for the proposed model, we also adopt the BIC. Since the models of different matched pairs have the same number of parameters and length of data set, to compare their BIC is equivalent to compare their

Table 3
Estimation results for proposed model

Sector A:	Consumer	Energy	Media	Transport
Sector B:	Media	Media	Consumer	Media
Proposed model				
a_0	0.0007	0.0004	0.0005	0.0013
a_1	0.0018	0.0033	0.0005	0.0012
a_2	0.0013	0.0018	0.0017	0.0026
a_3	0.0049	0.0032	0.0042	0.0052
Two-sector model [5]				
α_0	0.0013	0.0018	0.0005	0.0013
α_1	0.0043	0.0023	0.0033	0.0036

Table 4
The value of BIC for proposed model and two-sector model [5]

Sector A:	Consumer	Energy	Media	Transport
Sector B:	Media	Media	Consumer	Media
BIC (proposed model)	419.0813	215.4654	301.2534	2.1287
BIC (two-sector model [5])	434.6700	231.8225	321.0501	2.1460

Table 5
The value of BIC for matched pairs in Fig. 1 and other matched pairs

Matched pairs in Fig. 1				
Sector A:	Consumer	Energy	Media	Transport
Sector B:	Media	Media	Consumer	Media
Log-likelihood ratio	12.2717	12.6559	14.3757	4.4860
Other matched pairs				
Sector A:	Consumer	Energy	Media	Transport
Sector B:	Energy	Consumer	Energy	Consumer
Log-likelihood ratio	33.1330	7.3286	18.6264	1.9942
Sector A:	Consumer	Energy	Media	Transport
Sector B:	Transport	Transport	Transport	Energy
Log-likelihood ratio	10.7231	7.3495	14.6136	8.4934

log-likelihood ratio. Table 5 presents the log-likelihood ratios for the matched pairs in Fig. 1 against other matched pairs. We remark that all the log-likelihood ratios are positive which support the matched pairs in Fig. 1 for the proposed model.

Our proposed model aims at modeling causality of defaults in both direction. From the pair up results, one may find that the relation is not necessarily symmetric. This relation is only found symmetric for the sectors media and consumer, which means the causality of defaults from both direction is more reasonable for the media and consumer sector.

We provide a scatter plot to depict the correlation of defaults in the matched sectors. A simulation of defaults in matched sectors in our proposed model is also conducted. Figure 2 presents the number of surviving bonds in the matched sectors of empirical data and simulation.

To apply the two measures CRVaR and CRES in the proposed model, we consider some hypothetical values for the loss. The loss $L(W_T, T)$, for each $T = 1, 2, \ldots, X_0$ and $W_T = 0, 1, \ldots, X_0$, are as in Eq. (9). Then we present the value of CRVaR and CRES for the proposed model as well as the two-sector model Ching et al. [5] in Table 6. And the loss distribution are presented in Fig. 3.

$$\begin{cases} L(0, j) = j - 1 + 0.1, \\ \quad \text{for each } j = 1, 2, \ldots, X_0; \\ L(i, j) = L(0, j) + i - 1, \\ \quad \text{for each } i = 1, 2, \ldots, X_0 \\ \quad \text{and } j = 1, \ldots, X_0. \end{cases} \tag{9}$$

From Table 6, we see that for all of the four sectors, the existing two-sector model underestimates both the CRES and CRVaR. This reflects that failure to incorporate the contagion effect described in our proposed model leads to an underestimation of credit risk. This has an important consequence for credit risk management, such as inadequate capital charges for credit portfolios. Indeed, the loss distribution implied by the proposed model has a much fatter tail than that arising from the existing two-sector model. This explains why the proposed model provides more prudent estimates for the risk measures than the existing two-sector model via incorporating contagion. We also remark that the contagion model including the causality of defaults in both direction (i.e., looping defaults), has a significant impact on the loss distribution.

6. Concluding remarks

In this paper, we propose a general model framework for discrete-time default risk where default processes for all the entities are governed by predictable default probabilities. Existing literature [5,6,12] serve as our special cases. We give a general formula for the joint distribution of two important random variables featuring the severity of the crisis, i.e., the duration of crisis (T) and severity of the defaults (W_T). We propose a two-sector Markovian infectious model as a special case of the general framework. The proposed model incorporated two important features of credit contagion, namely, the chain reactions of defaults and the bi-lateral causality of defaults between

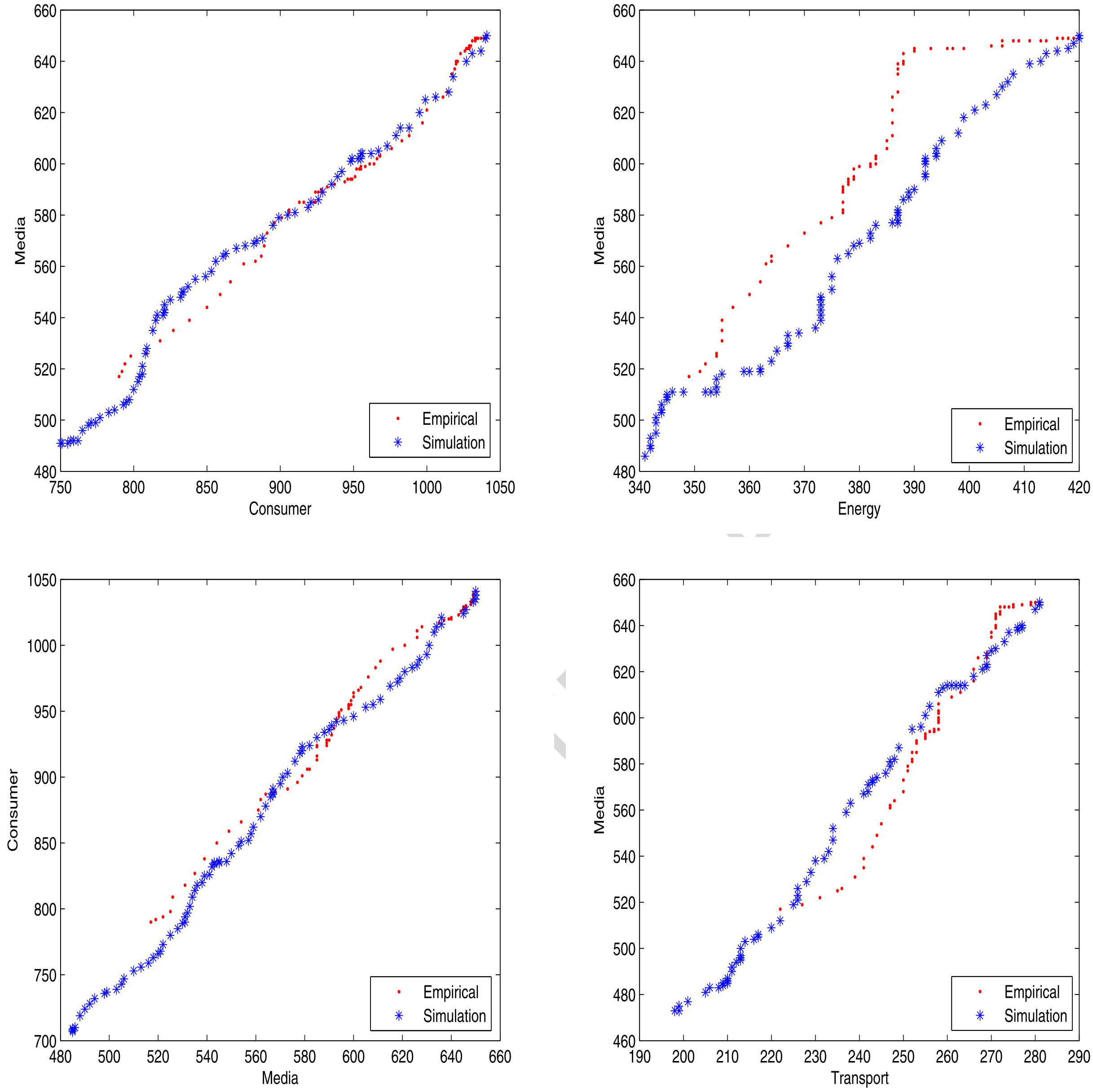

Fig. 2. Number of Surviving Bonds in Matched Sectors.

two industrial sectors. We capture the chain reactions of defaults by postulating that the future default probability switches over time according to the current number of defaults of two industrial sectors. The bi-lateral causality of defaults means that defaults in one sector are caused by defaults in another sector, and vice versa. This bi-lateral causality of defaults enriches the dependent structures of credit risk model. We provide an efficient estimation method of the proposed model based on the maximum likelihood estimation. Two important risk measures, namely, the CRVaR and the CRES, are evaluated under the proposed model.

We also conduct empirical studies on the credit risk models using real default data. We adopted the BIC to compare the proposed model with the existing two-sector model proposed in Ching et al. [5]. The numerical results reveal that the proposed two-sector model outperforms empirically the existing model. By comparing the risk measures evaluated from the proposed model and those evaluated from the existing two-sector

Table 6

CRVaR and CRES

Sector A:	Consumer	Energy	Media	Transport
Sector B:	Media	Media	Consumer	Media
		Proposed model		
CRVaR($\beta = 0.05$)	374.1	25.1	122.1	26.1
CRES($\beta = 0.05$)	424.7	33.8	150.4	33.8
CRVaR($\beta = 0.01$)	457.1	39.1	168.1	39.1
CRES($\beta = 0.01$)	495.1	47.5	192.4	46.5
		Two-sector model [5]		
CRVaR($\beta = 0.05$)	114.1	12.1	34.1	10.10
CRES($\beta = 0.05$)	146.1	17.1	45.7	14.1
CRVaR($\beta = 0.01$)	166.1	20.1	52.1	16.1
CRES($\beta = 0.01$)	195.6	24.5	63.3	20.2

model, we find that failure to incorporate the contagion effect described in the proposed model leads to an underestimation of risk measures. This provides some evidence to support the proposed model.

One possible topic for future research is to incorporate the impact of the number of defaults on the likelihood of future defaults via a different parametrization of the future default probability. In the current paper, we assume that the joint future default probability switches over time depending on the region where the current number of defaults falls in. Four parameters, namely, a_0, a_1, a_2 and a_3 were involved. To provide a more parsimonious way to incorporate the current number of defaults on the joint future default probability, one may consider the following parametrization for the default probability:

$$\alpha_t = a_0 + a_1 y_t^1 + a_2 y_t^2,$$

where y_t^1 and y_t^2 are the current numbers of defaults in the two industrial sectors. Using this parametrization, we can reduce the number of parameters by one and accounts for more information of the current number of defaults when evaluating the future default probability.

Acknowledgements

The authors would like to thank the anonymous referee and the editor for their helpful and constructive comments. This research work was supported by Research Grants Council of Hong Kong under Grant Number 17301214.

Appendix

A.1. Proof of Lemma 1

By the law of total probability and Markov property,

$$\begin{aligned}
&P_n(x_1, x_2, 0)\\
&\quad = P\{T_1 \geqslant n+1, X_n^1 = x_1,\\
&\qquad X_n^2 = x_2, I_{\{Y_n^2>0\}} = 0\}\\
&\quad = \sum_{s_1>x_1} \sum_{h=0,1} P\{T_1 \geqslant n, X_{n-1}^1 = s_1,\\
&\qquad X_{n-1}^2 = x_2, I_{\{Y_{n-1}^2>0\}} = h\}\\
&\qquad \times P\{T_1 \geqslant n+1, X_n^1 = x_1,\\
&\qquad X_n^2 = x_2, I_{\{Y_n^2>0\}} = 0 \mid T_1 \geqslant n, X_{n-1}^1 = s_1,\\
&\qquad X_{n-1}^2 = x_2, I_{\{Y_{n-1}^2>0\}} = h\}\\
&\quad = \sum_{s_1>x_1} \sum_{h=0,1} P_{n-1}(s_1, x_2, h)\\
&\qquad \times P\{Y_n^1 > 0, X_n^1 = x_1,\\
&\qquad X_n^2 = x_2, I_{\{Y_n^2>0\}} = 0 \mid T_1 \geqslant n, X_{n-1}^1 = s_1,\\
&\qquad X_{n-1}^2 = x_2, I_{\{Y_{n-1}^2>0\}} = h\}\\
&\quad = \sum_{s_1>x_1} \sum_{h=0,1} P_{n-1}(s_1, x_2, h)\\
&\qquad \times P\{Y_n^1 > 0, X_n^1 = x_1,\\
&\qquad X_n^2 = x_2, I_{\{Y_n^2>0\}} = 0 \mid Y_{n-1}^1 > 0, X_{n-1}^1 = s_1,\\
&\qquad X_{n-1}^2 = x_2, I_{\{Y_{n-1}^2>0\}} = h\}\\
&\quad = \sum_{s_1>x_1} \binom{s_1}{x_1} [P_{n-1}(s_1, x_2, 0)\\
&\qquad \times (a_1)^{s_1-x_1}(1-a_1)^{x_1}(1-b_2)^{x_2}\\
&\qquad + P_{n-1}(s_1, x_2, 1)\\
&\qquad \times (a_3)^{s_1-x_1}(1-a_3)^{x_1}(1-b_3)^{x_2}].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&P_n(x_1, x_2, 1)\\
&\quad = P\{T_1 \geqslant n+1, X_n^1 = x_1,\\
&\qquad X_n^2 = x_2, I_{\{Y_n^2>0\}} = 1\}
\end{aligned}$$

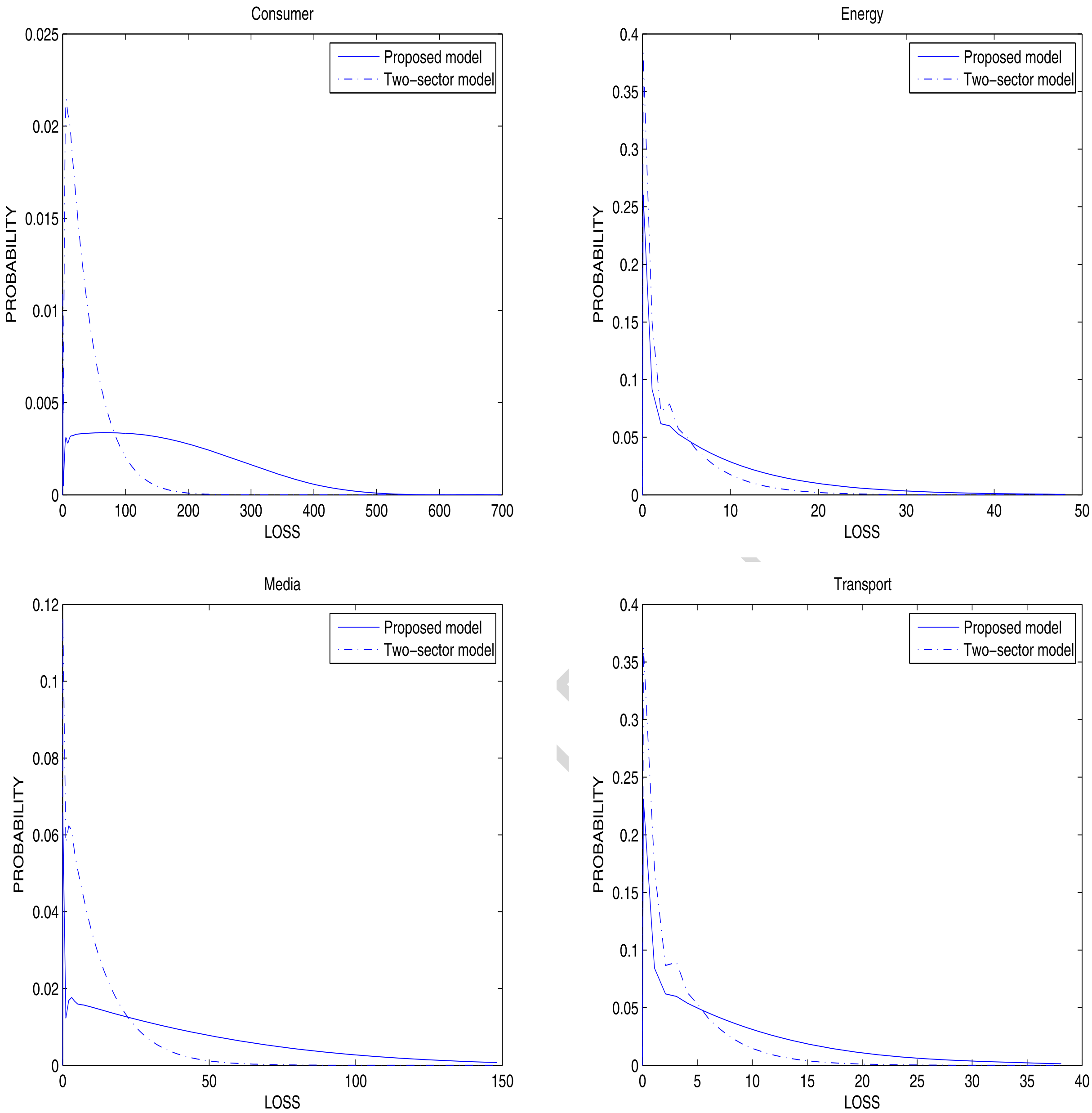

Fig. 3. Loss distribution for proposed model and two-sector model Ching et al. [5].

$$
\begin{aligned}
&= \sum_{s_1 > x_1} \sum_{s_2 > x_2} \sum_{h=0,1} P\big\{T_1 \geqslant n, X^1_{n-1} = s_1,\\
&\quad X^2_{n-1} = s_2, I_{\{Y^2_{n-1} > 0\}} = h\big\}\\
&\quad \times P\big\{T_1 \geqslant n+1, X^1_n = x_1,\\
&\quad X^2_n = x_2, I_{\{Y^2_n > 0\}} = 1 \mid T_1 \geqslant n, X^1_{n-1} = s_1,\\
&\quad X^2_{n-1} = s_2, I_{\{Y^2_{n-1} > 0\}} = h\big\}\\
&= \sum_{s_1 > x_1} \sum_{s_2 > x_2} \sum_{h=0,1} P_{n-1}(s_1, s_2, h)\\
&\quad \times P\big\{Y^1_n > 0, X^1_n = x_1,\\
&\quad X^2_n = x_2, I_{\{Y^2_n > 0\}} = 1 \mid T_1 \geqslant n, X^1_{n-1} = s_1,
\end{aligned}
$$

$$
\begin{aligned}
& X_{n-1}^2 = s_2, I_{\{Y_{n-1}^2>0\}} = h\big\} \\
&= \sum_{s_1>x_1}\sum_{s_2>x_2}\sum_{h=0,1} P_{n-1}(s_1, s_2, h) \\
&\quad \times P\big\{Y_n^1 > 0, X_n^1 = x_1, \\
&\quad X_n^2 = x_2, I_{\{Y_n^2>0\}} = 1 \mid Y_{n-1}^1 > 0, X_{n-1}^1 = s_1, \\
&\quad X_{n-1}^2 = s_2, I_{\{Y_{n-1}^2>0\}} = h\big\} \\
&= \sum_{s_1>x_1}\sum_{s_1>x_1} \binom{s_1}{x_1}\binom{s_2}{x_2}\big[P_{n-1}(s_1, s_2, 0) \\
&\quad \times (a_1)^{s_1-x_1}(1-a_1)^{x_1}(b_2)^{s_2-x_2}(1-b_2)^{x_2} \\
&\quad + P_{n-1}(s_1, s_2, 1) \\
&\quad \times (a_3)^{s_1-x_1}(1-a_3)^{x_1}(b_3)^{s_2-x_2}(1-b_3)^{x_2}\big].
\end{aligned}
$$

A.2. Proof of Proposition 3

$$
\begin{aligned}
& P\big\{\big(T_1, W_{T_1}^1\big) = (n, x)\big\} \\
&= P\big\{T_1 \geqslant n, Y_n^1 = 0, X_n^1 = x_0^1 - x\big\} \\
&= \sum_{x_2}\sum_{h=0,1} P\big\{T_1 \geqslant n, X_{n-1}^1 = x_0^1 - x, \\
&\quad X_{n-1}^2 = x_2, I_{\{Y_{n-1}^2>0\}} = h\big\} \\
&\quad \times P\big\{Y_n^1 = 0, X_n^1 = x_0^1 - x \mid \\
&\quad T_1 \geqslant n, X_{n-1}^1 = x_0^1 - x, \\
&\quad X_{n-1}^2 = x_2, I_{\{Y_{n-1}^2>0\}} = h\big\} \\
&= \sum_{x_2}\sum_{h=0,1} P_{n-1}\big(x_0^1 - x, x_2, h\big) \\
&\quad \times P\big\{Y_n^1 = 0, X_n^1 = x_0^1 - x \mid \\
&\quad Y_{n-1}^1 > 0, X_{n-1}^1 = x_0^1 - x, \\
&\quad X_{n-1}^2 = x_2, I_{\{Y_{n-1}^2>0\}} = h\big\} \\
&= \sum_{x_2} P_{n-1}\big(x_0^1 - x, x_2, 0\big)(1-a_1)^{x_0^1-x} \\
&\quad + \sum_{x_2} P_{n-1}\big(x_0^1 - x, x_2, 1\big)(1-a_3)^{x_0^1-x}.
\end{aligned}
$$

References

[1] T. Bielecki, A. Cousin and S. Crépey, Dynamic hedging of portfolio credit risk in a Markov copula model (Previous title: Dynamic modeling of portfolio credit risk with common shocks), Working papers in economics, 2011.

[2] T. Bielecki, A. Cousin, S. Crépey et al., Dynamic hedging of portfolio credit risk in a Markov copula model, *Journal of Optimization Theory Applications* **161** (2014), 90–102. doi:10.1007/s10957-013-0318-4.

[3] F. Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy* **81** (1973), 637–654. doi:10.1086/260062.

[4] F. Carapella, Banking panics and deflation in dynamic general equilibrium, FEDS Working Paper 2015-018, 2015. doi:10.17016/FEDS.2015.018.

[5] W. Ching, H. Leung, H. Jiang, L. Sun and T. Siu, A Markovian network model for default risk management, *International Journal of Intelligent Engineering Informatics* **1** (2010), 104–124. doi:10.1504/IJIEI.2010.033532.

[6] W. Ching, T. Siu, L. Li, T. Li and W. Li, On an infectious model for default crisis, Preprint, 2008, http://www.hku.hk/math/~imr/IMRPrePrintSeries/2007/IMR2007-21.pdf.

[7] J. Crook and F. Moreira, Checking for asymmetric default dependence in a credit card portfolio: A copula approach, *Journal of Empirical Finance* **18** (2011), 728–742. doi:10.1016/j.jempfin.2011.05.005.

[8] D. Daley and J. Gani, *Epidemic Modeling: An Introduction*, Cambridge University Press, 1999.

[9] M. Davis and V. Lo, Infectious defaults, *Quantitative Finance* **1** (2001), 382–387. doi:10.1080/713665832.

[10] Y. Dong and G. Wang, The dependence of assets and default threshold with thinning-dependence structure, *Journal of Industrial and Management Optimization* **8** (2012), 391–410. doi:10.3934/jimo.2012.8.391.

[11] G. Giampieri, M. Davis and M. Crowder, Analysis of default data using hidden Markov models, *Quantitative Finance* **5** (2005), 27–34. doi:10.1080/14697680500039951.

[12] J. Gu, W. Ching and T. Siu, A Markovian infectious model for dependent default risk, *International Journal of Intelligent Engineering Informatics* **1** (2011), 174–195. doi:10.1504/IJIEI.2011.040178.

[13] E. Hillebrand, A. Sengupta and J. Xu, Temporal correlation of defaults in subprime securitization, *Communications on Stochastic Analysis* **6** (2012), 487–511.

[14] R. Jarrow and S. Turnbull, Pricing derivatives on financial securities subject to credit risk, *Journal of Finance* **50** (1995), 53–85. doi:10.1111/j.1540-6261.1995.tb05167.x.

[15] R. Jarrow and F. Yu, Counterparty risk and the pricing of defaultable securities, *Journal of Finance* **56** (2001), 1765–1799. doi:10.1111/0022-1082.00389.

[16] M. Kijima, Monotonicities in a Markov chain model for valuing corporate bonds subject to credit risk, *Mathematical Finance* **8** (1998), 229–247. doi:10.1111/1467-9965.00054.

[17] D. Li, On default correlation: A copula function approach, *Journal of Fixed Income* **9** (2000), 43–54. doi:10.3905/jfi.2000.319253.

[18] S. Lu, Comparing the reliability of a discrete-time and a continuous-time Markov chain model in determining credit risk, *Applied Economics Letters* **16** (2009), 1143–1148. doi:10.1080/13504850701349153.

[19] R. Merton, On the pricing of corporation debt: The risk structure of interest rates, *Journal of Finance* **29** (1974), 449–470.

[20] F. Milne, Credit crises, risk management systems and liquidity modelling, Working paper, 2008.

[21] S. Mitra, A risk management framework and model for pension investment funds, *International Journal of Business Continuity Risk Management* **1** (2010), 301–316. doi:10.1504/IJBCRM.2010.038621.

[22] A. Saunders and L. Allen, *Credit Risk Measurement in and out of the Financial Crisis: New Approaches to Value-at-Risk and Other Paradigms*, 3rd edn, Wiley, New York, 2010.

[23] R. Savona and M. Vezzoli, Multidimensional distance-to-collapse point and sovereign default prediction, *Intelligent Systems in Accounting, Finance and Management* **19** (2012), 205–228.